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A note on a multiplicative function

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An ordered set of integers $a_1, \ldots, a_k$ is called a *k-vector* and is denoted by $\{a_i\}$. The set of all k-vectors $\{a_i\}$ where we take $a_i (\mathrm{mod}\, n)$ instead of a_i, is said to *constitute a complete residue system* $(\mathrm{mod}\, k, n)$. By the scalar multiple c of the vector $\{a_i\}$ we mean the vector $\{ca_i\}$. By the greatest common divisor (g.c.d.) of a vector $\{a_i\}$ we mean the g.c.d. of the constituents and denote it by (a_i). If $d_1 = 1, \ldots, d_t = n$ are all the positive divisors of n, then the complete residue system $(\mathrm{mod}\, k, n)$ can be divided into t classes $A(d_1), \ldots, A(d_t)$ such that the class $A(d_j)$ $(j = 1, \ldots, t)$ contains all those vectors $\{a_i\}$ of the complete residue system $(\mathrm{mod}\, k, n)$ which are such that $\big((a_i), n\big) = n/d_j$. The set of elements $A(n)$ is said to *constitute a reduced residue system* $(\mathrm{mod}\, k, n)$.

The object of this note is to study certain properties of a sum associated with the arithmetic function $f(m, n)$ whose values belong to the complex field and which has the following additional properties:

(i) $f(m, n) = f(m', n)$, whenever $m \equiv m' (\mathrm{mod}\, n)$,

(ii) $f(m, n) f(m', n') = f(mn' + m'n, nn')$.

In particular, if $c^{(k)}(m, n)$ is defined as $\sum f(ms, n)$, the summation being over all $s = s_1 + \ldots + s_k$, where $\{s_i\}$ runs over all the elements of $A(n)$, then we have

THEOREM.

$$c^{(k)}(m, n) = \frac{J_k(n)}{J_k(n/g)} \mu(g, n)$$

(*here* $J_k(n)$ *is the Jordan totient which is the number of elements of* $A(n)$ *and* $\mu(m, n)$ *is defined as* $\sum f(r, n)$, *the summation being over all* $r = r_1 + \ldots + r_k$ *with* $\{r_i\}$ *running over the elements of* $A(n/g)$, g *being the g.c.d. of* m, n).

For related literature on the function $f(m, n)$ with side conditions, reference may be made to the author [3], [4] and Venkataraman [5].

LEMMA 1. *If* $(n, n') = 1$ *and the vector* $\{a_i\}$ *ranges over the class* $A(d)$ $(\mathrm{mod}\, k, n)$ *and the vector* $\{a'_i\}$ *ranges over the class* $A(d')$ $(\mathrm{mod}\, k, n')$, *then the vector* $\{a_i n' + a'_i n\}$ *generates the class* $A(dd')$ $(\mathrm{mod}\, k, nn')$.

Proof. If we set $a = \{a_i\}$ and $a' = \{a'_i\}$, then the set of all $an' + a'n$ contains $J_k(d)J_k(d') = J_k(dd')$ elements and the elements are distinct $(\bmod k, nn')$ for different a's and a''s. Also $an' + a'n$ belongs to the class $A(dd')$ $(\bmod k, nn')$. Therefore we have the lemma as stated above.

LEMMA 2. *$\mu(m, n_1n_2) = \mu(m, n_1)\mu(m, n_2)$, whenever $(n_1, n_2) = 1$.*

Proof. From the definition it follows that $\mu(m, n) = \mu(g, n)$. Now the application of Lemma 1 to the definition of $\mu(m, n)$ and the use of properties (i) and (ii) of $f(m, n)$ yield the result.

LEMMA 3. *$\mu(m_1m_2, n_1n_2) = \mu(m_1, n_1)\mu(m_2, n_2)$, whenever $(m_1n_1, m_2n_2) = 1$.*

Proof. If $(m_1, n_1) = g_1$ and $(m_2, n_2) = g_2$, then $(m_1m_2, n_1n_2) = g_1g_2$. Now the left member of the equality in the lemma is equal to $\mu(g_1g_2, n_1n_2)$, which by Lemma 2 is $\mu(g_1g_2, n_1)\mu(g_1g_2, n_2)$, and this gives the result.

Proof of the Theorem. If $d|n$, the $J_k(n)$ elements of a reduced residue system $(\bmod k, n)$ can be decomposed into $J_k(n)/J_k(d)$ reduced residue systems $(\bmod k, d)$ (see Cohen [1], Lemma 7). Therefore

$$c^{(k)}(m, n) = \frac{J_k(n)}{J_k(n/g)} \sum f(ms, n),$$

where the summation is over all $s = s_1 + \ldots + s_k$ and $\{s_i\}$ runs over a reduced residue system $(\bmod k, n/g)$. Now put $r = ms$; then $m\ \{s_i\}$ runs over the class $A(n/g)$ $(\bmod k, n)$ as $\{s_i\}$ ranges over the reduced residue system $(\bmod k, n/g)$. The rest follows from the definition of $\mu(m, n)$.

Remark 1. The sum $c^{(k)}(m, n)$ has the following multiplicative properties, which are immediate consequences of Lemmas 1 and 2 and the multiplicative property of the Jordan totient $J_k(n)$:

$$c^{(k)}(m, n_1n_2) = c^{(k)}(m, n_1)c^{(k)}(m, n_2), \quad \text{if} \quad (n_1, n_2) = 1,$$

and

$$c^{(k)}(m_1m_2, n_1n_2) = c^{(k)}(m_1, n_1)c^{(k)}(m_2, n_2) \quad \text{whenever} \quad (m_1n_1, m_2n_2) = 1.$$

Remark 2. If $f(m, n) = \exp(2\pi im/n)$, then $c^{(k)}(m, n)$ reduces to the extension of the Ramanujan sum discussed in greater detail by Cohen in [1], § 3, and in [2].

References

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