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ROZPRAWY MATEMATYCZNE

L

E. S. THOMAS, Jr.

Monotone decompositions of irreducible continua

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INTRODUCTION

In what follows, "continuum" means "bicompact connected metric space".

A continuum is *of type A* provided it is irreducible between a pair of its points and admits a monotone upper semi-continuous decomposition, $\mathscr{D}$, whose quotient space is a non-degenerate arc. Such a decomposition is called *admissible.* If M is of type A and has an admissible decomposition, each of whose elements has void interior, then we say M is *of type A'*.

In Chapter 1 of this paper we treat the following questions:

(1) What conditions on a continuum M, irreducible between a pair of its points, imply that M is of type A or A'?

(2) If M is of type A or A' and $\mathscr{D}$ is an admissible decomposition of M then what structural properties of M can be obtained from properties of $\mathscr{D}$?

Before answering either question we prove a number of results which are used as tools in investigating irreducible continua and admissible decompositions. In particular, we show that if a continuum has an admissible decomposition, then it has a unique minimal decomposition (relative to the partial ordering by refinement). We also give a characterization of the admissible decomposition in terms of monotone, continuous functions with values in the unit interval.

Turning to question (1) we show that M is of type A' if and only if every subcontinuum of M with non-void interior is the union of two proper subcontinua (i.e. decomposable). Other conditions, sufficient for M to be of type A, are obtained by studying the collection of indecomposable subcontinua of M.

At this point several theorems relating to question (2) have already been obtained. More structure theorems are obtained by introducing the notion of aposyndicity. Indeed, using this notion, we completely characterize the elements of the minimal admissible decomposition and, in the process, get another answer to question (1).

As applications of the decomposition theory, we obtain some (known) characterizations of arcs and simple closed curves.

In Chapter 2 we investigate the class of those continua which are hereditarily of type A'. Such a continuum, M, can be successively decomposed and the pieces reassembled using inverse limits to obtain a new space, M_∞, which serves as an approximation to M. We show that M_∞ is metric if and only if it is Hausdorff, in which case M and M_∞ are homeomorphic. A necessary and sufficient condition for M_∞ to be Hausdorff is exhibited.

We next study the extent to which M_∞ characterizes M. The approach used here is to assume that M and N are continua hereditarily of type A' for which a homeomorphism exists between M_∞ and N_∞, and then to find conditions on M which imply that M and N are homeomorphic.

Theorems 11 and 12 tell how to construct continua hereditarily of type A' from other such continua using inverse limits, and this yields a continuum with very interesting properties.

We next show how to replace a decomposition element, D, of a given continuum, M, with another continuum, without disturbing $M - D$. The machinery needed for this also has applications at the end of Chapter 3.

In Chapter 3 we continue the study of the minimal decomposition, $\mathscr{D}$, of a continuum of type A', the emphasis here being on continuity properties.

To begin with, we show that $\mathscr{D}$ is continuous almost everywhere. An example, due to R. H. Bing and F. B. Jones, [4], shows that $\mathscr{D}$ may be continuous everywhere. The continuum of this example is snake-like and each element of its decomposition is indecomposable. By contrast, we show that if M is of type A', snake-like, and does not contain small indecomposable subcontinua, then, near every point where M is not locally connected, the minimal admissible decomposition for M is very discontinuous.

The main tool in establishing the above result is that if M is of type A', snake-like and does not contain small indecomposable subcontinua, then, joining any two open sets in M, there is a subcontinuum having a composant whose complement is a single point. This generalizes a result of G. W. Henderson, [6], and a result of L. K. Barrett, [1].

Finally, we develop a notion of "sidedness" for continua hereditarily of type A'. Besides being intrinsically interesting, this yields information on the structure of the minimal decomposition from an external point of view.

In Kuratowski's book, [12], will be found an investigation of irreducible continua; the development found there is carried in a different manner from ours and in some cases utilizes auxiliary results of a general nature. It is natural that there be some overlapping of results; perhaps the most interesting common result is our Theorem 10 of Chapter 1 and Theorem 3 on page 153 of [12]. In [12] can be found a bibliography of contributions to the theory developed there; we shall make our treatment as self-contained as possible.

With a few exceptions, which we will point out as they occur, the material presented in Chapters 2 and 3 is new.

The basic definitions and concepts used herein will be found in [7] and [10]. We shall assume a familiarity with the general topological concepts as presented in the first three chapters of Kelley, [10], and shall use several well-known results concerning compact metric continua which can be found in R. L. Moore's book on point-set topology, [14].

We use standard set-theoretic notation except that the void set is denoted by $\emptyset$, the union of the sets $A_1, A_2, \ldots, A_n$ is denoted by $A_1+A_2+ + \ldots + A_n$, and $A \subset B$ does not exclude the possibility that $A = B$.

The termination of a proof will be signified by two vertical lines, $\|$.

If A is a subset of a topological space S, then the closure of A in S, the interior of A in S and the boundary of A in S are denoted by $\mathrm{cl}_S(A)$, $\mathrm{int}_S(A)$ and $\partial_S(A)$, respectively, or by $\bar{A}$, A°, ∂A if it is not necessary to display S.

"Subcontinuum of" means "closed, connected subset of". A continuum is irreducible between two of its points provided no proper subcontinuum contains both points. A continuum is irreducible between two of its subsets provided no proper subcontinuum intersects both sets.

When we write "Let $M = P+Q$ be a decomposition of M ..." we mean that the continuum M is the union of the two proper subcontinua P and Q.

We also use "decomposition" in another sense. A decomposition, $\mathscr{D}$, of a set S is a collection of subsets of S whose union is S such that $D \in \mathscr{D}$, $E \in \mathscr{D}$ implies $D \cap E = \emptyset$ or $D = E$. Which of these meanings we intend will be clear from the context. We shall use script letters for decompositions of the latter sort.

Let $\mathscr{D}$ be a decomposition of the space S; $\mathscr{D}$ is *upper semi-continuous* provided that if U is open in S and contains $D \in \mathscr{D}$, then some open subset of U contains D and is the union of elements of $\mathscr{D}$. The function which assigns to a point of S the element of $\mathscr{D}$ containing it is called the *quotient map*, and the *quotient topology* for $\mathscr{D}$ is the strongest with respect to which the quotient map is continuous (where we consider $\mathscr{D}$ as a point set with its elements as points). The topological space thus obtained is called the *quotient space*. It is easy to see that a subset of the quotient space is open if and only if its preimage under the quotient map is open in S.

By "$S-T = A+B$, separated, ..." we mean that $S-T$ is the union of the non-void, separated (relative to S) sets, A and B.

In several places in Chapters 1 and 2 we shall use a result which appears as a remark on page 149 of Kelley's book, [10].

Remark on quotient spaces. *If X is a separable metric space and $\mathfrak{D}$ is an upper semi-continuous decomposition of X each of whose elements is bicompact, then the quotient space is metrizable.*

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CHAPTER 1

A function, f, from a space S to a space T is *monotone* provided the preimage under f of a subcontinuum of T is a subcontinuum of S. It is well known that if f is continuous and S and T are compact metric, then f is monotone if and only if $f^{-1}[t]$ is connected for each $t \in T$. Also if f is monotone and if S is irreducible from x to y, then $f(S)$ is irreducible from $f(x)$ to $f(y)$.

We shall begin by recording a number of facts about irreducible continua.

THEOREM 1. *Let M be a continuum irreducible between a pair of its points x and y. Then the following hold.*

(a) *If A is a subcontinuum of M which separates M, then we can write $M-A = X+Y$, separated, where $x \in X$, $y \in Y$; in particular, A contains neither x nor y. Moreover, X and Y are connected.*

(b) *If A is a subcontinuum of M, then A° is connected.*

(c) *Suppose that A is a subcontinuum of M such that $A = \overline{A^\circ}$. If A misses x and y, then A is irreducible between two points on ∂A. If A contains $x(y)$ and misses $y(x)$, then A is irreducible from $x(y)$ to ∂A.*

Proof. (a) Suppose $M-A = B+C$, separated and non-void. It is well known that $A+B$ and $A+C$ are connected and hence are proper subcontinua of M. By irreducibility of M it follows that neither set contains both x and y and in particular A does not contain either point. Thus we can relabel B and C as X and Y (or Y and X) so that $x \in X$, $y \in Y$.

The component C of X which contains x has a limit point in A. Then $C+A+Y$ is a subcontinuum of M containing x and y and therefore $C+A+Y = M$. This implies that $C = X$, i.e. X is connected; similarly, Y is connected.

(b) If $A^\circ = \emptyset$, the result is trivial. We deal with the case $A^\circ \neq \emptyset$ and $x \notin A$, $y \notin A$. Since $A^\circ \neq \emptyset$, we can write $M-A = X+Y$, separated, where $x \in X$, $y \in Y$. By part (a), X and Y are connected. Let $B = \overline{X}$, $C = \overline{Y}$; then B and C are disjoint subcontinua of M missing A°. Let D be a subcontinuum of A irreducible from $A \cap B$ to $A \cap C$. Then, as is well known, $D-[(A \cap B)+(A \cap C)]$ is connected. Denoting this last set by E, we have that E lies in some component of A° and $\overline{E} = D$ is irreducible from

A to B. It follows by irreducibility of M that $E = A^\circ$. In particular, $\bar{E} = \overline{A^\circ}$ is irreducible between a pair of points on its boundary.

In case A contains $x(y)$ but not $y(x)$ an obvious modification of the above argument establishes the desired result and also proves that $\overline{A^\circ}$ is irreducible from $x(y)$ to ∂A.

(c) As noted, we established this in the course of proving part (b).||

Definition 1. Let M be a continuum irreducible between a pair of its points, x and y. A decomposition, $\mathscr{D}$, of M is *admissible*, if $\mathscr{D}$ has more than one element, $\mathscr{D}$ is upper semi-continuous, every element of $\mathscr{D}$ is a subcontinuum of M, and every element of $\mathscr{D}$ not containing x or y separates M.

Note that by part (a) of Theorem 1, if M is irreducible from x to y and some subcontinuum A of M separates M, then A misses x and y. Hence the last criterion for admissibility does not depend on any particular choice of x and y. It also follows that the elements of $\mathscr{D}$ containing x and y do not separate M.

Theorem 2. *If $\mathscr{D}$ is an admissible decomposition of the continuum M, irreducible from x to y, then the induced quotient space, N, is an arc.*

Proof. By the remark on quotient spaces, N is metrizable. Being the continuous image of M under the quotient map, q, N is also bicompact and connected; hence N is a continuum. Let D be an element of $\mathscr{D}$ not containing x or y and write $M-D = X+Y$, separated, where $x \in X$, $y \in Y$. Then $N-\{D\} = q(X)+q(Y)$ and each set is non-void. Now $X+D$ is closed in M so that $q(X+D)$ is closed in N (because q is necessarily a closed map) and $q(Y) = N-q(X+D)$ is open in N; similarly, $q(x)$ is open in N. Thus, with at most two exceptions, every point of N separates N. As is well known, see p. 119 of [12], this implies that N is an arc.||

Definition 2. If $\mathscr{D}$ and $\mathscr{E}$ are admissible decompositions of the continuum M, then $\mathscr{D} \leqslant \mathscr{E}$ means that every element of $\mathscr{D}$ is contained in some element of $\mathscr{E}$, i.e. $\mathscr{D}$ refines $\mathscr{E}$.

Clearly $\leqslant$ defines a partial ordering on the family of admissible decompositions.

Theorem 3. *If the continuum, M, irreducible from x to y, has an admissible decomposition, then it has one which is minimal with respect to $\leqslant$.*

Proof. Let $\{D_\alpha | \alpha \in A\}$ be a chain of admissible decompositions, and for $z \in M$ and $\alpha \in A$, let Z_α be the element of $\mathscr{D}_\alpha$ containing z. For fixed $z \in M$, $\{Z_\alpha | \alpha \in A\}$ is a chain of continua and we denote by Z the intersection of this chain. Denoting by $\mathscr{D}$ the collection $\{Z | z \in M\}$, we see that $\mathscr{D}$ is a decomposition of M, each of whose elements is a continuum. It is easy to check that if $Z \in \mathscr{D}$ does not contain x or y, then Z separates. Suppose that U is open in M and contains $Z \in \mathscr{D}$. For some $\alpha \in A$, $Z_\alpha \subset U$, and since $\mathscr{D}_\alpha$ is upper semi-continuous, some open subset V of U contains Z_α and is the union of elements of $\mathscr{D}_\alpha$. A fortiori, V contains Z and is the union of elements of $\mathscr{D}$. Thus $\mathscr{D}$ is upper semi-continuous and therefore admissible. Since $\mathscr{D}$ refines each $\mathscr{D}_\alpha$, it is a lower bound for the chain. The proof is completed by applying Zorn's lemma.||

Before proving uniqueness of the minimal admissible decomposition we shall introduce some convenient notation and establish a few relevant facts.

DEFINITION 3. A continuum, M, is of *type A* provided it is irreducible between a pair of its points and has an admissible decomposition.

In what follows let M denote a continuum of type A; let Δ denote the collection of admissible decompositions of M; let $\mathscr{F}$ denote the collection of all monotone, continuous functions which map M onto the unit interval, I, and let Φ denote the family of homeomorphisms of I onto I.

For $f \in \mathscr{F}$ let Φf denote $\{\varphi \circ f \mid \varphi \in \Phi\}$; then $\Phi f \subset \mathscr{F}$.

Notice that if N is any continuum irreducible between a pair of its points and f is a monotone, continuous function from N onto I, then $\{f^{-1}[s] \mid s \in I\}$ is an admissible decomposition of N (hence N is of type A). The only non-trivial verification needed here is that the decomposition is upper semi-continuous. This follows easily from the fact that f is closed. In particular, if M is of type A and $f \in \mathscr{F}$, then $\mathscr{D}(f) = \{f^{-1}[s] | s \in I\} \in \Delta$.

Next notice that if $\mathscr{D} \in \Delta$, then there is an $f \in \mathscr{F}$ such that $\mathscr{D}(f) = \mathscr{D}$. This is essentially Theorem 1; for by that theorem there is a homeomorphism, h, of the quotient space onto I and we take f to be $h \circ q$ where q is the quotient map.

THEOREM 4. *The equation $w(\mathscr{D}) = \Phi f$, where f is any member of $\mathscr{F}$ for which $\mathscr{D} = \mathscr{D}(f)$, defines a function, w, from Δ into $\{\Phi f | f \in F\}$ which is one-to-one, and onto.*

Proof. Suppose that for $f \in \mathscr{F}$, $g \in \mathscr{F}$ we have $\mathscr{D}(f) = \mathscr{D}(g)$. It follows that the equation $\varphi(S) = f(g^{-1}[s])$ defines a function from I into I. Since f is onto, φ is onto; also φ is one-to-one since it has a well-defined inverse, $\varphi^{-1}[s] = g(f^{-1}[s])$. If F is closed in I, then $f^{-1}[F]$ is closed in M (because f is continuous) and therefore $\varphi^{-1}[F] = g(f^{-1}[F])$ is closed (g is a closed map). Thus φ is continuous and therefore a homeomorphism, i.e. $\varphi \in \Phi$. By construction, $f = \varphi \circ g$ and this implies $\Phi f \subset \Phi g$; dually, $\Phi g \subset \Phi f$ so that $\Phi f = \Phi g$. This shows that w is well defined.

That w is onto is the content of the second remark preceding this theorem.

Finally, w is one-to-one because, if $\mathscr{D}$ and $\mathscr{E}$ are in Δ, then $w(\mathscr{D}) = w(\mathscr{E})$ means that there is f in $\mathscr{F}$ such that $\mathscr{D} = \mathscr{D}(f) = \mathscr{E}$.||

COROLLARY. *If f and g are in $\mathscr{F}$, then $\mathscr{D}(f) \leqslant \mathscr{D}(g)$ if and only if there is $\varphi \in \Phi$ such that $f = \varphi \circ g$.*

THEOREM 5. *Let M be a continuum of type A, $\mathscr{D} \in \Delta$, $f \in \mathscr{F}$ such that $\mathscr{D}(f)$, and suppose that K is a subcontinuum of M such that $f(K)$ is an interval $[r, s]$ where $r < s$, i.e., such that K meets at least two elements of $\mathscr{D}$. Then (disregarding those $D \in \mathscr{D}$ such that $D \cap K = \emptyset$) $\{D \cap K | D \in \mathscr{D}\}$ is a non-trivial, upper semi-continuous decomposition of K each element of which is connected. For $r < t < s$, $f^{-1}[t]$ lies in and separates K while $f^{-1}[r] \cap K$, $f^{-1}[s] \cap K$ do not separate K. In particular, if K is irreducible between a pair of its points, then it is of type A and $\{D \cap K | D \in \mathscr{D}\}$ is an admissible decomposition.*

Proof. The non-void elements of $\{D \cap K | D \in \mathscr{D}\}$ are just the elements $\{f^{-1}[t] \cap K | r \leqslant t \leqslant s\}$. Certainly the decomposition is non-trivial, upper semi-continuous, and its elements are closed.

Let $t \in (r, s)$ and take $\varepsilon > 0$ so that $r < t - \varepsilon < t + \varepsilon < s$ and denote by L_1 and L_2 the sets $f^{-1}([0, t-\varepsilon])$ and $f^{-1}([t+\varepsilon, 1])$, respectively. Since f is in $\mathscr{F}$, it is monotone, so that L_1 and L_2 are subcontinua of M. Hence $L_1 + K + L_2 = M$ and this implies that $f^{-1}[(t-\varepsilon, t+\varepsilon)] \subset K$, therefore $f^{-1}[t] \subset K$. Since $f^{-1}[t]$ separates M if $r < t < s$, it is evident that $f^{-1}[t]$ separates K, and, further, that if x and y are points between which K is irreducible, then one of x, y lies in $f^{-1}[r] \cap K$ and the other in $f^{-1}[s] \cap K$.

To complete the proof of the theorem, we have only to show that $f^{-1}[r] \cap K$ and $f^{-1}[s] \cap K$ are connected. Let $K_0 = \overline{f^{-1}[(r, s)]}$ and $K_1 = \bigcap_{n=1}^{\infty} \overline{f^{-1}[(r, r+1/n)]}$; then K_1 is a subcontinuum of K_0 and K_0 is a subcontinuum of K. Let $x \in f^{-1}[r] \cap K$; then it is easy to prove that the component, C, of $f^{-1}[r] \cap K$ which contains x meets K_0. Since $C \subset f^{-1}[r]$, it follows that C meets K_1 which implies that $f^{-1}[r] \cap K$ is connected.||

Note that in the course of proving Theorem 5 we showed that if a subcontinuum K of M meets two elements of some $\mathscr{D} = \mathscr{D}(f)$ in Δ, then $K^\circ \neq \emptyset$; indeed, K contains the complete preimage under f of an open interval in I.

Using the basic ideas in the proof of Theorem 5 we are able to prove our uniqueness theorem.

THEOREM 6. *Let M be a continuum of type A; then Δ contains a unique minimal element.*

Proof. Let $\mathscr{D}$ and $\mathscr{E}$ be in Δ and suppose that some element K of $\mathscr{E}$ meets two elements of $\mathscr{D}$. We will show that $\mathscr{E}$ is not minimal. This will show that a minimal element of Δ refines every element of Δ and therefore will prove the theorem.

Using the notation of Theorem 5, let $\mathscr{D} = \mathscr{D}(f)$ and write $f(K) = [r, s]$

where $r < s$ in I. We assert that the following collection of sets, denoted by $\mathscr{E}'$, is an element of Δ:

$$\mathscr{E}-\{K\}+\{f^{-1}[t] \mid t \in (r, s)\}+\{f^{-1}[r] \cap K\}+\{f^{-1}[s] \cap K\}.$$

Denote the last two sets by A and B, respectively.

Several facts are immediate: $\mathscr{E}'$ is surely a decomposition of M; the elements of $\mathscr{E}'$ are continua; and upper semi-continuity of $\mathscr{E}'$ everywhere, except possibly at A and B, is clear. Moreover, as far as the separating condition on elements of $\mathscr{E}'$ is concerned, we need only show that if M is irreducible from x to y and $A(B)$ contains neither point, then $A(B)$ separates M.

We deal first with the latter argument. Suppose that A contains neither x nor y. This implies that $0 < r < 1$. Now f, being monotone, maps the set $\{x, y\}$ onto the set $\{0, 1\}$ in some order, say $f(x) = 0, f(y) = 1$. Let L_1 and L_2 be subcontinua of M irreducible from x to K and y to K respectively. Since $0 < r$, L_1-K is non-void. Now L_1 and L_2 are disjoint and $M = L_1+K+L_2$. Note that L_1-K, being the complement in M of $K+L_2$, is open and $(K+L_2)-L_1$ being the complement of L_1, is open. Thus $M-(L_1 \cap K)$ is separated, being the sum of the non-void, disjoint open sets L_1-K and $(K+L_2)-L_1$. Since L_1 is irreducible from x to K, $f(L_1)$ is irreducible from $f(x) = 0$ to $f(K) = [r, s]$. It follows that $L_1 \cap K \subset f^{-1}[r] \cap K = A$ and, since $L_1 \cap K$ separates, so does A.

We now verify upper semi-continuity at A. We shall deal with the case $0 < r$, so that in the notation of the previous paragraph $L_1-K = L_1-A$ is non-void. Suppose that U is an open subset of M containing A. The union of U with $K+L_2$ is an open subset, U_1, of M which contains the element K of $\mathscr{E}$. Hence there is an open set V_1 which is the union of elements of $\mathscr{E}$, contains K and lies in U_1. Let V be the union of those elements of $\mathscr{E}$ lying in $V_1 \cap L_1$. Next, by Theorem 5, there is a subset W of K, open relative to K, containing A and lying in $K \cap U$, which is the union of the set A together with sets of the form $f^{-1}[t]$ where $r < t < s$. Thus $V+W$ contains A, lies in U and is the union of elements of $\mathscr{E}'$. Let $x \in V+W$; if $x \in (V-A)+(W-A)$, then $V+W$ is surely a neighborhood of x. Suppose $x \in A$ and let $\{x_n\}$ be a sequence in M converging to x. By choice of $V, (W)$, the collection of points of the sequence lying in $M-K, (K)$, is either finite or forms a subsequence which is ultimately in $V, (W)$. Since every point of the sequence, $\{x_n\}$, is in $M-K$ or in K, the sequence is ultimately in $V+W$. Thus $V+W$ is a neighborhood of each of its points and so is open.

In the above we were assuming $0 < r$. If $r = 0$, the argument is easier. We simply use W in place of $V+W$.

Thus $\mathscr{E}'$ is admissible and since it properly refines $\mathscr{E}$, $\mathscr{E}$ is not minimal, q.e.d.||

At this point we give two simple examples of continua of type A. Since we will refer many times to these examples, we give them descriptive names rather than numbers.

The simplest continuum of type A is, of course, an arc. Perhaps the next simplest is "the sin $(1/x)$ curve" which is our first example. This is the set of points (x, y) in the plane whose coordinates satisfy:

(1) $x = 0,\ -1 \leqslant y \leqslant 1$;

(2) $0 < x \leqslant 1,\ y = \sin\dfrac{1}{x}$.

We now give an intuitive description of the second example.

Let C denote the standard Cantor "middle third" set on the line segment $\{(x, y) | 0 \leqslant x \leqslant 1, y = 0\}$ in the plane. This is obtained by deleting an open interval I_{11}, of length one third; then deleting two open intervals, I_{21} and I_{22}, of length one ninth; then four open intervals, $I_{31}, \ldots, I_{34}$, of length 1/27, and so on.

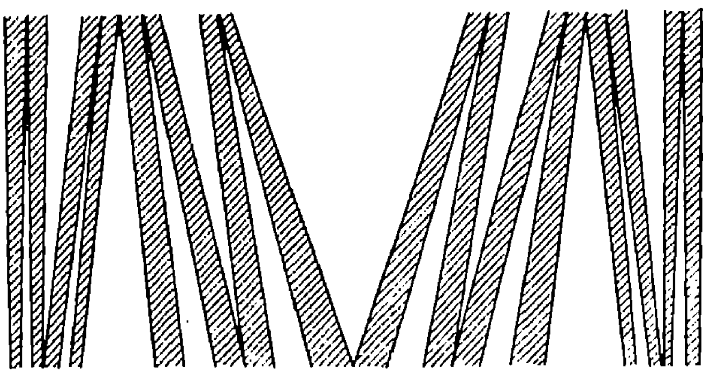

Fig. 1

Above each point of C erect a vertical line segment of height 1 and denote the set so obtained by P. Deform P by "pulling" the two endpoints of I_{11} to the midpoint of I_{11}, thus replacing two vertical line segments with a "$\vee$". Next "pull" the points whose x-coordinates are the endpoints of I_{21} and whose y-coordinates are 1 to the point whose x-coordinate is the midpoint of I_{21} and whose y-coordinate is 1; do the same for the interval I_{22}. Thus, this step replaces two pairs of vertical lines with "$\wedge$'s." At the next step we will identify four pairs of points, corresponding to the endpoints of $I_{31}, \ldots, I_{34}$ and all having y-coordinates 0, each pair being "pulled" to the midpoint of its respective interval.

The continuum obtained by continuing this process is originally due to Knaster and is well known. For a rigorous definition, see page 132 of [12]. It can be pictured as in Fig. 1.

Throughout this thesis the term "accordionlike continuum" will mean this continuum (or any homeomorph of it).

Evidently the minimal admissible decomposition of the $\sin(1/x)$ curve consists of the vertical line segment together with the collection of singleton point sets $\{\{(x, y)\} \mid 0 < x \leqslant 1,\ y = \sin(1/x)\}$.

The accordionlike continuum provides an example of a continuum of type A in which every element of the minimal decomposition is non-degenerate. The elements of this decomposition are the "$\vee$'s" and "$\wedge$'s" together with uncountably many straight line segments (these correspond to the points of the Cantor set which are not endpoints of deleted intervals).

DEFINITION 4. A continuum is of *type* A' provided it is of type A and has an admissible decomposition each of whose elements has void interior.

Thus the arc, $\sin(1/x)$ curve and accordionlike continuum are all of type A'. With the next few theorems we show that a continuum of type A fails to be of type A' only if it contains large pathological subcontinua.

A continuum is called *decomposable* provided it is the union of two proper subcontinua, otherwise it is called *indecomposable*. The basic facts concerning this notion are to be found in [13], however for completeness we include the following result.

THEOREM 7. *Suppose that the continuum M is irreducible between the closed subsets A and B and that every subcontinuum of M with non-void interior is decomposable. If $J = \{x \in M \mid M$ is irreducible from A to $x\}$, then J is a subcontinuum of M and $J^\circ = \emptyset$.*

(This strengthens Theorem 148 on page 61 of [14].)

Proof. We first show that J is closed. Suppose that the point $x \in M-J$ is a limit point of J. Let L be a proper subcontinuum of M joining H to x; being proper, L misses J. Let $y \in J$ and let N be a subcontinuum of M irreducible from x to y. Since $y \in J$, $L+N = M$ so that N contains $M-L$ and has non-void interior. Thus N is decomposable, say $N = N_x + N_y$ where $x \in N_x$, $y \in N_y$. Since $J \subset M-L \subset N$ and since $x \in \bar{J}$, N_x contains a point z of J. But then $L+N_x$ would be a proper subcontinuum of M joining H to the point z of J, a contradiction. This shows that J contains all of its limit points, i.e., J is closed.

If J is not connected, then we may write $J = P+Q$ where P and Q are disjoint closed subsets of J. Let U be open in M such that $P \subset U \subset \bar{U} \subset M-Q$. Some component, C, of U contains a point x of P and a point y of ∂U. Since ∂U misses J, there is a subcontinuum, N, of M joining H to z and missing J. Then $N+\bar{C}$ is a subcontinuum of M joining H to x but missing Q, a contradiction. Thus J is connected and therefore a subcontinuum of M.

Finally, since M is irreducible from H to J (by definition of J), M misses J° which is merely a way of saying $J^\circ = \emptyset$.||

The following result will be used frequently without explicit mention.

THEOREM 8. *Let M be of type A' and let $f \in \mathcal{F}$ be such that $\mathcal{D}(f)$ is the minimal admissible decomposition of M. Then for $0 \leqslant r < s \leqslant 1$, $\overline{f^{-1}[(r, s)]} = K$ is a subcontinuum of M irreducible from every point of $K \cap f^{-1}[r] = K_r$ to every point of $K \cap f^{-1}[s] = K_s$. Also, K_r and K_s are subcontinua of K with void interior relative to K.*

Proof. By definition, K_r and K_s have no interior relative to K. Thus, if H is a proper subcontinuum of K joining K_r to K_s, then $K-H$ contains some point z such that $r < f(z) < s$. But then $H+f^{-1}[[0, r]]+ +f^{-1}[[s, 1]]$ is a subcontinuum of M joining $f^{-1}[0]$ to $f^{-1}[1]$ and still missing z, which contradicts irreducibility of M. That K_r and K_s are subcontinua of K follows from Theorem 7.||

THEOREM 9. *Let M denote a continuum which is irreducible between two closed subsets H and K such that every subcontinuum of M with non-void interior is decomposable. Then the following hold.*

(a) *There is a decomposition of M, $M = M_H+M_K$, where $H \subset M_H$, $K \subset M_K$ and $\overline{M_H-M_K} \cap M_K$ is connected.*

(b) *If U and V are open subsets of M such that $H \subset U \subset \overline{U} \subset V \subset M-K$ and ∂V is connected, then there is an open set W of M such that $\overline{U} \subset W \subset \overline{W} \subset V$ and ∂W is connected.*

Proof. (a) Let $M = P+Q$ be any decomposition of M; then H lies in one of $P-Q$, $Q-P$ and K in the other, say $H \subset P-Q$ and $K \subset Q-P$. Now $P-Q = M-Q$ is connected and its closure, M_H, is therefore a subcontinuum of M containing H and irreducible from H to Q. Let $J = \{x \in M_H | M_H \text{ is irreducible from } H \text{ to } x\}$.

By Theorem 7, J is a sub-continuum of M_H with void interior relative to M_H, hence a subcontinuum of M with void interior in M.

Now let $M_K = Q+J$; M_K is a continuum since J meets Q. Clearly $\overline{M_H-M_K} \cap M_K = J$ which is connected.

(b) With U and V as in the statement of part (b), let L be a subcontinuum of M irreducible from $\overline{U}$ to ∂V. Since $H \subset V$ and since V is connected, part (a) of Theorem 1 implies that $M-\overline{V}$ is connected. Since ∂V is connected, $\overline{U}+L+\partial V+\overline{M-\overline{V}}$ is connected and, since it joins H to K, must be all of M. This means that $L-(\overline{U}+\partial V)$ must be all of $V-\overline{U}$. Let $L = L_1+L_2$ be a decomposition of L with $\overline{U} \cap L \subset L_1-L_2$ and $\partial V \cap L \subset L_2-L_1$. By part (a) of this theorem we may also require that $\overline{L_1-L_2} \cap L_2$ be connected.

We put $W = \overline{U}+L_1-L_2$; then W is connected, open in M, and $\overline{U} \subset W$. Since L is irreducible from $\overline{U}$ to ∂V, $\overline{L_1-L_2}$ misses ∂V and hence lies in V. It follows that $\partial W = \overline{W} \cap (\overline{M-W}) = (\overline{U}+\overline{L_1-L_2}) \cap (L_2+\overline{M-V}) = \overline{L_1-L_2} \cap L_2$ and this is connected, q.e.d.||

THEOREM 10. *Let M be a continuum irreducible between a pair of points x and y. A necessary and sufficient condition that M be of type A' is that every subcontinuum of M with non-void interior be decomposable.*

Proof of sufficiency. Using part (a) of Theorem 9, we decompose M as follows: $M = M_x+M_y$ where $x \in M_x$, $y \in M_y$ and $\overline{M_x-M_y} \cap M_y$ is connected. Let $W_{1/2} = M_x-M_y$; then $W_{1/2}$ is open, connected and has connected boundary. Also, by part (c) of Theorem 1, $\overline{W}_{1/2}$ is irreducible from x to $\partial W_{1/2}$. Applying part (a) of Theorem 9 again, we decompose $\overline{W}_{1/2}$ as follows: $\overline{W}_{1/2} = P+Q$, where $x \in P$, $\partial W_{1/2} \subset Q$ and $\overline{P-Q} \cap Q$ is connected. Let $W_{1/4} = P-Q$; then $W_{1/4}$ is open in M, connected and has connected boundary. Also $\overline{W}_{1/4} \subset W_{1/2}$ and $W_{1/2}-\overline{W}_{1/4}$ is connected. At the next step, apply part (a) of Theorem 9 to $\overline{W}_{1/4}$ and apply part (b) of Theorem 9 to the gap between $W_{1/4}$ and $W_{1/2}$ to get a pair of open, connected sets, $W_{1/8}$ and $W_{3/8}$, with connected boundaries such that $x \in W_{1/8} \subset \overline{W}_{1/8} \subset W_{1/4} \subset \overline{W}_{1/4} \subset W_{3/8} \subset \overline{W}_{3/8} \subset W_{1/2}$ and if r and s are any of the numbers 1/8, 1/4, 3/8, 1/2 with $r < s$, then $W_s-\overline{W}_r$ is connected. Continuing this construction in the obvious way we obtain a family $\{W_r | \ r$ is a diadic rational in $(0, 1/2]\}$ with the following properties: each W_r is an open, connected subset of M with connected boundary; for $r < s$, $\overline{W}_r \subset W_s$ and $W_s-\overline{W}_r$ is connected. Since $M-\overline{W}_{1/2}$ is connected, we may extend the process to get a family $\{W_r | r$ a diadic rational in $(0, 1)\}$ with the same properties as above. Finally, we put $W_1 = M$.

Define a function f from M to the unit interval as follows: for $z \in M$, $f(z) = \inf\{t | z \ \in W_t\}$. By Lemma 3 on page 114 of [10], a function defined *via* a family of sets such as ours is continuous. We now show f is monotone. Suppose $0 < r < 1$; for each positive integer n, let $a_n = r-1/n$ and $b_n = r+1/n$. Then, for any $z \in M$, $f(z) = r$ if and only if (for n sufficiently large) $z \in W_{b_n}-\overline{W}_{a_n}$, which is connected. Since $\overline{W_{b_n}-\overline{W}_{a_n}} \subset \overline{W_{b_{n-1}}-W_{a_{n-1}}}$, $f^{-1}[r]$ is the intersection of a family of nested continua and is therefore connected. This implies that f is monotone (see the remark preceding Theorem 1).

It is obvious that $\{f^{-1}[r] | \ r \in I\}$ is an admissible decomposition for M, i.e. M is of type A. Let $\mathscr{E}$ be any admissible decomposition for M. If some element K of $\mathscr{E}$ has non-void interior then $\bar{K}^\circ$ is a continuum to which the preceding results apply; in particular, $\bar{K}^\circ$ is of type A. Let $\mathscr{D}$ denote an admissible decomposition for $\bar{K}^\circ$. Then, arguing as we did in the proof of Theorem 6 (the uniqueness theorem), we combine $\mathscr{D}$ and $\mathscr{E}$

to obtain an admissible decomposition of M properly refining $\mathscr{E}$. This shows that if $\mathscr{E}$ is the minimal decomposition for M, then no element of $\mathscr{E}$ has non-void interior. Thus M is of type A'.

Proof of necessity. Suppose that M is of type A' and K is a subcontinuum with non-void interior. Since no element of the minimal admissible decomposition, $\mathscr{D}$, of M has interior, K is contained in no element of $\mathscr{D}$. By Theorem 5, K has a (non-trivial) admissible decomposition (namely, $\{D \cap K|\ D \in \mathscr{D}\}$) and this surely implies that K is decomposable. ||

Remark. Our Theorem 10 and Theorem 3 on page 153 of Kuratowski's book, [12], are essentially the same result, although it takes some effort to resolve the notational differences. Our development seems to be more intuitive and, perhaps for that reason not so elegant as that of Kuratowski. At this point our work and his diverge.

Until further notice, let M denote a continuum irreducible between a pair of its points and let $\mathscr{I}$ denote the family of indecomposable subcontinua of M with non-void interior. By Theorem 10, $\mathscr{I}$ is empty if and only if M is of type A'. We are interested in finding conditions on $\mathscr{I}$ which imply that M is of type A.

The following well-known facts may be found in [13]. If N is any continuum and $x \in N$, we define the *x-composant of N* to be the set of points $y \in N$ such that some proper subcontinuum of N contains x and y. Every composant of N is dense in N and if N is indecomposable, then each of its composants has void interior. N is indecomposable if and only if every proper subcontinuum of N has void interior in N.

An indecomposable continuum is very far from being of type A, as is illustrated by the following

Remark. *If the continuum N is the monotone, continuous image, under the function f, of the indecomposable continuum M, then N is indecomposable.* Since if P is a proper subcontinuum of N, then $f^{-1}(N)$ is a proper subcontinuum of M and therefore has void interior in M, which implies that N has void interior in P.

We now proceed with our investigation.

LEMMA. (a) *If $F \in \mathscr{I}$, then $F = \overline{F^\circ}$.*

(b) *Let $\mathscr{A}$ be a collection of subsets of M such that for each $A \in \mathscr{A}$, $A = \overline{A^\circ}$ and let $B = \bigcup \{A | A \in \mathscr{A}\}$. Then $B \subset \overline{B^\circ}$ and if B is closed, $B = \overline{B^\circ}$.*

Proof. (a) If $F \in \mathscr{I}$, then $F^\circ \neq \emptyset$. By part (b) of Theorem 1, F° is connected, hence $\overline{F^\circ}$ is a subcontinuum of F with non-void interior relative to F and, as we noted, this implies that $F = \overline{F^\circ}$.

(b) We have: $B = \bigcup \{A | A \in \mathscr{A}\} = \bigcup \{\overline{A^\circ} | A \in \mathscr{A}\} \subset \overline{\bigcup \{A^\circ | A \in \mathscr{A}\}} \subset \overline{B^\circ}$; and, if B is closed, then the reverse containment holds. ||

THEOREM 11. *If E and F are distinct elements of $\mathscr{I}$, then $E^\circ \cap F = \emptyset$. The family $\mathscr{I}$ is at most countable.*

Proof. By part (a) of the lemma, $\overline{E^\circ} = E$ and $\overline{F^\circ} = F$; thus, by part (b) of the lemma, $\overline{(E+F)^\circ} = E+F$. If $E^\circ \cap F \neq \emptyset$, then $E+F$ is a continuum and by part (c) of Theorem 1 must be irreducible between two of its points, x and y. Clearly x lies in one of $E-F$, $F-E$ and y in the other, say $x \in E-F$, $y \in F-E$. Let C be the y-composant of F, then, since C is dense in F, C meets the relatively open subset $E^\circ \cap F$ of F. Hence there is a subcontinuum K of F lying in C joining y to some point of $E^\circ \cap F$. But then $E+K$ is a proper subcontinuum of $E+F$ joining x to y, contradicting choice of x and y. Thus $E^\circ \cap F = \emptyset$ as asserted.

From the preceding it follows that $\{F^\circ |\ F \in \mathscr{I}\}$ is a family of pairwise disjoint open subsets of M. Since M is compact metric, this family, and hence $\mathscr{I}$ itself, is countable.‖

DEFINITION 5. Let R be the subset of $M \times M$ consisting of all pairs (x, y) such that $x = y$ or there exist finitely many elements $F_1, F_2, \ldots, F_n$ of $\mathscr{I}$ such that $F_1+F_2+\ldots+F_n$ is connected and contains x and y. It is a simple matter to verify that R is an equivalence relation on M. We denote by $\mathscr{D}(R)$ the decomposition of M into the equivalence classes determined by R. Note that the elements of $\mathscr{D}(R)$ are connected.

THEOREM 12. *The following statements are related as follows*: (a) *and* (b) *are equivalent and either implies* (c).

(a) *Every element of $\mathscr{D}(R)$ is closed.*

(b) *Every element D of $\mathscr{D}(R)$ contains* (*intersects*) *at most finitely many elements of $\mathscr{I}$.*

(c) *If $\mathscr{J}$ is a subfamily of $\mathscr{I}$ and the union of the elements of $\mathscr{J}$ is a subcontinuum of M, then $\mathscr{J}$ is finite.*

Proof. We begin by noting that since every member of $\mathscr{I}$ lies in some element of $\mathscr{D}(R)$, an element D of $\mathscr{D}(R)$ intersects a member F of $\mathscr{I}$ if and only if $F \subset D$. Thus, "contains" and "intersects" are equivalent in (b). Next notice that if D is in $\mathscr{D}(R)$, $x \in D$, and D intersects $F \in \mathscr{I}$, then D contains a subcontinuum of M of the form $F_1+\ldots+F_n$ where $x \in F_1$, $F = F_n$ and each F_i is in $\mathscr{I}$. Thus, if an element D of $\mathscr{D}(R)$ intersects at least one member of $\mathscr{I}$, then D is the union of those members of $\mathscr{I}$ which it intersects.

In view of this, (b) clearly implies (a).

We now prove that (a) implies (b). Let D be a closed member of $\mathscr{D}(R)$ and suppose that D meets infinitely many distinct members, F_1, $F_2, \ldots$, of $\mathscr{I}$. By the preceding remark, $D = \bigcup_{i=1}^{\infty} F_i$. By the lemma preceding Theorem 11 and the assumption that D is closed, $\overline{D^\circ} = D$. By part

(c) of Theorem 1, D is irreducible between a pair of its points, x and y. Since $(x, y)\in R$, there exist $E_1, \ldots, E_n$ in $\mathscr{I}$ such that $E_1+\ldots+E_n$ is a subcontinuum of M containing x and y. Each E_i is some F_j, say $E_i = F_{j_i}$. Thus $F_{j_1}+\ldots+F_{j_n}$ is a subcontinuum of M, and hence of D, joining x to y. By irreducibility of D, $D = F_{j_1}+\ldots+F_{j_n}$. This implies that, for some n, $F_n \subset F_{j_1}+\ldots+F_{j_n}$. Since F_n has non-void interior relative to M, we conclude that for some i, $F_n^\circ \cap F_{j_i} \neq \emptyset$. This contradicts the first statement in Theorem 11.

We next prove that (b) implies (c). Let $\mathscr{J}$ be a subfamily of $\mathscr{I}$ the union of whose members is a subcontinuum of M and suppose $\mathscr{J}$ is infinite. If (b) holds, then we can write $\mathscr{J} = \{F_1, F_2, \ldots\}$ and choose infinitely many distinct, and hence disjoint, elements, $D_1, D_2, \ldots$, of $\mathscr{D}(R)$ in such a way that there exist positive integers $1 < n_1 < n_2 < \ldots$ for which $F_1+\ldots+F_{n_1} \subset D_1$, $F_{n_1+1}+\ldots+F_{n_2} \subset D_2$ and in general, $F_{n_i+1}+\ldots+F_{n_{i+1}} \subset D_{i+1}$. Since no continuum is the union of countably many closed, pairwise disjoint, non-void subsets, see Theorem 56 on page 23 of [14], we have obtained a contradiction and therefore $\mathscr{J}$ is finite. ||

Later we shall give an example to show that (c) does not imply (a) or (b). However, a strengthened version of (c) turns out to be equivalent to (a). This is the content of the next theorem and its corollary.

THEOREM 13. *If the elements of $\mathscr{D}(R)$ are closed and $D_1, D_2, \ldots$ is any collection of at least two distinct elements of $\mathscr{D}(R)$ each of which is closed and intersects a member of $\mathscr{I}$, then $\bigcup_{i=1}^{\infty} D_i$ is not connected.*

(Note that this reduces to a special case of the last theorem if "not connected" is replaced by "not a subcontinuum of M".)

Proof. For non-triviality, we assume that there are infinitely many D_i's. Suppose that $D = \bigcup_{i=1}^{\infty} D_i$ is connected; we shall obtain a contradiction. By the lemma preceding Theorem 11, $D_i = \overline{D_i^\circ}$ for each i, so again by that lemma, $D \subset \overline{D^\circ}$ and therefore, if K denotes the continuum $\bar{D}$, then $K = \overline{K^\circ}$. Thus, by part (c) of Theorem 1, K is irreducible between a pair of its points, x and y. We shall assume that each of x and y belongs to some D_i and show how to remove this assumption later on. So, reordering if necessary, assume $x\in D_1$ and $y\in D_2$. Denote by $\mathscr{D}$ the collection $\{D_1, D_2, \ldots\}$ and note that for $i \geqslant 3$, D_i separates K, $K-D_i = X_i+Y_i$, separated, where $x\in X_i$, $y\in Y_i$ and X_i and Y_i are connected. (This is just part (a) of Theorem 1.)

Using the above fact we now define an ordering on as follows: For every i, $D_1 \leqslant D_i$ and $D_i \leqslant D_2$. For every i and j, $D_i \leqslant D_j$ if and only if $D_i = D_j$ or $D_i \subset X_j$. (Intuitively, "$D_i \subset X_j$" means "D_i is on the x side of D_j".)

Let $i, j \geqslant 3$ and suppose $D_i \subset X_j$. Then, since $\bar{X}_j$ is irreducible from x to a point on ∂X_j (part (c) of Theorem 1) and since $\partial X_j \subset D_j$, $\bar{X}_j - D_i = A + B$, separated, where x is in A, B meets ∂X_j and A and B are connected (part (a) of Theorem 1). But then we have: $K - D_i = A + (B + + D_j + Y_j)$, separated, where $x \in A$, $y \in B + D_j + Y_j$. Since D_i separates K uniquely into X_i and Y_i, we conclude that $A = X_i$ and $B + D_j + Y_j = Y_i$.

So if $D_i \neq D_j$ $(i, j \geqslant 3)$, then $D_i \subset X_j$ implies $D_j \subset Y_i$. Reversing the roles of x and y we get the dual statement and we conclude that for $D_i \neq D_j$ $(i, j \geqslant 3)$, the following are equivalent: $D_i \leqslant D_j$, $D_i \subset X_j$, $D_j \subset Y_i$. Moreover, from the last sentence of the preceding paragraph we see that if $D_i \neq D_j$ $(i, j \geqslant 3)$ and $D_i \leqslant D_j$, then $D_j + Y_j \subset X_i$, and, of course, the dual of this statement holds.

If, for $i, j \geqslant 3$, we have $D_i \leqslant D_j$ and $D_j \leqslant D_i$ but $D_i \neq D_j$, then by the above results, $D_j \subset Y_i$ and $D_j \subset X_i$ which is absurd. Thus, omitting the obvious arguments in case i or j take on the values 1 or 2, $D_i \leqslant D_j$ and $D_j \leqslant D_i$ imply $D_i = D_j$.

If $D_i \leqslant D_j$ and $D_j \leqslant D_k$ $(i, j, k \geqslant 3)$ then, since $D_k \subset Y_j$ and $Y_j + D_j \subset Y_i$, we have $D_k \subset Y_i$ which, as we have noted, is equivalent to $D_i \leqslant D_k$. Again omitting the arguments involving D_1 and D_2, we conclude that $\leqslant$ is transitive.

We have proved that $\leqslant$ is a linear ordering on $\mathscr{D}$. We now show there are no gaps. Suppose $D_i \neq D_j$ and $D_i \leqslant D_j$. The sets $X_i + D_i$ (or just D_i if $i = 1$) and $Y_j + D_j$ (or just D_j if $j = 2$) are closed, connected, and disjoint in K; hence, if S denotes their union, $K - S$ is non-void and open in K. It follows that for some k, $D_k \subset K - S$ and that $D_i < D_k < D_j$.

From the last fact, it follows immediately that, for $k \geqslant 3$, $\bigcap \{Y_i \cap X_j | D_i \leqslant D_k \leqslant D_j, D_i \neq D_k \neq D_j\} = \bigcap \{\overline{Y_i \cap X_j} | D_i \leqslant D_k \leqslant D_j, D_i \neq D_k \neq D_j\}$ and that D_k is the unique element of $\mathscr{D}$ contained in this intersection. With the obvious modifications, similar statements hold for $k = 1$ and 2.

Now let U be a subset of D open relative to D containing the element D_k of $\mathscr{D}$ $(k \geqslant 3)$. Let V be open in K such that $U = V \cap D$; we can choose D_i and D_j in $\mathscr{D}$ such that $D_i \leqslant D_k \leqslant D_j$, $D_i \neq D_k \neq D_j$ and $D_k \subset Y_j \cap X_i \subset V$. But then, $U = V \cap D$ contains $Y_j \cap X_i \cap D$ and this last set contains D_k, is open in D, and the union of elements of $\mathscr{D}$. A similar argument in case $k = 1$ or 2 shows that $\mathscr{D}$ is an upper semi-continuous decomposition of the set D and the elements of $\mathscr{D}$ are compact. Since D is separable metric, the quotient space D induced by $\mathscr{D}$ is metric (by the remark on quotient spaces). The quotient space is also connected, since D was assumed connected, and countable, since $\mathscr{D}$ is countable. No metric space is both countable and connected, so we have the desired contradiction.

Recall that we assumed that x and y belonged to elements of $\mathscr{D}$. If $x(y)$ belongs to no element of $\mathscr{D}$, then add to $\mathscr{D}$ another element $D_x = \{x\}$

($D_y = \{y\}$) and let D_x (D_y) play the role of D_1 (D_2) in our proof. Only trivial modifications are needed to use the above proof. ||

COROLLARY 1. *If the elements of* $\mathscr{D}(R)$ *are closed and* $D_1, D_2, \ldots$ *is a countable collection of at least two elements of* $\mathscr{D}(R)$, *then* $D = \bigcup_{i=1}^{\infty} D_i$ *is not connected.*

Proof. Let $\mathscr{E}$ be the collection of those D_i's which are not points, i.e. which meet some member of $\mathscr{I}$. For non-triviality we may assume $\mathscr{E}$ has at least two elements and we may then write: $\bigcup \{D_i|\ D_i \in \mathscr{E}\} = A + B$, separated. Then $A, B, \{D_i|\ D_i \notin \mathscr{E}\}$ is an upper semi-continuous decomposition of D whose quotient space is countable and metric and therefore not connected, whence D is not connected. ||

COROLLARY 2. *A necessary and sufficient condition that the elements of* $\mathscr{D}(R)$ *be closed is that if* $\mathscr{J}$ *is a subfamily of* $\mathscr{I}$ *the union of whose members is connected, then this union lies in some element of* $\mathscr{D}(R)$ *and* $\mathscr{J}$ *is finite.*

Proof. Suppose that the elements of $\mathscr{D}(R)$ are closed. Let $\mathscr{J}$ be a subfamily of $\mathscr{I}$ and denote the union of its members by G. Certainly G is contained in the union of all members of $\mathscr{D}(R)$ which it meets. Now, by Theorem 13, no union of two or more elements of $\mathscr{D}(R)$ is connected, so if G is connected, then G lies in some element of $\mathscr{D}(R)$ and therefore, by part (b) of Theorem 12, G meets at most finitely many elements of $\mathscr{I}$, i.e. $\mathscr{J}$ is finite.

Conversely, suppose that $D \in \mathscr{D}(R)$ and D is not a single point; then D contains every member of $\mathscr{I}$ which it meets. (See the initial paragraph in the proof of Theorem 12.) If the above condition on subfamilies $\mathscr{J}$ of $\mathscr{I}$ holds, then, since D is connected, it meets (contains) only finitely many members of $\mathscr{I}$. In this case D is the finite union of closed sets and is closed. ||

Notice that Theorems 12 and 13 and the corollary just proved are, to a large extent, concerned with finding conditions which imply or are equivalent to the condition, "every element of $\mathscr{D}(R)$ is closed". In a moment we shall see that this last condition implies that M is of type A. Before proving this, we need another decomposition theorem which, in some sense, is one step removed from Theorem 2.

Using a portion of the arguments of Theorem 3 it is easy to prove that *there is a decomposition,* $\mathscr{D}$, *of* M *which is minimal with respect to being upper semi-continuous and to being refined by* $\mathscr{D}(R)$. It is possible for $\mathscr{D}$ to be trivial, i.e., consist of the single set M (see the first example following Theorem 16).

THEOREM 14. *Let* $\mathscr{D}$ *be as above. The elements of* $\mathscr{D}$ *are subcontinua of* M *and if* $\mathscr{D}$ *is non-trivial, then the quotient space of* M *relative to* $\mathscr{D}$ *is a continuum of type* A', *and* M *itself is of type* A.

Proof. Let $\mathscr{E}$ be the decomposition whose elements are the components of members of $\mathscr{D}$. Since the elements of $\mathscr{D}(R)$ are connected, $\mathscr{D}(R)$ refines $\mathscr{E}$. We want to prove $\mathscr{E} = \mathscr{D}$ and, for this, it suffices to prove that $\mathscr{E}$ is upper semi-continuous (by minimality of $\mathscr{D}$). Notice that since $\mathscr{D}$ is upper semi-continuous, its elements are closed and therefore so are the elements of $\mathscr{E}$.

Suppose that the open subset U of M contains the element E of $\mathscr{E}$. Pick $D \in \mathscr{D}$ so that E is a component of D; then we may write $D = A+B$, separated, where $E \subset A$ and $D-U \subset B$. Each of A, B is closed in M so we can find open sets V and W in M such that $A \subset V \subset U$, $B \subset W$ and $\overline{V} \cap \overline{W} = \emptyset$. Then $V+W$ is an open set in M containing D and by upper semi-continuity of $\mathscr{D}$ there is an open set U_1, which is the union of members of $\mathscr{D}$, such that $D \subset U_1 \subset V+W$. Since V and W are separated, the set $U_1 \cap V$ is open in M, contains D, is contained in U and is the union of all those components of members of $\mathscr{D}$ which it meets, i.e. the union of members of $\mathscr{E}$. This proves that $\mathscr{E}$ is upper semi-continuous.

We now turn to the main part of the theorem. Assume that $\mathscr{D}$ is non-trivial, so that the quotient space N is a non-trivial compact continuum. Since the elements of $\mathscr{D}$ are continua, the quotient map q is monotone and the quotient space is irreducible between a pair of its points. Now let K be a subcontinuum of N such that $\overline{K^\circ} = K$; then $q^{-1}[K] = L$ satisfies $\overline{L^\circ} = L$ and is therefore irreducible between a pair of its points x, y. If K contains no separating points, no member of $\mathscr{D}$ lies in L unless it contains exactly one of x, y. So L contains at most two elements of $\mathscr{D}$ and L is not separated by either of these nor by their union. An obvious argument then shows that $q[L] = K$ is decomposable and, by Theorem 10, N is of type A'.

To finish the proof, let f be a monotone, continuous function mapping N onto I. Then the composition, $f \circ q$, is monotone, continuous, and maps M onto I, which, as we have noted before (see the remarks preceding Theorem 4) implies that M is of type A.||

We now prove that a decomposition $\mathscr{D}$ such as we have considered above is unique.

THEOREM 15. *There is just one minimal upper semi-continuous decomposition,* $\mathscr{D}$, *refined by* $\mathscr{D}(R)$.

Proof. For the first statement of the theorem, let $\mathscr{E}_1$ and $\mathscr{E}_2$ be upper semi-continuous decompositions refined by $\mathscr{D}(R)$ and define $\mathscr{E}$ to be the collection of all sets of the form $E_1 \cap E_2$ where $E_i \in \mathscr{E}_i$. If $E_1 \cap E_2$ intersects $E_1' \cap E_2'$, then $E_1 \cap E_2' \neq \emptyset$ and $E_2 \cap E_2' \neq \emptyset$ which implies that $E_1 = E_1'$, $E_2 = E_2'$ and $E_1 \cap E_2 = E_1' \cap E_2'$. It follows that $\mathscr{E}$ is a decomposition of M. Certainly $\mathscr{E}$ is refined by $\mathscr{D}(R)$. A simple argument, based on normality of M and following the argument used in Theorem 14, shows that $\mathscr{E}$ is upper semi-continuous. Thus the collection of upper semi-continuous decompositions refined by $\mathscr{D}(R)$ is directed downwards by refinement and a minimal element must be unique. ||

THEOREM 16. *If $\mathscr{D}(R)$ has at least two elements and every element of $\mathscr{D}(R)$ is closed, then M is of type A.*

Proof. Let x and y be points between which M is irreducible, and let $\mathscr{E}$ be the collection of non-degenerate elements of $\mathscr{D}(R)$; $\mathscr{E}$ is countable. Define the relation $\leqslant$ on $\mathscr{E}$ as in Theorem 13 (here, M will play the role of the continuum K used in Theorem 13); $\leqslant$ is a linear ordering of $\mathscr{E}$.

Suppose that there exist distinct elements E_1 and E_2 of $\mathscr{E}$ such that $E_1 \leqslant E_2$ and no E in $\mathscr{E}$ distinct from E_1 and E_2 satisfies $E_1 \leqslant E \leqslant E_2$. Then, in the notation of Theorem 13, the following collection of sets forms a non-trivial upper semi-continuous decomposition refined by $\mathscr{D}(R)$:

$$X_1+E_1, \quad E_2+Y_2, \quad \{\{x\} | x \in Y_1 \cap Y_2\}.$$

Thus, if $\mathscr{E}$ has gaps relative to $\leqslant$, then Theorem 14 can be applied to show that M is of type A.

Suppose, then, that between every pair of distinct elements of $\mathscr{E}$ there lies a third element. In this case we shall bypass Theorem 14 and construct a monotone, continuous function mapping M onto I. The method of construction is that used in the proof of Theorem 10. Assume, for convenience, that there exist elements E_1 and E_2 of $\mathscr{E}$ such that $x \in E_1$ and $y \in E_2$. Take E_3 distinct from E_1 and E_2, so that $E_1 \leqslant E_2 \leqslant E_3$ and define $W_{1/2} = X_3$. Take E_4 and E_5 distinct from E_1, E_2 and E_3 and such that $E_1 \leqslant E_4 \leqslant E_3 \leqslant E_5 \leqslant E_2$ and define $W_{1/4} = X_4$ and $W_{3/4} = X_5$. Continuing in this way, and defining $W_1 = M$, we get a collection of open connected sets, $\{W_r | r$ is a diadic rational in $(0,1]\}$, such that for $r < s$, $\overline{W_r} \subset W_s$ and $W_s - \overline{W_r}$ is connected. As in Theorem 10, the function, f, defined on M by $f(z) = \inf\{t | z \in W_t\}$ is continuous, monotone and maps M onto I as required. ||

The aim of the preceding work has been to find conditions on a continuum M which imply or are equivalent to the condition that M be of type A or type A'. We next give some simple examples relating to this work, and finish the chapter with some results on the structure of admissible decompositions.

In what follows a simple closed curve plus its interior will be used to represent an indecomposable continuum. If M and N are indecomposable continua and there exist composants C_M and C_N of M and N, respectively, such that $M \cap N \subset C_M \cap C_N$, then it is easy to see that $M+N$ is irreducible between each pair of points, x and y, such that $x \in M - C_M$ and $y \in N - C_N$. Two or more indecomposable continua, $M_1, M_2, \ldots$, intersecting in this way, will be represented by the corresponding number of simple closed curves, $C_1, C_2, \ldots$; the fact that $M_i \cap M_j \neq \emptyset$ will be represented by drawing C_i touching C_j.

Consider the continuum, M, obtained by running a string of indecomposable continua, each touching its neighbors as above, up to the interval $J = [(0, 0), (0, 1)]$ in the plane, so that the result resembles a $\sin(1/x)$ curve. Clearly M is irreducible between every point of J and

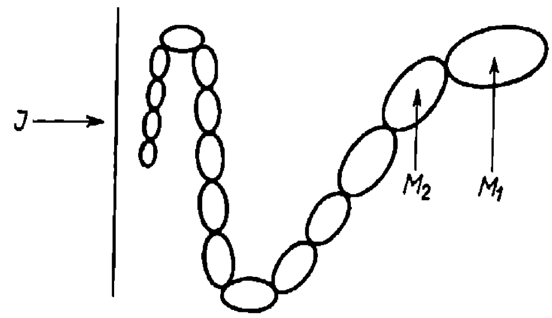

Fig. 2

every point of M_1 not on the composant of M_1 which meets M_2. $\mathscr{D}(R)$ has only one non-degenerate element, namely $M - J$, and the other elements of $\mathscr{D}(R)$ are the sets $\{x\}$ where $x \in J$. This example shows that, in Theorem 12, condition (c) implies neither (a) nor (b), and indicates why condition (c) had to be strengthened to the condition in Corollary 2 of Theorem 13 in order to obtain the reverse implication. Also, in this example $\mathscr{D}$ is trivial.

As an example of a continuum of type A in which $\mathscr{D}(R)$ has an abundance of non-degenerate elements consider the set in E^2 obtained as follows. Let I denote the interval $[(0, 0), (1, 0)]$ and, abusing the notation, re-

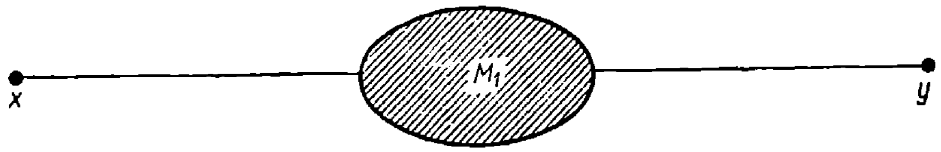

Fig. 3

place the interval $[1/3, 2/3]$ on I with an indecomposable continuum M_1 of diameter $1/3$, so that $[0, 1/3] + M_1 + [2/3, 1]$ is irreducible from $x = (0, 0)$ to $y = (1, 0)$.

Next, replace $[1/9, 2/9]$ and $[7/9, 8/9]$ on I by two indecomposable continua M_2 and M_3 of diameter $1/9$ so that the result is irreducible between x and y. The continuum obtained by continuing this process may be pictured as in Fig. 4.

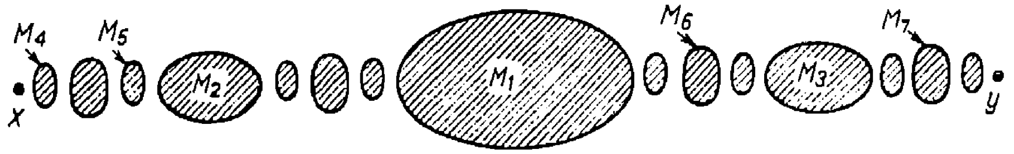

Fig. 4

In this example, the collection of non-degenerate elements of $\mathscr{D}(R)$ is precisely the family $\mathscr{I}$ since no two elements of $\mathscr{I}$ intersect. Hence the elements of $\mathscr{D}(R)$ are closed and by Theorem 16, M is of type A. Notice that between any two M_i's there is a third; i.e., there are no gaps relative to the partial ordering established in Theorem 16. Thus the construction used in the proof of Theorem 16 applies to this continuum and actually provides us with a "nice" function mapping M onto I; that is, M_1 is squeezed to the point $1/2 \in I$, M_2 and M_3 to $1/4$ and $3/4$, respectively, and so on. The degenerate elements of $\mathscr{D}(R)$ correspond to points of the Cantor set which are not endpoints of deleted intervals and these are mapped onto the complement in I of the diadic rationals.

If a subset, A, of a continuum M has the property that every pair of points of A can be joined by a subcontinuum of M lying in A, then A is called *strongly connected* (*relative to M* or *in M*). For example, in the $\sin(1/x)$ curve, M, the subset consisting of the points off the vertical line segment plus the midpoint of the segment, is not strongly connected relative to M, although this subset is connected.

THEOREM 17. *Let M be of type A and $f \in \mathscr{F}$ (i.e. f is a monotone, continuous function from M onto I). Then the following hold.*

(a) *If C is connected in I, then $f^{-1}[C]$ is strongly connected in M.*

(b) *If $r \neq s$ in I, then $f^{-1}[r]$* and *$f^{-1}[s]$ are contained in strongly connected open sets whose closures are disjoint.*

Proof. (a) If C is connected in I, then there exist sequences $\{r_i\}$, $\{s_i\}$ in I such that $\ldots \leqslant r_3 \leqslant r_2 \leqslant r_1 \leqslant s_1 \leqslant s_2 \leqslant s_3 \leqslant \ldots$ and $C = \bigcup_{i=1}^{\infty}[r_i, s_i]$. Since f is monotone, each $f^{-1}\big[[r_i, s_i]\big]$ is a subcontinuum of M; thus, $f^{-1}[C]$, being the increasing union of subcontinua of M, is strongly connected in M.

(b) If $r \neq s$ in I let U, V be open intervals containing r and s respectively, such that $\overline{U} \cap \overline{V} = \emptyset$. Then $f^{-1}(U)$ and $f^{-1}(V)$ are strongly connected open sets in M containing $f^{-1}[r]$ and $f^{-1}[s]$, respectively. Since f is continuous, $\overline{U} \cap \overline{V} = \emptyset$ in I implies $\overline{f^{-1}(U)} \cap \overline{f^{-1}(V)} = \emptyset$ in M. ||

COROLLARY 1. *With M and f as above, suppose that $f(x)=0$ and $f(x)=1$, where M is irreducible from x to y. Let r be a point of $(0,1)$ and X, Y the separated sets whose union is $M-f^{-1}[r]$ with $x\in X, y\in Y$. Then $X=f^{-1}[[0,r)]$ and $Y=f^{-1}[(r,1]]$.*

Proof. Since $f^{-1}[[0,r)]$ is (strongly) connected and meets X and since Y is separated from x, $f^{-1}[[0,r)]$ must lie in X. Similarly for the other half. ||

We now introduce an important concept which is due to F. B. Jones, see [8] and [9]. Let x and y be distinct points of a continuum M. We say that *M is aposyndetic at x with respect to y* provided there is a subcontinuum of M containing x in its interior and not containing y. If this condition fails, i.e., if every subcontinuum of M which contains x in its interior contains y, then M is *non-aposyndetic at x with respect to y*. If M is the $\sin(1/x)$ curve and z lies on the vertical interval, then M is non-aposyndetic at z with respect to every other point on the vertical interval and M is aposyndetic at z with respect to every point of the vertical interval.

Given a continuum M and a point z in M, let us denote by $K(z)$ the collection of points $y\in M$ such that $y=z$ or M is non-aposyndetic at z with respect to y and by $L(z)$ the collection of $y\in M$ such that $y=z$ or M is non-aposyndetic at y with respect to z. It is well known and easy to prove that, for each $z\in M$, $K(z)$ and $L(z)$ are closed in M and $L(z)$ is connected and hence a subcontinuum of M.

COROLLARY 2 (of Theorem 17). *With M and f as in Theorem 15, if $r\in I$ and $z\in f^{-1}[r]$ then $K(z)+L(z)\subset f^{-1}[r]$.*

Proof. If $y\in M$ and $y\notin f^{-1}[r]$, then $y\in f^{-1}[s]$ where $s\in I, s\neq r$. By part (b) of Theorem 17, there are open connected subsets V and W such that $f^{-1}[r]\subset V$, $f^{-1}[s]\subset W$ and $\bar{V}\cap\bar{W}=\emptyset$. Thus $\bar{V}$ is a subcontinuum of M with z in its interior missing y, which implies that $y\notin K(z)$. Similarly the properties of $\bar{W}$ imply that $y\notin L(z)$. Thus, $y\notin f^{-1}[r]$ implies $y\notin K(z)+\alpha(z)$ as asserted. ||

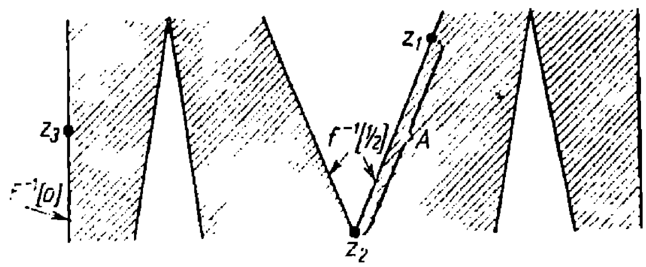

Fig. 5

Let M denote the accordionlike continuum with points and decomposition elements labeled as in Fig. 5. Here, A is one side of the decomposition element $f^{-1}[1/2]$, and z_2 is the vertex of $f^{-1}[1/2]$, while

z is a point of $A-\{z_2\}$. From the definitions it is easy to see that $K(z_1) = L(z_1) = A$. On the other hand, $K(z_2) = L(z_2) = f^{-1}[1/2]$ and $K(z_3) = L(z_3) = f^{-1}[0]$. Thus, in a sense, z_2 and z_3 completely determine the decomposition elements containing them, while z_1 does not. The natural question raised by this observation is answered in the next result and the example following.

THEOREM 18. *Let M be of type A', irreducible from x to y, and let f be a function such that $\mathscr{D}(f)$ is the minimal admissible decomposition. For every $r \in I$ there is a point, z, of $f^{-1}[r]$ such that $f^{-1}[r] = K(z) = L(z)$.*

Proof. We deal first with the case $0 < r < 1$. For convenience assume $f(x) = 0$ and $f(y) = 1$. With the usual notation, we have: $M - f^{-1}[r] = f^{-1}[[0, r)] + f^{-1}[(r, 1]] = X + Y$. Since $f^{-1}[r]$ has no interior, there is a point z lying in $\bar{X} \cap \bar{Y}$. Thus $z \in f^{-1}[r]$ and, by Corollary 2 of Theorem 17, $L(z) + K(z) \subset f^{-1}[r]$. Let L be any subcontinuum of M with $z \in L^\circ$. Then, by choice of z, L meets both X and Y. These sets are strongly connected (by part (a) of Theorem 16) and it follows that $L+X+Y = M$ which implies $f^{-1}[r] \subset L$. This shows that $f^{-1}[r]$ lies in every subcontinuum of M containing z as an interior point. Since $K(z)$ is obviously the intersection of all such continua, $f^{-1}[r] \subset K(z)$. With the reverse containment having already been established, we conclude that $K(z) = f^{-1}[r]$.

Next suppose $w \in f^{-1}[r]$; we wish to show that z is in $K(w)$ for this clearly implies $w \in L(z)$. If $w \in \bar{X} \cap \bar{Y}$, then, by the result just proved, $K(w) = f^{-1}[r]$ and $z \in K(w)$. Thus suppose $w \notin \bar{X} \cap \bar{Y}$; then w lies in one of $\bar{X} - \bar{Y}$, $\bar{Y} - \bar{X}$, say $w \in \bar{X} - \bar{Y}$. Now let L be a subcontinuum of M with $w \in L^\circ$. If L meets both X and Y, then, as before, L contains $f^{-1}[r]$. The other possibility is that $L \subset X + f^{-1}[r] = \bar{X} + f^{-1}[r]$. In this case, $\overline{L^\circ}$ is a subcontinuum of $\bar{X}$ containing w in its interior. Since $\bar{X}$ is irreducible from x to $\bar{X} \cap f^{-1}[r]$ and since $w \in \bar{X} \cap f^{-1}[r]$, it follows that L contains all of $\bar{X} \cap f^{-1}[r]$ and hence $z \in L$. Thus, in either case, $z \in L$, and we have proved that every subcontinuum of M containing w in its interior contains z, hence z lies in $K(w)$, as desired.

Our orginal assumption was that $0 < r < 1$. If $r = 0$ or 1, then a simplification of the above proof shows that for every $z \in f^{-1}[r]$, $K(z) = L(z) = f^{-1}[r]$. ||

In the preceding theorem we cannot replace the hypothesis "M is of type A'" by "M is of type A". To see this, let M consist of three indecomposable continua M_1, M_2, M_3 and two arcs A_1, A_2 joined together as in Fig. 6 so that M is irreducible from x to y. It is easy to see that M is of type A and that the minimal admissible decomposition has exactly one non-degenerate element, namely, $M_1+M_2+M_3$. However, for any $z \in M_1+M_2+M_3$, $K(z)+L(z)$ is the sum of at most two of M_1, M_2, M_3.

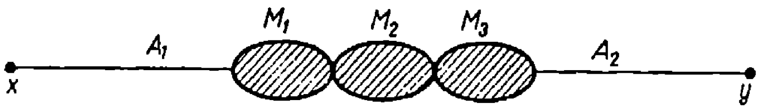

Fig. 6

We now give another characterization of continua of type A' in terms of aposyndicity.

THEOREM 19. *Let M denote a continuum irreducible from x to y. Then the following are equivalent.*

(a) *M is of type A'.*

(b) *For each $z \in M$, $K(z)^\circ = \emptyset$.*

(c) *For each $z \in M$, $L(z)^\circ = \emptyset$.*

Proof. We first show that (a) implies (b) and (c). If M is of type A' and $z \in M$, then $K(z)+L(z)$ lies in some element of the minimal admissible decomposition of M (by Corollary 2 of Theorem 17). Since each such element has void interior, the same is true of $K(z)$ and $L(z)$.

To prove that (b) implies (a), suppose that M contains an irreducible subcontinuum N such that $N^\circ \neq \emptyset$ (in view of Theorem 10, this is precisely the assumption that M is not of type A'). We want to find z in M such that $K(z)^\circ = \emptyset$. If $N = M$, then, for any $z \in M$, $K(z) = N = M$, so we may assume N is proper in M. By part (a) of the lemma preceding Theorem 11, $\overline{N^\circ} = N$.

There are two possible cases to handle.

Suppose, first, that $x \in N$ so that $M-N = Y$ is a connected set containing y, $\overline{Y}$ is irreducible from y to $N \cap \overline{Y}$, and N is irreducible from x to $N \cap \overline{Y}$. Let $w \in N \cap \overline{Y}$ and let z be any point of N°. If L is any subcontinuum of M and $z \in L^\circ$, then we can choose subcontinua H and K of M lying in the x-composant and w-composant of N repectively such that H joins x to L and K joins w to L. Then $H+L+K$ joins x to w whence $H+L+K+\overline{Y} = M$. Since H and K have void interior relative to N and since $N^\circ \cap \overline{Y} = \emptyset$, we have $N^\circ \subset L^\circ$. This shows that $N^\circ \subset K(z)$ and, therefore, $K(z)^\circ \neq \emptyset$. Our second case is that N contains neither x nor y. Here $M-N = X+Y$, separated, $x \in X$, $y \in Y$ and $\overline{X}$ is irreducible from x to $N \cap \overline{X}$, $\overline{Y}$ from y to $N \cap \overline{Y}$, and N is irreducible between two points $u \in N \cap \overline{X}$ and $v \in N \cap \overline{Y}$. If $z \in N^\circ$ and L is a subcontinuum of M with $z \in L^\circ$, then, as above, $N^\circ \subset L^\circ$. Thus for any $z \in N^\circ$, $N^\circ \subset K(z)^\circ$.

So, if (a) fails, then (b) fails and (b) implies (a). The above arguments also show that if (a) fails and N is an irreducible subcontinuum of M with non-void interior, then for any $z \in N^\circ$, $N^\circ \subset L(z)$, i.e., (c) fails. Thus (c) implies (a). ||

COROLLARY. *A continuum M is of type A' if and only if M is irreducible between a pair of its points and every open subset of M contains a pair of points x and such that M is aposyndetic at x with respect to y.*

Recall that a continuum is *locally connected at the point z* provided every neighborhood of z contains a connected neighborhood of z. Aposyndicity is a weak form of local connectedness as is the following concept due to G. T. Whyburn, see [9]. A continuum is *semi-locally-connected at the point z* provided that if U is an open set containing z, then there is an open set V such that $z \in V \subset U$ and finitely many components of the complement of V cover the complement of U.

The following result relates the preceding concepts to the minimal decomposition in irreducible continua.

THEOREM 20. *Let the continuum M be irreducible from x to y. Then for each $z \in M$,* (a), (b) *and* (c) *are equivalent. If M is of type A and $\mathcal{D} \in \Delta$, then* (d) *implies* (a), (b) *and* (c). *If M is of type A' and $\mathcal{D}$ is the minimal decomposition, then all four conditions are equivalent.*

(a) *M is aposyndetic at z with respect to every other point of M.*

(b) *M is semi-locally-connected at z.*

(c) *M is locally connected at z.*

(d) $\{z\} \in \mathcal{D}$.

Proof. Suppose that M is aposyndetic at z with respect to every other point of M and let U be an open set containing z. For each $y \in M-U$, let Hy be a subcontinuum of M such that $z \in Hy^\circ \subset Hy \subset M-y$. Then $M-\overline{Hy^\circ}$ has at most two components, one of which, Ky, contains y. Now Ky is a continuum satisfying $y \in Ky^\circ \subset Ky \subset M-x$. By compactness there exist $y_1, \ldots, y_n \in M-U$ so that $Ky_1^\circ, \ldots, Ky_n^\circ$ covers $M-U$. Then $M-(Ky_1+\ldots+Ky_n) = V$ is an open set containing z, lying in U such that a finite number of components of $M-V$ cover $M-U$, i.e., M is semi-locally-connected at z. Thus (a) implies (b).

Next we show that (b) implies (c). Assume that (b) holds for $z \in M$ and let U be an open set containing z. We may assume that U is a proper subset of M. Choose an open subset V of M such that $z \in V \subset \bar{V} \subset U$ and $M-U$ is covered by a finite number, $K_1, \ldots, K_n$, of components of $M-V$. We may assume each K_i has non-void interior relative to M. In the following construction we shall apply various parts of Theorem 1 without explicit mention at each step. $M-K_1$ is the sum of at most two connected open sets one of which, Z_1, contains z, and $\bar{Z}_1$ is irreducible between some point and $\bar{Z}_1 \cap K_1$. Notice that $\partial Z_1 \subset K_1$, $\bar{Z}_1 \cap K_1 \subset \bar{V} \subset U$ and $\bar{Z}_1 \cap K_1^\circ = \emptyset$. Thus $\bar{Z}_1$ meets at most $n-1$ of the sets $K_1^\circ, \ldots, K_n^\circ$; also, for each i, Z_1 meets K_i° if and only if $K_i \subset \bar{Z}_1$. If $\bar{Z}_1$ meets no K_i, then $Z_1 \subset U$ and Z_1 is a connected open set containing z and lying in U as required. If $\bar{Z}_1$ meets some K_i°, say K_2°, then $\bar{Z}_1 - K_2 - K_1$ is the sum of at most two connected open (in M) sets and we let Z_2 be the one containing z. Then $\bar{Z}_2$ is irreducible between some pair of points and misses $K_1^\circ + K_2^\circ$. Continuing in this way we obtain, in not more than n steps, a connected open set Z_n, containing z with the property that $\bar{Z}_n$ misses $K_1^\circ + \ldots + K_n^\circ$. Thus, Z_n is a connected open set, lying in U and containing z as required. So, (b) implies (c).

In any continuum (c) implies (a).

Suppose that M is of type A, $\mathscr{D} \in \Delta$ and $\{z\} \in \mathscr{D}$. We show that M is locally connected at z. Choose $f \in \mathscr{F}$ such that $\mathscr{D} = \mathscr{D}(f)$ and let $f(z) = r \in I$, so that $\{z\} = f^{-1}[r]$. By an argument analogous to that used in proving part (b) of Theorem 17, it is easy to prove that if U is an open set containing $\{z\}$, then there exist an open connected set W in I containing r such that $V = f^{-1}(W)$ lies in U. By part (a) of Theorem 17, V is connected. This shows that M is locally connected at z.

Finally, suppose that M is of type A' and $\mathscr{D}$ is the minimal admissible decomposition. Suppose for some $z \in M$ that (a) holds, i.e., $K(z) = \{z\}$. If D is the element of $\mathscr{D}$ containing z, then, by Theorem 18, there is $w \in D$ such that $D = L(w) = K(w)$. But $z \in L(w)$ implies $w \in K(z)$ which implies $w = z$. Thus $D = K(z) = \{z\}$.||

Consider the continuum consisting of the boundary of the unit square in the plane plus the line segments $L_i = [(1/i, 0), (1/i, 1)]$ and let $z = (0, 1/2)$. Then (a) and (b) hold at z but the continuum is not locally connected at z. Thus, unless M is irreducible, (a) and (b) together need not imply (c).

Let M consist of a chain of indecomposable continua "converging to a point z" plus an arc, A, joined together as follows:

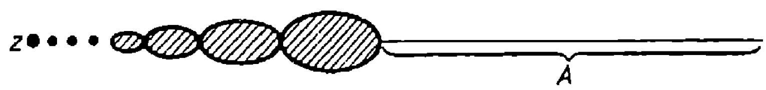

Fig. 7

Then M is of type A and is locally connected at z but $\{z\}$ is not an element of any admissible decomposition for M. Thus, (c) may fail to imply (d) even for the minimal decomposition of a continuum of type A, unless M is of type A'.

Perhaps the most obvious application of the theory we have developed so far is to characterizations of arcs and simple closed curves. We shall give two examples of such applications; the first one is evident.

THEOREM 21. *A necessary and sufficient condition that the continuum M be an arc is that M be of type A and that, for some $\mathscr{D} \in \Delta$, every element of $\mathscr{D}$ is degenerate.*

Proof. Necessity is obvious. Conversely, if for some $\mathscr{D}$ in Δ every element of $\mathscr{D}$ is a single point, then choosing $f \in \mathscr{F}$ such that $\mathscr{D} = \mathscr{D}(f)$, we see that f is one-to-one and hence a homeomorphism of M onto I. ||

Before stating the next theorem we recall a few definitions. A continuum M is a *simple closed curve* provided M is the union of two arcs having only their endpoints in common. A subset N of a continuum M *cuts between the pair of points* x *and* y of M provided every subcontinuum of M containing x and y intersects N. If it is not important to display a pair of points between which N cuts, we shall merely say, "N cuts".

THEOREM 22. *Let M be of type A' and $\mathscr{D} = \mathscr{D}(f)$ $(f \in \mathscr{F})$ the minimal admissible decomposition of M. If p and q are distinct points of $f^{-1}[0]$ $(f^{-1}[1])$ then p cuts q from every point of $M - f^{-1}[0]$ $(M - f^{-1}[1])$. If $0 < r < 1$ and $p \in \overline{f^{-1}[[0, r)]} \cap \overline{[f^{-1}(r, 1]]}$, then p cuts every other point of $f^{-1}[r]$ from every point of $M - f^{-1}[r]$.*

Proof. Suppose $p, q \in f^{-1}[0]$ and let $z \in M - f^{-1}[0]$. Then $f(z) = r > 0$. If K were a subcontinuum of M joining q to z missing p, then $K + f^{-1}[[r, 1]]$ would be a proper subcontinuum of M joining q to a point y of $f^{-1}[1]$. Since M is irreducible from q to y (by Theorem 8), this is impossible. Thus p cuts q from z.

Suppose $0 < r < 1$, p is as above, $q \in f^{-1}[r]$ and $z \in M - f^{-1}[r]$, say $f(z) = s$, where $0 \leqslant s < r$. Now if K is a subcontinuum of M joining z to q, then K contains $f^{-1}[(s, r)]$ and thus contains $\overline{f^{-1}[(s, r)]}$ which contains p, and, again, p cuts q from z. ||

We shall use the following fact, due to F. B. Jones, see [8].

If M is a continuum no point of which cuts, then there is a dense G_δ set G in M such that, for each $x \in G$, M is aposyndetic at x with respect to every other point of M and M is semi-locally-connected at x.

THEOREM 23. *The following are equivalent for a continuum M:*

(a) *If x, y, z are distinct points of M, then x does not cut z from y, while if x, y are distinct points of M, then there exist points z and w of M such that x, y, z and w are distinct and $\{x, y\}$ cuts z from w in M. (Briefly, no point cuts and every pair of points cuts.)*

(b) *No point cuts and no subcontinuum of M separates M.*

(c) *No subcontinuum N of M cuts between a pair of points in $M - N$*

(d) *M is a simple closed curve.*

Proof. The following relationships are clear: (c) implies (b) and (d) implies (a), (b) and (c). We show that (a) implies (b) and (b) implies (d).

Suppose that A is a subcontinuum of M and $M - A = U + V$ separated and non-void. Let $x \in U$, $y \in V$. If (a) holds, then there exist z and w such that x, y, z, w are distinct and every subcontinuum of M joining z and w contains x or y. Suppose $z \in U$; since x is not a cut point, there is a subcontinuum K of M joining z to some point of the continuum $A+V$ missing x. Let H be a subcontinuum of K irreducible from z to $A+V$. Then H misses V (because $V \subset (A+V)^{\circ}$) and meets A (because $\partial(A+V) \subset A$). Thus we have a subcontinuum H of M joining z to A missing x and y. Such a continuum, H, also exists if $z \in V$ or (trivially) if $z \in A$. The same argument shows there is a subcontinuum, L, of M joining w to A and not containing x or y. Then $H+L+A$ is a subcontinuum of M joining z to w, missing x, and y, which is a contradiction. Thus, if (a) holds, no subcontinuum of M separates M, i.e., (a) implies (b).

Now assume (b) holds for M. Suppose first that M is semi-locally-connected at the point x. If U is an open set containing x, then let V be an open subset of M containing x such that a finite number of components, $K_1, \ldots, K_n$, of $M-V$ cover $M-U$. Since x is not a cut point, there is a subcontinuum K, of M such that K misses x and K a point of each K_i. Then $K+K_1+\ldots+K_n = H$ is a subcontinuum of M missing x. Since H does not separate M, $M-H$ is an open subset of M containing x. Since the K_i cover $M-U$, $M-H \subset U$. This shows that if (b) holds and M is semi-locally-connected at x, it is locally connected at x. Thus M is locally connected at every point of some dense G_δ set.

Let x and y be points of M at which M is locally connected, and let $\{U_i\}$, $\{V_i\}$ be sequences of open connected sets in M such that $\overline{U}_1 \cap \overline{V}_1 = \emptyset$, $U_1 \supset \overline{U}_2 \supset U_2 \supset \ldots, V_1 \supset \overline{V}_2 \supset V_2 \supset \ldots, \bigcap_{i=1}^{\infty} U_i = \{x\}$, and $\bigcap_{i=1}^{\infty} V_i = \{y\}$.

Suppose that, for some i, $M-(\overline{U}_i+\overline{V}_i)$ were connected. Then $\overline{M-(\overline{U}_i+\overline{V}_i)}$ would be a subcontinuum of M which separates, a contradiction. Thus, for each i, $M-(\overline{U}_i+\overline{V}_i)$ is disconnected. There is a component C of $M-(\overline{U}_i+\overline{V}_i)$ such that $\overline{C}$ meets both $\overline{U}_i$ and $\overline{V}_i$. Since $\overline{C}+\overline{U}_i+\overline{V}_i$ is a subcontinuum of M, it does not separate M and this implies that $M-(\overline{U}_i+\overline{V}_i)$ has only one component besides C. Thus, for each i, $M-(\overline{U}_i+\overline{V}_i)$ is the sum of two disjoint connected open sets and we can label these sets so that $M-(\overline{U}_i+\overline{V}_i) = A_i+B_i$, open and connected, where $A_1 \subset A_2 \subset A_3 \subset$ and $B_1 \subset B_2 \subset B_3 \subset \ldots$ Since $\bigcap_{i=1}^{\infty} U_i = \{x\}$ and $\bigcap_{i=1}^{\infty} V_i = \{y\}$, we see that $M-\{x, y\}$ is the sum of the two disjoint open connected sets $A = \bigcup_{i=1}^{\infty} A_i$ and $B = \bigcup_{i=1}^{\infty} B_i$.

To complete the proof of the theorem, we show that $A+\{x, y\}$ is an arc with endpoints x and y. (Since the same result will hold for $B+\{x, y\}$, M will be a simple closed curve.) Since B is open, $\overline{A} = A+\{x, y\}$ and, dually, $\overline{B} = B+\{x, y\}$. Let C be irreducible in $\overline{A}$ from x to y. Then $M-C = (\overline{A}-C)+B$ and, since C does not separate M, we conclude that $\overline{A}-C = \emptyset$, whence $\overline{A} = C$ and $\overline{A}$ is irreducible from x to y.

Because M is locally connected on a dense G_δ it contains no indecomposable subcontinuum with non-void interior. Since $\bar{A}$ inherits this property, it is, by Theorem 10, of type A'.

Let $\mathscr{D}$ be the minimal admissible decomposition for $\bar{A}$ and let f be a function on $\bar{A}$ to I such that $\mathscr{D} = \mathscr{D}(f)$. For convenience, assume $f(x) = 0$ and $f(y) = 1$. If $f^{-1}[0]$ contains a point $z \neq x$, then we know from Theorem 22 that x cuts z from y in $\bar{A}$. Since $\bar{B} \cap \bar{A} = \{x, y\}$, it is obvious that x would also cut z from y in M, a contradiction. Thus $f^{-1}[0] = \{x\}$ and, similarly, $f^{-1}[1] = \{y\}$. In the same way, if, for $0 < r < 1$, $f^{-1}[r]$ were non-degenerate, then by Theorem 22 there would be distinct points z, w in $f^{-1}[r]$, such that z cuts w from both of x and y in $\bar{A}$. From this it follows readily that z would cut w from both of x and y in M. This contradiction implies that $f^{-1}[r]$ is degenerate. Thus every element of $\mathscr{D}$ is degenerate, and, by Theorem 21, $\bar{A}$ is an arc. ||

In [2], R. H. Bing gives several characterizatios of arcs and simple closed curves. Most of the characterizations in that paper can be obtained by applying our decomposition theory. In particular, Theorem 10 of his paper is the statement that (b) is equivalent to (d) in our Theorem 23, while his Theorem 11 is the statement that (a) is equivalent to (d).

We remark that weaker forms of our Theorem 10 appear in the literature. In a study of unicoherent continua, [13], H. C. Miller proves that an irreducible continuum containing no indecomposable subcontinua has an admissible decomposition. In an earlier paper, [15], W. A. Wilson proves the existence of admissible decompositions for irreducible continua having a dense set of points at which the oscillation of the continuum is zero (if x is such a point, then $\{x\}$ is a member of the minimal admissible decomposition). Although these conditions are sufficient for a continuum to be of type A', they are far from being necessary.

CHAPTER 2

DEFINITION 1. A continuum M is *hereditarily of type A'* if and only if every non-degenerate subcontinuum of M is of type A'.

With each continuum hereditarily of type A' we shall associate a sequence of decompositions. This sequence will be indexed with a subset of the set of ordinals not greater than the first uncountable ordinal. It suffices for our purposes to use the description of this set given on page 29 of [10]. Thus, by Ω' we mean an uncountable set which is linearly ordered by a relation $\leqslant$, and having a largest element Ω, such that $\leqslant$ well orders Ω' and for every $\alpha \neq \Omega$ in Ω' there are at most countably many β in Ω such that $\beta \leqslant \alpha$. We will use small Greek letters for elements of Ω'; the ordinal Ω is called the *first uncountable ordinal* and the other elements of Ω' are called *countable ordinals*. Since $\leqslant$ well orders Ω', every countable ordinal α has an immediate successor, denoted by $\alpha+1$. Elements of Ω' which have no immediate predecessor are called *limit ordinals*. The first few ordinals are denoted by $1, 2, 3, \ldots$ and their supremum by ω_0.

For convenience, the minimal admissible decomposition of a continuum M of type A will be denoted by $\mathscr{D}(M)$. If M is single point, then $\mathscr{D}(M)$ will mean M itself. Moreover, we shall drop the adjective "admissible" so that, unless otherwise stated, "decomposition" means "admissible decomposition". Where convenient we shall use the Δ and $\mathscr{F}$ notation of Chapter 1.

Let M be hereditarily of type A'; for each ordinal, α, in Ω' define a decomposition, $\mathscr{D}_\alpha$, of M in the following way. First of all, $\mathscr{D}_1 = \mathscr{D}(M)$. Let $\alpha \in \Omega'$ and suppose that the decomposition $\mathscr{D}_\beta$ has been defined for each β in Ω' for which $\beta < \alpha$. Then $\mathscr{D}_\alpha$ is defined in one of two ways. If α is not a limit ordinal, then there is $\beta \in \Omega'$ such that $\beta+1 = \alpha$. In this case, $\mathscr{D}_\alpha = \bigcup \{\mathscr{D}(D) |\ D \in \mathscr{D}_\beta\}$. If α is a limit ordinal, then for each $z \in M$ and for each $\beta < \alpha$, let Z_β be the element $\mathscr{D}_\beta$ containing z. Let $Z_\alpha = \bigcap \{Z_\beta |\ \beta < \alpha\}$ and let $\mathscr{D}_\alpha = \{Z_\alpha |\ z \in M\}$. (In this case, $\mathscr{D}_\alpha$ is the infimum, relative to refinement, of the decompositions preceding it.)

Using transfinite induction (i.e., the fact that Ω' is well ordered) it is easy to verify that the above process actually does define a sequence of decompositions of M. One also obtains the following facts:

(1) *If $\alpha, \beta \in \Omega'$ and $\alpha < \beta$, then $\mathscr{D}_\beta$ refines $\mathscr{D}_\alpha$.*

(2) *The elements of each $\mathscr{D}_\alpha$ are subcontinua of M.*

DEFINITION 2. The sequence $\{\mathscr{D}_\alpha|\ \alpha \in \Omega'\}$ is called the *decomposition sequence of M* (or, *associated with M*).

If $z \in M$ and $\alpha \in \Omega'$, then the element of $\mathscr{D}_\alpha$ containing z will frequently be denoted by Z_α.

THEOREM 1. *For each $z \in M$ there exists a countable ordinal, α, such that for each β in Ω' with $\beta \geqslant \alpha$, $Z_\alpha = Z_\beta = \{z\}$. There exists an ordinal $\alpha \in \Omega'$ such that, for each $z \in M$, $Z_\alpha = \{z\}$, i.e., $\mathscr{D}_\alpha = \{\{z\}|\ z \in M\}$.*

Proof. Fix $z \in M$ and observe that, by definition, $Z_\Omega = \bigcap \{Z_\alpha|\ \alpha < \Omega\}$. Since the Z_α are closed and M is compact metric, we can choose a countable subcollection of $\{Z_\alpha|\ \alpha < \Omega\}$ whose intersection is Z_Ω, say $Z_\Omega = \bigcap_{i=1}^{\infty} Z_{\alpha_i}$, where, for each i, $\alpha_i < \Omega$. Let α_0 be the supremum of the ordinals $\{\alpha_i|\ i = 1, 2, \ldots\}$. Then α_0 is countable and $Z_{\alpha_0} = \bigcap_{i=1}^{\infty} Z_{\alpha_i} = Z_\Omega$. If Z_{α_0} were non-degenerate then Z_{α_0+1} would be a proper subcontinuum of Z_{α_0} containing Z_Ω, a contradiction. Thus Z_{α_0} is degenerate, i.e. $Z_{\alpha_0} = \{z\}$, as asserted. For the ordinal, α, of the second sentence of the theorem we may take the supremum over all $z \in M$ of the ordinals obtained above. Since each of the latter is countable, their supremum is not greater than Ω.‖

DEFINITION 3. *The order of a point $z \in M$*, denoted by $O(z)$, is the least ordinal α such that $Z_\alpha = \{z\}$. The *order of M*, $O(M)$, is the least ordinal α such that for each $z \in M$, $Z_\alpha = \{z\}$. Thus, by Theorem 1, $O(z)$ is countable for each $z \in M$ and $O(M) = \sup\{O(z)|\, z \in M\} \in \Omega'$.

In the $\sin(1/x)$ curve $O(z) = 1$, if z is not on the vertical line segment, while $O(z) = 2$ otherwise. In the accordionlike example every point has order 2.

We now construct a continuum M such that $O(M) = \omega_0$. This continuum will also provide examples of other behavior of interest to us later on. Let S_1 denote the $\sin(1/x)$ curve in the plane. Let C be a homeomorph of $(0, 1]$ in the plane whose closure in the plane is the disjoint union of C and a copy of S_1; call this closure S_2. Thus, S_2 is S_1 with the vertical interval replaced by a $\sin(1/x)$ curve. In general, the continuum S_n is obtained from S_{n-1} by replacing the vertical interval in S_1 by a copy of S_{n-1}. It is easy to see that, for each i, S_i is hereditarily of type A' and $O(S_i) = i+1$. Moreover, the set of points of S_i which have order $i+1$ is an arc, A_i.

We now string copies of the S_i together in the following way. Let B_0 be the interval $[(0, 0), (0, 1)]$ in the plane, and for, $i \geqslant 1$, let B_i be the interval $[(1/i, 0), (1/i, 1)]$. For $i \geqslant 1$, place a copy of S_i in the closed rectangle bounded by the lines $y = 0$, $y = 1$, $x = 1/i$, $x = \frac{1}{2}(1/i + 1/(i-1))$ (replace $1/(i-1)$ by 1 in the last expression if $i = 1$) in such a way that

A_i coincides with B_i. Thus, we have placed the S_i so that the arcs $A_i = B_i$ converge homeomorphically to B_0 and, since each point of S_i is within $\frac{1}{2}(1/(i-1))$ of A_i $(i \geqslant 2)$, the S_i also converge to B_0. Let S denote the compact subset of the plane obtained so far.

Now, for each i, the minimal decomposition, $\mathscr{D}(S_i)$, of S_i has exactly one non-degenerate element (the copy of S_{i-1}). Among the other elements of $\mathscr{D}(S_i)$ there is exactly one, $\{y_i\}$, which does not separate S_i. The last step in our construction is to join S_i to S_{i+1} with an arc, C_i, such that $C_i \cap S = \{y_{i+1}, (1/i, 0)\}$, and such that C_i lies in the closed rectangle bounded by $y = 0$, $y = 1$, $y = 1/i$, $y = 1/(i+1)$.

The continuum, M, obtained in this way can be pictured as in Fig. 8.

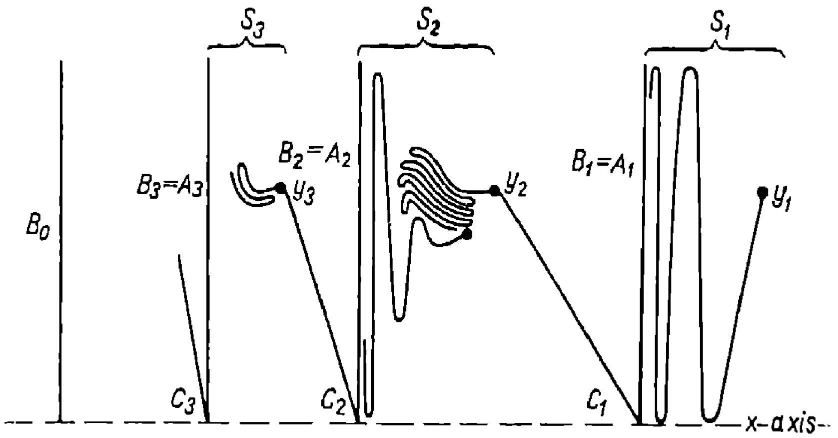

Fig. 8

This continuum is irreducible from y_1 to every point of B_0. It is hereditarily of type A'; in particular, $\mathscr{D}(M)$ has a countable number of non-degenerate elements, the arc B_1, the copy of S_1 lying in S_2, the copy of S_2 lying in S_3, etc., and the arc B_0. For each $i \geqslant 1$, the order in M of the arc, B_i, is $i+1$, hence $\mathrm{O}(M) = \omega_0$. We shall refer to the continuum of this example as *the continuum with increasing oscillations.*

In investigating the relationship between a continuum and its decomposition sequence, we will use the notion of an inverse limit. We shall develop the basic definitions and facts needed below. A discussion of inverse limits, in a somewhat more restricted setting than ours, can be found in [7]. We shall also assume a knowledge of Moore-Smith limits as found in [10].

To begin with, let $\{X_a | \ a \in A\}$ be a collection of (non-void) sets. The Cartesian product of the X_a will be denoted by $\prod\{X_a | \ a \in A\}$, or, if no confusion will result, by Π. Let P_a be the usual projection map of Π onto X_a. By $\langle x_a \rangle$, we mean that point, x, of Π such that $P_a(x) = x_a$ for each $a \in A$. If each X_a is a topological space, then the weakest topology for Π with respect to which each P_a is continuous is called the *product topology* for Π.

Now let A be a set linearly ordered by a relation, $\leqslant$, and, for each $a \in A$, suppose that X_a is a non-void topological space. Suppose also that, for $a, b \in A$ and $a \leqslant b$, there exists a continuous function f_{ba} from X_b into X_a such that f_{aa} is the identity map on X_a and, if $a < b < c$, then $f_{ca} = f_{ba} \circ f_{cb}$. Let X_∞ be the subset of Π consisting of all points, $\langle x_a \rangle$, such that, for $a, b \in A$ and $a < b, f_{ba}(x_a) = x_b$. The maps f_{ba} are called *bonding maps* and the space X_∞, with the relativized product topology, is called *the inverse limit space* $\{X_a\}$ *with bonding maps of* $\{f_{ba}\}$. Unless some restrictions are put on the X_a and the f_{ba}, the inverse limit space may well be empty. The restriction to X_∞ of the projection map P_b will be denoted p_b. Notice that, automatically, *each* p_b *is continuous on* X_∞. Also, if $a < b$, then $p_a = f_{ba} \circ p_b$ on X_∞.

Adding some restrictions we obtain the following useful

REMARK ON INVERSE LIMITS. *Suppose, in the above, that each* X_a *is non-void, bicompact, and* T_1 *(but not necessarily Hausdorff) and that each bonding map is onto. Then, for each* $b \in A$, p_b *maps* X_∞ *onto* X_b; *in particular,* X_∞ *is non-void. Moreover, the topology for* X_∞ *has a base consisting of all sets of the form* $p_b^{-1}[U]$ *where* $b \in A$ *and* U *is open in* X_b. *Each* p_b *is an open map.* X_∞ *is closed in* Π, *bicompact and* T_1.

Proof. Fix $b \in A$ and $y \in X_b$. Let $\mathscr{C}$ be the collection of finite subsets, C, of A whose least element is b. For each $C \in \mathscr{C}$, let $F(C)$ be the collection of points $\langle x_a \rangle$ in Π such that, if $c, d \in C$ and $c < d$, then $f_{dc}(x_d) = x_c$ where we define x_b to be the point y. Using the fact that each bonding map is onto, it is easy to see that, for each $C \in \mathscr{C}$, $F(C)$ is non-void in Π. Also, if $C, D \in \mathscr{C}$, then $F(C+D) \subset F(C) \cap F(D)$. Finally, suppose that $\{x^\lambda\}$ is a net in $F(C)$ converging in Π to the point x. Write $x^\lambda = \langle x_a^\lambda \rangle$; then if $c, d \in C$ and $c < d$, $x_c = \lim\limits_\lambda x_c^\lambda = \lim\limits_\lambda f_{dc}(x_d^\lambda) = f_{dc}(x_d)$. Thus $x \in F(C)$ and we have shown that $F(C)$ is closed. Since $\{F(C) |\ C \in \mathscr{C}\}$ is a family of closed non-void subsets of Π with the finite intersection property and since Π is bicompact, there is a point $x \in \bigcap \{F(C) |\ C \in \mathscr{C}\}$. Then clearly $x \in X_\infty$ and $p_b(x) = y$. This shows p_b is onto.

There is a base for the topology of X_∞ consisting of all sets of the form $V \cap X_\infty$ where $V = \bigcap\limits_{i=1}^{n} P_{b_i}^{-1}[U_i]$, $b_1, \ldots, b_n \in A$, and U_i open in X_{b_i} for $i = 1, \ldots, n$. Suppose that V is of this form; we may assume that $b_1 \leqslant \ldots \leqslant b_n$. Let $U = \bigcap\limits_{i=1}^{n} f_{b_n b_i}^{-1}[U_i]$; then U is open in X_{b_n}. A simple computation shows that $V \cap X_\infty = P_{b_n}^{-1}[U] \cap X_\infty$ and this latter set is, by definition, $p_{b_n}^{-1}[U]$ This proves the assertion about the base for the topology of X_∞.

Finally, fix $b \in A$. To prove that p_b is open, it is enough to prove that the image under p_b of every member of some base for the topology of X_∞ is open. Let $\mathscr{V}$ be the base described above and let V be in $\mathscr{V}$. Write $V = p_a^{-1}[U]$ for some $a \in A$ and some open set, U, in X_a. Then, since p_b is onto, $p_b(V) = X_b$, if $a \neq b$, and U, if $a = b$; in either case, $p_b(V)$ is open.

That X_∞ is closed in Π follows from a simple convergence argument, such as the one used above to prove $F(C)$ closed in Π. Thus, X_∞, being a closed subset of a bicompact space, is bicompact. X_∞ is T_1 because any product of T_1 spaces (and any subspace of a T_1 space) is T_1.‖

We now return to M and its decomposition sequence, $\{\mathscr{D}_\alpha | \ \alpha \in \Omega'\}$. For each ordinal, α, such that $\alpha \leqslant \mathrm{O}(M)$ let M_α denote the quotient space of M with respect to the decomposition $\mathscr{D}_\alpha$ and let q_α denote the quotient map of M onto M_α. We shall retain the use of the symbols $\{Z_\alpha | z \in M\}$ to denote the points of M_α. If $\alpha \leqslant \beta \leqslant \mathrm{O}(M)$, then there is a natural mapping $f_{\beta\alpha}$ of M_β onto M_α defined as follows: for $Z_\beta \in M_\beta$, $f_{\beta\alpha}(Z_\beta) = Z_\alpha$. If $\alpha \leqslant \beta$, then $\mathscr{D}_\beta$ refines $\mathscr{D}_\alpha$, so *the function $f_{\beta\alpha}$ is well defined and maps M_β onto M_α. Moreover, if $\alpha \leqslant \beta \leqslant \gamma \leqslant \mathrm{O}(M)$, then $f_{\gamma\alpha} = f_{\gamma\beta} \circ f_{\beta\alpha}$. Finally, if $\alpha \leqslant \beta \leqslant \mathrm{O}(M)$, then $f_{\beta\alpha}$ is continuous.* To see this last fact first notice that, on X_∞, $q_\alpha = f_{\beta\alpha} \circ q_\beta$. By definition of the quotient topology, a subset U of M_α is open in M_α if and only if $q_\alpha^{-1}[U]$ is open in M. This happens if and only if $q_\beta^{-1}[f_{\beta\alpha}^{-1}[U]]$ is open in M and, again by definition of the quotient topology, this is equivalent to the statement that $f_{\beta\alpha}^{-1}[U]$ is open in M_β. Thus, not only have we proved $f_{\beta\alpha}$ continuous, but also: *if $\alpha \leqslant \beta \leqslant \mathrm{O}(M)$, then M_α has the quotient topology induced by the map $f_{\beta\alpha}$ of M_β onto M_α.* We make one more observation, namely, *if $\alpha \leqslant \mathrm{O}(M)$, then the functions $f_{\mathrm{O}(M)\alpha}$ and q_α coincide on M.*

THEOREM 2. *Assume the above notation. Then, for $\alpha \leqslant \mathrm{O}(M)$, M_α is bicompact, T_1 and connected and the following are equivalent:* (a) *M_α is metrizable*; (b) *M_α is Hausdorff*; (c) *$\mathscr{D}_\alpha$ is upper semi-continuous, and* (d) *q_α is a closed map. If $\alpha \leqslant \beta \leqslant \mathrm{O}(M)$, then $f_{\beta\alpha}$ is monotone. (In particular, if $\alpha < \mathrm{O}(M)$, q_α is monotone.)*

Proof. Since q_α is continuous and maps M onto M_α, M_α is bicompact and connected. It is T_1 because the elements of $\mathscr{D}_\alpha$ are closed in M.

Trivially, (a) implies (b). If $\mathscr{D}_\alpha$ is not upper semi-continuous, then there exists a sequence $\{D_i | \ i = 1, 2, \ldots\}$ of elements of $\mathscr{D}_\alpha$, an element $D \in \mathscr{D}_\alpha$, and two sequences of points, $\{x_i\}$ and $\{y_i\}$, where, for each $i \geqslant 1$, $x_i \in D_i$ and $y_i \in D_i$, such that $\{x_i\}$ converges to a point of D and $\{y_i\}$ converges to a point, y, of $M - D$. Pick E in $\mathscr{D}_\alpha$ such that $y \in E$. Then, in the quotient space M_α, every open set containing D (or E) contains almost all the D_i. Thus any two open sets containing D and E intersect, and M_α is not Hausdorff. This shows that (b) implies (c). The above argument is easily reversed to show that (c) implies (b). Virtually the same proof establishes the equivalence of (c) and (d). Finally, by the remark on quotient spaces, (c) implies (a).

Suppose $a \leqslant \beta \leqslant O(M)$; note that for $Z_a \in M_a$, $f_{\beta a}^{-1}[Z_a] = q_\beta(q_a^{-1}[Z_a])$. Hence the preimage in M_β of a point in M_a under the map $f_{\beta a}$ is a subcontinuum of M_β. (We need to argue further because $f_{\beta a}$ is not necessarily closed.) Let K be a closed subset of M_a such that $f_{\beta a}^{-1}[K] = E+F$, separated, in M_β. Then E and F are closed and, by the above remark, each consists of complete preimages of points of K. Now because M_a has the quotient topology induced by $f_{\beta a}$, $f_{\beta a}(E)$ and $f_{\beta a}(F)$ are closed in M_β. Since these sets are disjoint and K is their union, K is not connected. This proves that $f_{\beta a}$ is monotone.||

Notice that M_1 and $M_{O(M)}$ are metric, indeed M_1 is an arc and $M_{O(M)}$ is homeomorphic to M, but none of the M_a between M_1 and $M_{O(M)}$ need be metric. This is the case in the example of the continuum with increasing oscillations. If M denotes the continuum of that example and x and y are distinct points of the left hand vertical interval, B_0, then for every integer $i > 2$, X_i and Y_i (which are $\{x\}$ and $\{y\}$, respectively) cannot be separated with open subsets of M_i. More specifically, for each $i \geqslant 1$, let x^i be the point $(1/i, 0)$ of A_i in M. Then, for any integer $j \geqslant 2$, the sequence $\{X_j^i|\ i = 1, 2, \ldots\}$ converges in M_j to the distinct points X_j and Y_j of M_j.

Definition 4. Let M be hereditarily of type A' and suppose $O(M)$ is a limit ordinal. Assume the notation established above. Then *the inverse limit space associated with* M, denoted by M_∞, is the inverse limit space of the sequence $\{M_a|\ a < O(M)\}$, with bonding maps $\{f_{\beta a}|\ a \leqslant \beta < O(M)\}$.

Theorem 3. *If M is hereditarily of type A' and $O(M)$ is a limit ordinal, then M_∞ is bicompact, T_1 and connected. There is a one-tó-one, onto continuous function, φ_M, from M onto M_∞ defined as follows: for $z \in M$, $\varphi_M(z) = \langle Z_a \rangle$. The following are equivalent:* (a) *M_∞ is metrizable*; (b) *φ_M is a homeomorphism* (c) *φ_M is closed (open), and,* (d) *M_∞ is Hausdorff.*

Proof. By Theorem 2, each M_a $(a < O(M))$ is bicompact and T_1 and by the remark on inverse limits, this implies M_∞ is bicompact and T_1. That M is connected will follow from the fact that φ_M is continuous, so we now turn to the assertions concerning φ_M.

First of all, for $z \in M$ we have $O(z) \leqslant O(M)$ so that $\{z\} = \bigcap \{Z_a|\ a < O(M)\}$; thus, if z and w are points of M, then $z = w$ if and only if $z \in W_a$ for each $a < O(M)$, i.e., if and only if $Z_a = W_a$ for each $a < O(M)$. This shows that φ_M is well defined and one-to-one. Let y be a point of M_∞; then, for a, $\beta \in \Omega'$ and $a \leqslant \beta < O(M)$, $p_a(y) = f_{\beta a}(p_\beta(y))$ (recall that p_a is the projection of M_∞ onto M_a). This translates into the statement that, in M, $p_\beta(y)$ is a subcontinuum of the continuum $p_a(y)$. Thus, for $y \in M_\infty$, $\{p_a(y)|\ a < O(M)\}$ is a monotone collection of subcontinua of M and therefore there is a point z, belonging to each $p_a(y)$. Since $p_a(y)$ is an element of $\mathscr{D}_a$, we have, by definition, that $Z_a = p_a(y)$ for each $a < O(M)$, and we may write: $\varphi_M(z) = \langle Z_a \rangle = \langle p_a(y) \rangle = y$. This proves that φ_M is onto.

A function into a subset of a product space is continuous if and only if the composition with each coordinate projection map is continuous. If $a < O(M)$ and z is a point of M, then $p_a(\varphi_M(z)) = p_a(\langle z_a \rangle) = Z_a = q_a(z)$, i.e., $p_a \circ \varphi_M$ is just the quotient map, q_a, of M onto M_a. Since each q_a is continuous, φ_M is continuous..

Since the continuous image of a connected space is connected, $\varphi_M(M) = M_\infty$ is connected.

Any one-to-one, continuous function from a compact metric space onto a compact metric space is a homeomorphism, hence (a) implies (b). Any homeomorphism is a closed map, so (b) implies φ_M is closed and, since φ_M is one-to-one, it preserves complementation and is closed if and only if it is open. Thus (b) implies (c). Suppose that x and y are distinct points of M and let U and V be disjoint open sets containing x and y, respectively. If φ_M is open, then $\varphi_M(U)$ and $\varphi_M(V)$ are disjoint, open and contain $\varphi_M(x)$ and $\varphi_M(y)$, respectively. Since φ_M is one-to-one, this shows that (c) implies (d). If M_∞ is Hausdorff, then it is metric by the well-known proposition that the continuous image of a compact metric space is metrizable if it is Hausdorff. (As a matter of fact, the latter proposition is a simple consequence of the remark on quotient spaces.)||

If M is the continuum with increasing oscillations, then M_∞ is not Hausdorff; for let $x_i = (1/i, 0)$ $(i = 1, 2, \ldots)$, then $\{\varphi_M(x_i)\}$ converges to $\varphi_M(x)$ for every point x of B_0. However it is easily seen that if we modify M by squeezing B_0 to a point, the new continuum is homeomorphic with its associated inverse limit space.

After developing some facts about sequential convergence we shall prove that φ_M is monotone. The methods used previously to prove a function monotone do not work here because, in the most interesting cases, φ_M is not closed nor does M_∞ have the quotient topology induced by φ_M.

From now on, we shall use the term "standard notation" to mean the collection of symbols $\mathscr{D}_a$, M_a, Z_a, q_a, p_a, $f_{\beta a}$, and φ_M whose meanings are defined in the preceding paragraphs. Here we understand that a ranges over the set of ordinals less than $O(M)$.

Perhaps it is appropriate to point out at this place that the decomposition sequence, bonding maps, and the inverse limit space associated with a continuum hereditarily of type A' are topological invariants. To begin with we prove the following result.

LEMMA. *Let M and N be continua such that M is irreducible between some pair of points and N is of type A. If h is a monotone, continuous map of M onto N and $\mathscr{D}$ is an admissible decomposition of N, then $\{h^{-1}[D] \mid D \in \mathscr{D}\}$ is an admissible decomposition of M, whence M is of type A.*

Proof. Let f be a monotone continuous function from N onto I such that $\mathscr{D} = \mathscr{D}(f)$. Then the composition, $g = f \circ h$, is monotone, continuous and maps M onto I. By the remarks preceding Theorem 4 of Chapter 1, M is of type A and $\mathscr{D}(g) = \{h^{-1}[D] \mid D \in \mathscr{D}\}$ is an admissible decomposition of M.

Using the Lemma, one easily establishes the following two results:

THEOREM 4. *Suppose that h is a homeomorphism of the topological space M onto the space N. Then one M, N is a continuum of type A' if and only if both are, in which case, $\mathscr{D}(M) = \{h^{-1}[D] \mid D \in \mathscr{D}(N)\}$.*

THEOREM 5. *Let M and N be continua hereditarily of type A' and let h be a homeomorphism of M onto N. Then $\mathrm{O}(M) = \mathrm{O}(N)$ and for each $z \in M$, $\mathrm{O}(z) = \mathrm{O}(h(z))$. Suppose that $\mathrm{O}(M)$ is a limit ordinal; for each $\alpha < \mathrm{O}(M)$ there is a homeomorphism, h_α, of M_α onto N_α. If $\alpha \leqslant \beta < \mathrm{O}(M)$ and $f_{\beta\alpha}$, $g_{\beta\alpha}$ denote the bonding maps for M and N, respectively, then the following diagram commutes:*

$$\begin{tikzcd} M_\alpha \arrow[d, "h_\alpha"'] & M_\beta \arrow[l, "f_{\beta\alpha}"'] \arrow[d, "h_\beta"] \\ N_\alpha & N_\beta \arrow[l, "g_{\beta\alpha}"'] \end{tikzcd}$$

The functions $\{h_\alpha \mid \alpha < \mathrm{O}(M)\}$ induce a homeomorphism, $\hat{h}$, of M_∞ onto N_∞.

In connection with Definition 4 and Theorem 3, notice that we require that $\mathrm{O}(M)$ be a limit ordinal and that the ordinal subscripts for the inverse limit sequence yielding M_∞ go up to, but do not include, $\mathrm{O}(M)$. The reason for these restrictions is this. We want to approximate M in some way by reassembling its successive decompositions. Clearly, if $\alpha < \beta \leqslant \mathrm{O}(M)$, then M_β is a better approximation to M than M_α. However, it is not fair to use $\mathscr{D}_{\mathrm{O}(M)}$ because the quotient space is just M itself. Having agreed not to use $\mathscr{D}_{\mathrm{O}(M)}$, we have problems if $\mathrm{O}(M)$ is not a limit ordinal; for, if there is α in Ω' such that $\alpha+1 = \mathrm{O}(M)$, then the best available approximation to M will be M_α. In this case the quotient mapping q_α of M onto M_α (which is the natural map to use for comparison purposes) must map at least one non-degenerate arc in M onto a point of M_α. As an illustration, let $S_1, S_2,$ be the continua we string together to get the continuum with increasing oscillations; then, for each $i \geqslant 1$, $\mathrm{O}(S_i) = i+1$. Let $(S^i)_1, (S^i)_2, \ldots, (S^i)_{i+1}$ be the first $i+1$ quotient spaces of S_i so that $(S^i)_1$ is an arc and $(S^i)_{i+1}$ is (homeomorphic with) S_i. Then it is easy to verify that, for $i \geqslant 2$, $(S^i)_i$ is homeomorphic with S_{i-1}.

Before continuing our investigation, we prove a useful fact about continua hereditarily of type A'. A continuum T is called a *triod* provided there is a subcontinuum S of T such that $T-S = S_1+S_2+S_3$ where the S_i are pairwise disjoint, non-void and open in T. A continuum, T, is *unicoherent* provided that, if P and Q are subcontinua of T and $T = P+Q$, then $P \frown Q$ is connected; T is *hereditarily unicoherent* provided every subcontinuum of T is unicoherent.

THEOREM 6. *If the continuum M is hereditarily of type A', then M is hereditarily unicoherent and contains no triod.*

Proof. Suppose that T is a triod and S is a subcontinuum of T such that $T-S = S_1+S_2+S_3$, non-void, disjoint and open in T. For each i, S_i+N is a subcontinuum of T. If x and y are points of T, then there exist i and j such that S_i+N+S_j contains x and y. Thus T is irreducible between no pair of its points. Thus no continuum of type A' is a triod and a continuum hereditarily of type A' contains no triod.

Now suppose that P and Q are proper subcontinua of M and $P+Q = M$. If $P \frown Q = A+B$, separated and non-void, then choose p in A and q in B. Let H and K be subcontinua of A and B, respectively, each irreducible from p to q. Note that $H-K \neq \emptyset$ since $H \not\subset B$ and, similarly, $K-H \neq \emptyset$. Now $H+K$ is irreducible between a pair of its points, say x and y. Not both of x and y lie in H, for if so, then $H+K = H$ and $K-H = \emptyset$. Thus for suitable relabeling, $x \in H-K$ and $y \in K-H$. Let D_x be the element of $\mathscr{D}(H+K)$ containing x; then D_x is contained in $H-K$. For if not, then $D_x+K = H+K$ and $H-K \subset D_x$, and since $H-K$ is open in $H+K$ and D_x has void interior relative to $H+K$, this is impossible. Thus D_x is a subcontinuum of $H-K$ with void interior in $H+K$ and hence also with interior relative to H. It follows that D_x lies in some element, D, of $\mathscr{D}(H)$. Next, let D_p and D_q be the elements of $\mathscr{D}(H)$ containing p and q, respectively; then D_x misses D_p and D_q. For if $D_x \frown D_p \neq \emptyset$, then $D_x+D_p+K = H+K$ and $H-K \subset D_x+D_p$, and, again, this is impossible since both D_x and D_p have void interior relative to $H+K$. Now let L_p be a subcontinuum of H irreducible from D_x to D_p. Then $D_x+L_p+K = H+K$ which implies $H-(D_x+L_p) \subset K-H$. Let $E = \overline{H-(D_x+L_p)}$; then E is a subcontinuum of $H \frown K$ joining q to the element D of $\mathscr{D}(H)$ in which D_x lies (and whose existence we established several lines above).

Summarizing, we have a continuum, E, lying in $H \frown K$ and joining q to D. Similary, we can get a continuum, F, lying in $H \frown K$ joining p to D. But then $E+F+D = H$ which implies $H-K \subset D$ and this is impossible. So, after all, $P \frown Q$ is connected and the proof is complete. ||

COROLLARY 1. *Let M be hereditarily of type A' and let x and y be points of M. Then there is a unique subcontinuum of M irreducible from x to y.*

Proof. Let H and K be irreducible from x to y. Then $H+K$ is unicoherent; hence $H \cap K$ is connected. This implies $H \cap K = H = K$.||

Corollary 2. *Let M be hereditarily of type A', K a subcontinuum of M and $a \in \Omega'$. Let $\mathscr{K}_a = \{D \in \mathscr{D}_a | \ D \cap K \neq \emptyset\}$; then all but at most two elements of $\mathscr{K}_a$ are contained in K.*

Proof. If D_1, D_2 and D_3 are in $\mathscr{D}_a$ and $D_i - K \neq \emptyset$ for $i = 1, 2$ and 3, then $K+D_1+D_2+D_3$ would be a triod in M, contradicting the theorem.||

We now characterize sequential convergence in the M_a and in M_∞ in the following theorem and its corollaries.

Theorem 7. *Let M be hereditarily of type A' with $O(M)$ a limit ordinal, and assume the standard notation. Let $\{x^i | \ i = 1, 2, \ldots\}$ be a convergent sequence in M, w a point of M and a an ordinal such that $a < O(M)$. Then $\{X_a^i\}$ converges to W_a in M_a if and only if $\limsup_{i} {}_M X_a^i$ intersects W_a.*

(Recall that $\limsup_{i} {}_M X_a^i = \bigcap_{i=1}^{\infty} [\mathrm{cl}_M(\bigcup_{j=1}^{\infty} X_a^j)]$. It is well known that if $K_1, K_2, \ldots$ are subcontinua of a continuum M and there exists a convergent sequence of points $\{y_i\}$, where $y_i \in K_i$, then $\limsup_{i} {}_M K_i$ is a subcontinuum of M.)

Proof. If $\limsup_{i} {}_M X_a^i$ contains $z \in W_a$, then, for each i, we can pick $z_i \in X_a^i$ so that $\{z_i\}$ converges to z. Then $\{q_a(z_i)\}$ converges to $q_a(z)$ in M_a. Since $q_a(z_i) = X_a^i$ and $q_a(z) = W_a$ this proves the "if" part of the theorem.

Conversely, suppose that $(\limsup_{i} {}_M X_a^i) \cap W_a = \emptyset$; then there is an integer j such that $[\mathrm{cl}_M(\bigcup_{i=j}^{\infty} X_a^i)] \cap W_a = \emptyset$. Let K be the continuum $\limsup_{i} {}_M X_a^i$ and let $\mathscr{K}_a = \{D \in \mathscr{D}_a | D \cap K \neq \emptyset\}$. Now, by Corollary 2 of Theorem 6, $\bigcup \{D | D \in \mathscr{K}_a\}$ is a subcontinuum, L, of M; L contains K and, since $W_a \notin \mathscr{K}_a$, misses W_a. Let $F = L + \bigcup_{i=j}^{\infty} X_a^i$; then F is closed in M, F is the union of elements of $\mathscr{D}_a$ and F misses W_a. Hence $q_a(F)$ is closed in M_a because M_a has the quotient topology relative to q_a. Since $q_a(F)$ contains almost all the X_a^i and misses W_a and is closed in M_a, $\{X_a^i\}$ does not converge to W_a in M_a.||

Corollary 1. *Let M, $\{x^i\}$, and w be as in Theorem 7. For each $a < O(M)$, let $K_a = \limsup_{i} {}_M X_a^i$. Then $\{\varphi_M(x^i)\}$ converges to $\varphi_M(w)$ in M_∞ if and only if $w \in \bigcap \{K_a | \, a < O(M)\}$.*

Proof. If $w \in K_a$ for each $a < O(M)$, then, by the theorem, $\{X_a^i\}$ converges to W_a for each $a < O(M)$ and therefore $\{\langle X_a^i \rangle\}$ converges to $\langle W_a \rangle$ in M_∞.

Conversely, if $\{\varphi_M(x^i)\}$ converges to $\varphi_M(w)$, then, for each $a < \mathrm{O}(M)$, $\{X_a^i\}$ converges to W_a and, by the theorem, $K_a \cap W_a \neq \emptyset$, for each $a < \mathrm{O}(M)$. Fix $a < \mathrm{O}(M)$; then, for $a \leqslant \beta < \mathrm{O}(M)$, $K_\beta \subset K_a$ and $K_a \cap W_\beta \supset K_\beta \cap W_\beta \neq \emptyset$. Thus $\bigcap\{K_a \cap W_\beta | a \leqslant \beta < \mathrm{O}(M)\} = K_a \cap \{w\} \neq \emptyset$, i.e., $w \in K_a$.||

COROLLARY 2. *Let M and $\{x^i\}$ be as in Theorem 7. For each $a < \mathrm{O}(M)$, the set C_a of points w of M such that $\{X_a^i\}$ converges to W_a in M_a is a subcontinuum of M and $q_a(C_a)$ is a subcontinuum of M_a. The set C of points w of M such that $\{\varphi_M(x^i)\}$ converges to $\varphi_M(w)$ in M_∞ is a subcontinuum of M and $\varphi_M(C)$ is a subcontinuum of M_∞.*

Proof. For $a < \mathrm{O}(M)$, let K_a be the set defined in Corollary 1, so that $w \in C_a$ if and only if $W_a \cap K_a = \emptyset$. Thus, $C_a = \bigcup\{D \in \mathscr{D}_a | D \cap K_a \neq \emptyset\}$. By Corollary 2 of Theorem 6, this last set is a subcontinuum of M. By definition, $q_a(C_a)$ is closed in M_a and, since q_a is continuous, $q_a(C_a)$ is connected. For the second assertion, we note that $C = \bigcap\{C_a | a < \mathrm{O}(M)\}$ and since the C_a are nested continua, C is a subcontinuum of M. By definition $\varphi_M(C)$ is closed in M_∞ and therefore, since φ_M is continuous, $\varphi_M(C)$ is a subcontinuum of M_∞.||

It is natural to try to replace sequential convergence by net convergence (i.e. convergence in general) in Theorem 7. The straightforward attempt at generalization would read as follows: "Let M be hereditarily of type A' with $\mathrm{O}(M)$ a limit ordinal and let $\{x^\lambda | \lambda \in \Lambda\}$ be a convergent net in M, w a point of M, and $a < \mathrm{O}(M)$. Then the net $\{X_a^\lambda\}$ converges to W_a in M_a if and only if $\limsup_{\lambda} {}_M X_a^\lambda$ intersects W_a."

Here $\limsup_{\lambda} {}_M X_a^\lambda = \bigcap_{\lambda \in \Lambda} [\mathrm{cl}_M(\bigcup_{\mu \geqslant \lambda} X_a^\mu)]$ may not be connected. In any case, the "if" part of the generalization goes through easily. However, the "only if" part fails, as the following example shows. Let M be the continuum with increasing oscillations and let $x = (0, 0)$. Let H be the collection of points z of M such that $\mathrm{O}(z) = 1$ and the y-coordinate of z is not less than $1/2$. Then every point of M whose y-coordinate is not less than $1/2$ is a limit point of H. If $a < \mathrm{O}(M)$ and U is a neighborhood of X_a in M_a, then $p_a^{-1}[U]$ contains almost all of the vertical intervals $\{B_i | i = 1, 2, \ldots\}$ and in particular $p_a^{-1}[U]$ meets H. From this, and the characterization of the base for the topology of M_∞, it follows that $\varphi_M(x)$ is a limit point of $\varphi_M(H)$ in M_∞. Let $N = \{y^\lambda | \lambda \in \Lambda\}$ be a net in H such that $\{\varphi_M(y^\lambda) | \lambda \in \Lambda\}$ converges in M_∞ to $\varphi_M(x)$. Let $\{x^\mu | \mu \in \Lambda'\}$ be a subnet of N which is convergent in M; then $\{\varphi_M(x^\mu) | \mu \in \Lambda'\}$ still converges to $\varphi_M(x)$ and, therefore, for every $a < \mathrm{O}(M)$, $\{X_a^\mu\}$ converges to X_a. But each x^μ is in H, so, for $a > 1$, $X_a^\mu = \{x^\mu\}$. Hence for every a such that $1 < a < \mathrm{O}(M)$, $\limsup_{\mu} {}_M X_a^\mu$ misses $X_a = \{x\}$, and the generalization fails badly.

THEOREM 8. *Let M be hereditarily of type A' with $\mathrm{O}(M)$ a limit ordinal and assume the standard notation. The function φ_M is monotone.*

Proof. Suppose that K is a closed subset of M_∞ and $\varphi_M^{-1}[K] = E+F$, separated, in M. Since φ_M is continuous, E and F are closed and since φ_M is one-to-one, K is the disjoint union of the sets $\varphi_M(E)$ and $\varphi_M(F)$. To complete the proof, we need only show $\varphi_M(E)$ and $\varphi_M(F)$ are separated in M_∞. Suppose this is false, then there is a sequence $\{y^i\}$ of points in E and a point y of F such that, in M_∞, $\{\varphi_M(y^i)\}$ converges to $\varphi_M(y)$. Since E is closed in M, there is a subsequence $\{x^i\}$ of $\{y^i\}$ converging in M to a point $w \in E$. Thus $\{\varphi_M(x^i)\}$ converges to $\varphi_M(w)$ and to $\varphi_M(y)$. Let C denote the set of points, z, of M such that $\{\varphi_M(x^i)\}$ converges to $\varphi_M(z)$. Then $\varphi_M(C) \subset K$, because K is closed, and, thus $C \subset \varphi_M^{-1}[K]$. But, by Corollary 2 of Theorem 7, C is a subcontinuum of M and since C contains $w \in E$ and $y \in F$, we have a contradiction.||

We will now give a necessary and sufficient condition for M_∞ to be Hausdorff (i.e. for φ_M to be a homeomorphism). Recall that a topological space, S, *satisfies the first axiom of countability* provided there is a countable base for the neighborhood system of each point of S and, in this case, a point, x, of S is a limit point of a subset, T, of S if and only if some sequence in T converges to x. In this connection, notice that, in the example preceding Theorem 8, no sequence in $\varphi_M(H)$ converges to $\varphi_M(x)$ (otherwise Theorem 7 would fail) and hence M_∞ (M is the continuum with increasing oscillations) does not satisfy the first axiom of countability. Also, for $\alpha < \mathrm{O}(M)$, there exists $x \in M$ such that X_α has diameter 1. These two observations suggest the following theorem.

THEOREM 9. *Let M be hereditarily of type A' with $\mathrm{O}(M)$ a limit ordinal. Then M_∞ is Hausdorff if and only if*: (1) *M_∞ satisfies the first axiom of countability*; *and*, (2) *for each positive integer n, there is an ordinal, α_n, such that $\alpha_n < \mathrm{O}(M)$ and, for every $x \in M$*, $\operatorname{diam} X_{\alpha_n} \leqslant 1/n$. (If A is a subset of a metric space with metric d, then $\operatorname{diam} A = \sup\{d(x, y) \mid x \in A, y \in A\}$.)

Proof. Suppose that M_∞ is Hausdorff; then it is metric and (1) holds automatically. If condition (2) fails, then there is a positive integer n and, for each $\alpha < \mathrm{O}(M)$, a point $x^\alpha \in M$ such that $\operatorname{diam}(X^\alpha)_\alpha > 1/n$. Let $A = \{\alpha \mid \alpha < \mathrm{O}(M)\}$; we consider $\{x^\alpha \mid \alpha \in A\}$ as a net in M. Now M is bicompact, so there is a subnet $\{y^\lambda \mid \lambda \in \Lambda\}$ converging to a point of M. Since $\{y^\lambda \mid \lambda \in \Lambda\}$ is a subnet, there is a function g mapping Λ into A such that $y^\lambda = x^{g(\lambda)}$ and for each $\beta \in A$ there is $\mu \in \Lambda$ such that $\lambda \geqslant \mu$ and $\lambda \in \Lambda$ imply $g(\lambda) \geqslant \beta$.

For $\alpha \in A$, let $L_\alpha = \limsup_{\lambda} {}_M (Y^\lambda)_\alpha$; then the net $\{\varphi_M(y^\lambda) \mid \lambda \in \Lambda\}$ converges in M_∞ to $\varphi_M(w)$ for every point w of M for which $w \in \bigcap \{L_\alpha \mid \alpha \in A\}$.

Fix $\alpha \in A$ and pick $\mu \in \Lambda$ such that $\lambda \geqslant \mu$ implies $g(\lambda) \geqslant \alpha$. Then

$$L_\alpha = \limsup_{\lambda} {}_M(Y^\lambda)_\alpha \supset \limsup_{\lambda \geqslant \mu} {}_M(Y^\lambda)_\alpha = \limsup_{\lambda \geqslant \mu} (X^{g(\lambda)})_\alpha.$$

But $g(\lambda) \geqslant \alpha$ implies $(X^{g(\lambda)})_\alpha \supset (X^{g(\lambda)})_{g(\lambda)}$; hence, for $\lambda \geqslant \mu$, $\operatorname{diam}(X^{g(\lambda)})_\alpha \geqslant 1/n$. Therefore L_α contains the limit superior of sets of diameter at least $1/n$ and therefore $\operatorname{diam} L_\alpha \geqslant 1/n$. This holds for each $\alpha \in A$ and, since the L_α are nested, we conclude that $\bigcap \{L_\alpha | \ \alpha \in A\}$ has diameter at least $1/n$. Thus the net $\{\varphi_M(y^\lambda) | \ \lambda \in \Lambda\}$ converges to at least two points of M_∞ and so M_∞ is not Hausdorff.

Now suppose that condition (1) holds; then M_∞ is Hausdorff if and only if every convergent sequence in M_∞ converges to exactly one point of M_∞. Let $\{x^i\}$ be a sequence in M and w a point of M such that $\{\varphi_M(x^i)\}$ converges to $\varphi_M(w)$. By choosing a subsequence of the x_i if necessary, we may require that $\{x^i\}$ be convergent in M. Given $\varepsilon > 0$, let n be a positive integer such that $2/n < \varepsilon$ and let α_n be the ordinal corresponding to n given in condition (2). Then $L_{\alpha_n} = \lim_i \sup (X^i)_{\alpha_n}$ has diameter not greater than $2/n$. This result, together with Corollary 1 of Theorem 7, implies that $\{\varphi_M(x^i)\}$ converges to exactly one point, namely $\varphi_M(w)$, of M_∞. ||

Corollary. *If M_∞ is Hausdorff (i.e. if φ_M is a homeomorphism), then* $O(M)$ *is countable.*

Proof. If M_∞ is Hausdorff, then (2) holds. Let α be the supremum of the α_n given in condition (2). Then α is countable (since no sequence in Ω' has Ω as its supremum) and clearly $O(M) = \alpha$. ||

We have seen, Theorems 3 and 8, that M_∞ is a reasonable approximation to M in the sense that there is a continuous, one-to-one, monotone function from M onto M_∞ and Theorem 9 gives necessary and sufficient conditions for this function to be a homeomorphism. The question naturally arises as to whether M_∞ characterizes M. More precisely, if M and N are hereditarily of type A' and M_∞ and N_∞ are homeomorphic, then are M and N homeomorphic? We do not know the answer to this question. The results which we do obtain along this line assume a homeomorphism between M_∞ and N_∞ and an extra condition on one of M, N. (This condition is satisfied by every continuum we know of which is hereditarily of type A' and whose order is a limit ordinal.)

For the next theorem we shall assume that M and N are hereditarily of type A' and that h a is homeomorphism from M_∞ onto N_∞. The induced function $g = \varphi_N^{-1} \circ h \circ \varphi_M$ is well defined, one-to-one and maps M onto N. If it is continuous, then it is a homeomorphism.

Theorem 10. *In addition to the above, assume that D is a dense subset of M with the property that for each positive integer n there is an ordinal,*

a_n, such that $a_n < \mathrm{O}(M)$ and for each $x \in D$, $\operatorname{diam} X_{a_n} \leqslant 1/n$. Then the function, g, defined above is continuous.

Proof. If $\{x^i\}$ is a sequence in M converging to $x \in M$ and if $\{g(x^i)\}$ converges to $g(y) \in N$; then, in M_∞, $\{\varphi_M(x^i)\}$ converges to both $\varphi_M(x)$ and $\varphi_M(y)$. This follows from the facts that $h^{-1} \circ \varphi_N$ is continuous from N to M_∞ and that $\varphi_M = h^{-1} \circ \varphi_N \circ g$. Using Corollary 1 of Theorem 7, and following the argument of the last paragraph in the proof of Theorem 9, we can then show easily that if $\{x^i\}$ is a sequence in D, converging to $x \in M$, then $\{g(x^i)\}$ converges to $g(x)$ in N.

We now prove that g is continuous. Let $\{z^i\}$ be a sequence in M converging to $z \in M$ and, extracting a subsequence if necessary, let y be a point of M such that $\{g(z^i)\}$ converges to $g(y)$. We wish to show $z = y$. Let d be a metric for M and, for each pair of positive integers i and j, let x^i_j be a point of D such that $d(x^i_j, z^i) \leqslant 1/2^{(i+j)}$. Thus, for each i, $\{x^i_j \mid j = 1, 2, \ldots\}$ converges to z^i. By the above remark, $\{g(x^i_j) \mid j = 1, 2, \ldots\}$ converges to $g(z^i)$ for each i. Since $\{g(z^i)\}$ converges to $g(y)$, a diagonalization process yields a sequence of the form $\{g(z^i_{k_i}) \mid i = 1, 2, \ldots\}$ which converges to $g(y)$. But $\{z^i_{k_i}\}$ converges to z in M, because, for each i, $d(z, z^i_{k_i}) \leqslant d(z, z^i) + 1/2^{(i+k_i)}$. By our opening remark, this implies $g(z) = g(y)$, i.e., $z = y$. ||

COROLLARY. *If M and N are hereditarily of type A' and h is a homeomorphism of M_∞ onto N_∞, then each of the following implies M is homeomorphic with N.*

(a) *$\mathrm{O}(M)$ is uncountable, i.e., $\mathrm{O}(M) = \Omega$.*

(b) *For each positive integer n, there is an ordinal a_n such that $\operatorname{diam} X_{a_n} \leqslant 1/n$, for every $x \in M$.*

(c) *M has a dense set D such that for some ordinal a, $a < \mathrm{O}(M)$ and $X_a = \{x\}$, for every $x \in D$.*

Proof. Condition (b) and (c) are stronger forms of the condition in Theorem 8. If condition (a) holds, then let D be any countable dense subset of M. For each $x \in D$, $\mathrm{O}(X)$ is a countable ordinal (by Theorem 1 of this chapter) and therefore, if we define $a = \sup\{\mathrm{O}(x) \mid x \in D\}$, then a is also countable, i.e., $a < \mathrm{O}(M)$. Hence D satisfies condition (c). ||

We now prove two theorems about inverse limits of continua hereditarily of type A' which, together, yield a method for constructing more complicated examples.

If we have a countable collection $X_1, X_2, \ldots$ of spaces and continuous mappings f_i from X_i onto X_{i-1} $(i \geqslant 2)$, then, defining $g_{ji} = f_{i+1} \circ \ldots \circ f_{j-1} \circ f_j$ for $j > i$ and $g_{jj} =$ identity on X_j, we obtain a family of bonding maps and can form the corresponding inverse limit, X_∞. In this case, we say X_∞ is the *inverse limit* of $X_1 \xleftarrow{f_2} X_2 \xleftarrow{f_3} \ldots$; we shall retain the use of p_i for the projection map of X_∞ onto X_i and we shall write a point, x, of M_∞ in its coordinate notation, $x = \langle x_i \rangle$ where $x_i = p_i(x)$, for i = 1, 2, ...

THEOREM 11. *Let $M_1, M_2, \ldots$ be a sequence of continua, each hereditarily of type A', and for each $i > 1$, let f_i be a continuous, monotone function from M_i onto M_{i-1}. Then the inverse limit, M_∞, of $M_1 \overset{f_2}{\leftarrow} M_2 \overset{f_3}{\leftarrow} \ldots$ is hereditarily of type A'; indeed, if K is a non-degenerate subcontinuum of M_∞ and i_0 the least integer, i, such that $p_i(K)$ is non-degenerate in M_i, then $\{p_{i_0}^{-1}[D] \cap K \mid D \in \mathscr{D}(p_{i_0}(K))\}$ is an admissible decomposition for K. If D is an element of $\mathscr{D}(p_i(K))$ which separates $p_{i_0}(K)$, then $p_{i_0}^{-1}[D] \subset K$.*

Proof. To begin with, the cartesian product of the M_i is metric because there are only countable many coordinate spaces. Hence M_∞ is a bicompact metric space.

Fix $i \geqslant 1$; we will show that the projection, p_i, is monotone. Note that, since p_i is continuous and M_∞ and M_i are bicompact metric, p_i is a closed map, so it suffices to show that, for $x \in M_i$, $p_i^{-1}[x]$ is connected in M_∞. For each $j > 1$, $p_j(p_i^{-1}[x])$ is connected in M_j. (For the non-trivial cases, i.e., $j > i$, this reduces to the fact that the bonding maps, being compositions of the f_i, are monotone). Hence if $p_i^{-1}[x] = A+B$, non-void and closed in M_∞, then, for each $j \geqslant 1$, $p_j(A) \cap p_j(B) \neq \emptyset$. Thus, for each $j \geqslant 1$, the set $C_j = p_j^{-1}[p_j(A) \cap p_j(B)]$ is a non-void closed subset of M_∞. Also if $1 \leqslant j \leqslant k$, then $C_j \supset C_k$ and, by compacteness, there is a point $z = \langle z_i \rangle$ in $\bigcap_{j=1}^{\infty} C_j$. This implies that for each $j \geqslant 1$, $z_j \in p_j(A)$ and, since A is closed, this means that $z \in A$; similarly $z \in B$. Thus $A \cap B \neq \emptyset$ and this implies $p_i^{-1}[x]$ is connected. In particular, $M_\infty = p_1^{-1}[M]$ is connected and, hence, is a continuum.

Let K and i_0 be as in the statement of the theorem. For each $j \leqslant i_0$, $p_j(K)$ is of type A' and irreducible between a pair of elements of $\mathscr{D}(p_j(K))$; call these X_j and Y_j. Let L be any subcontinuum of $p_j(K)$ which meets $f_{j+1}(X_{j+1})$ and $f_{j+1}(Y_{j+1})$ (these sets are contained in $p_j(K)$); then $f_{j+1}^{-1}[L]$ is a subcontinuum of M_{j+1} joining X_{j+1} to Y_{j+1}. By unicoherence and the fact that $p_{j+1}(K)$ is irreducible from X_{j+1} to Y_{j+1}, we conclude that $p_{j+1}(K) \subset f_{j+1}^{-1}[L]$. Therefore, $p_j(K) = f_{j+1}(p_{j+1}(K)) \subset L$ and $p_j(K) = L$. So, $p_j(K)$ is irreducible from $f_{j+1}(X_j)$ to $f_{j+1}(Y_j)$ and, for suitable relabeling, $X_{i_0} \supset f_{i_0+1}(X_{i_0+1}) \supset f_{i_0+1}(f_{i_0+2}(X_{i_0+2})) \supset \ldots$ and similarly for the Y's. Thus, in M_∞, we can pick

$$x \in \bigcap_{j=i_0}^{\infty} p_j^{-1}[X_j] \quad \text{and} \quad y \in \bigcap_{j=i_0}^{\infty} p_j^{-1}[Y_j].$$

It is clear that x and y are points of K between which K is irreducible.

We now show that K has an admissible decomposition. Let $\hat{K} = p_{i_0}^{-1}[p_{i_0}(K)]$. Then the restriction of p_{i_0} to $\hat{K}$ is monotone and, by the lemma preceding Theorem 4, $\mathscr{D} = \{p_{i_0}^{-1}[D] \mid D \in \mathscr{D}(p_{i_0}(K))\}$ is an admissible decomposition of $\hat{K}$. Let X and Y denote the two elements of $\mathscr{D}$ which do not separate $\hat{K}$; then K joins X and Y (indeed $p_{i_0}(X) = X_{i_0}$ and $p_{i_0}(Y) = Y_{i_0}$ if we label properly). From Theorem 5 of Chapter 1 it follows that $\{D \cap K \mid D \in \mathscr{D}\}$ is an admissible decomposition of K and that if $D \in \mathscr{D}$ separates $\hat{K}$, then $D \subset K$. The elements of $\mathscr{D}$ which separate $\hat{K}$ are precisely the preimages under p_{i_0} of the elements of $\mathscr{D}(p_{i_0}(K))$ which separate $p_{i_0}(K)$. This completes the proof. ||

In what follows, we shall use the term "i-th decomposition of M", where M is hereditarily of type A', to denote the decomposition which in standard notation for M we have denoted $\mathscr{D}_i$. The reason for this is that we will be dealing with decompositions of several continua simultaneously.

THEOREM 12. *Suppose, in addition to the hypotheses of Theorem* 11, *that the following conditions hold.*

(a) *M_1 is a non-degenerate arc and, for $i > 1$ and $x \in M_{i-1}$, either $p_i^{-1}[x]$ is a single point or else $f_i^{-1}[x]$ is a non-degenerate arc in M_i.*

(b) *If $2 \leqslant i < j$ and E is a nondegenerate element of the $(i-1)$-st decomposition of M_i, then for each $x \in E$, $(f_{i+1} \circ \ldots \circ f_{j-1} \circ f_j)^{-1}[x]$ has no interior relative to $(f_{i+1} \circ \ldots \circ f_{j-1} \circ f_j)^{-1}[E]$.*

Then for $1 \leqslant i < j$, the i-th decomposition of M_j is $\{(f_{i+1} \circ \ldots \circ f_{j-1} \circ f_j)^{-1}[x] \mid x \in M_i\}$, and, for $1 \leqslant i$, the i-th decomposition of M_∞ is $\{p_i^{-1}[x] \mid x \in M_i\}$.

Proof. Suppose that, for some $i \geqslant 1$, we have proved the second assertion of the theorem. Fix $j > i$ and, letting f denote the identity map of M_j onto itself, consider the sequence: $M_1 \xleftarrow{f_2} M_2 \xleftarrow{f_3} \ldots M_{i-1} \xleftarrow{f_j} M_j \xleftarrow{f} M_j \xleftarrow{f} M_j \leftarrow \ldots$. It is clear that this sequence satisfies all the hypotheses of Theorems 11 and 12. Hence, if N denotes the associated inverse limit space of the new sequence, then the i-th decomposition of N is $\{\pi_i^{-1}[x] \mid x \in M_i\}$, where for $i \geqslant 1$, π_i denotes the projection of N onto M_i. But it is easy to verify that π_j is a homeomorphism and hence the i-th decomposition of M_j is just: $\{\pi_j(\pi_i^{-1}[x]) \mid x \in M_i\} = \{(f_{i+1} \circ \ldots \circ f_{j-1} \circ f_j)^{-1}[x] \mid x \in M_i\}$. So, if, for some $i \geqslant 1$, the second assertion has been proved, then for each $j > i$, the first assertion also holds.

We now verify the second assertion for $i = 1$. By Theorem 11, and the fact that M_1 is an arc, $\{p_1^{-1}[x] \mid x \in M_1\}$ is an admissible decomposition of M_∞. Condition (b), plus the fact that the projections of M_∞ onto the M_i are open, implies that, for $x \in M_1$, $p_1^{-1}[x]$ has void interior in M_∞. Thus, this decomposition must be the minimal decomposition (first decomposition) for M_∞.

Suppose that the second assertion has been proved for all i such that $1 \leqslant i \leqslant n-1$. Let D be a member of the $(n-1)$-st decomposition of M_∞;

then there is $x \in M_{n-1}$ such that $D = p_{n-1}^{-1}[x]$. Assume that D is non-degenerate and let $E = f_n^{-1}[x]$, so that the non-degenerate arc, E, is an element of the $(n-1)$-st decomposition of M_n and $D = p_n^{-1}[E]$.

Define a new sequence of continua and bonding maps as follows. Let $E_1 = E$, $E_2 = f_{n+1}^{-1}[E_1]$, $E_3 = f_{n+2}^{-1}[E_2]$, etc., and let $g_2 = f_{n+1}|E_2$, $g_3 = f_{n+2}|E_3$, etc. Then the sequence $E_1 \xleftarrow{g_2} E_2 \xleftarrow{g_3} E_3 \leftarrow \ldots$ satisfies the hypotheses of Theorem 11 and the associated inverse limit space is D. Let π_i denote the projection of D onto E_i. We assert that the sequence also satisfies the hypotheses of Theorem 12. Condition (a) is easy to verify, so we will proceed with condition (b). Suppose that $2 \leqslant i < j$ and F is a non-degenerate element of the $(i-1)$-st decomposition of E_i and $x \in F$. E_i is an element of the $(n-1)$-st decomposition of M_{n+i-1} and, therefore, F is an element of the $(n+i-2)$-nd decomposition of M_{n+i-1}. By condition (b), $(f_{n+i} \circ \ldots \circ f_{n+j-2} \circ f_{n+j-1})^{-1}[x]$ has void interior in $(f_{n+i} \circ \ldots \circ f_{n+j-2} \circ f_{n+j-1})^{-1}[F]$. In terms of the g_i's this is the statement that $(g_{i+1} \circ \ldots \circ g_{j-1} \circ g_j)^{-1}[x]$ has void interior in $(g_{i+1} \circ \ldots \circ g_{j-1} \circ g_j)^{-1}[F]$, q.e.d.

We can now apply the result proved in the case $i = 1$ for our original sequence to this new sequence to conclude that $\mathscr{D}(D) = \{\pi_1^{-1}[x] | \ x \in E_1\}$. This is precisely the statement that $\mathscr{D}(D) = \{p_n^{-1}[x] | \ x \in E_1\}$, which completes the induction step and proves the theorem.

For emphasis, we restate Theorems 11 and 12 as follows:

Let $M_1 \xleftarrow{f_2} M_2 \xleftarrow{f_3} M_3 \ldots$ *satisfy the hypotheses of Theorems* 11 *and* 12. *For* $1 \leqslant i < j$ *define* f_{ji} *to be the map* $f_{i+1} \circ \ldots \circ f_{j-1} \circ f_j$ *and, for* $1 \leqslant j$, *define* f_{jj} *to be the identity map of* M_j *onto itself. Then* M_∞ *is hereditarily of type* A' *and the collection of spaces* $\{M_i | \ i = 1, 2, \ldots\}$, *together with the family of functions* $\{f_{ji} | \ 1 \leqslant i \leqslant j\}$, *is the decomposition sequence associated with* M_∞. *In particular,* $\mathrm{O}(M) = \omega_0$ *and* $(M_\infty)_\infty = M_\infty$.

Before proceeding with our work we introduce a new notion. A continuum N is said to be *snake-like* provided that for each $\varepsilon > 0$ there is a finite collection, $U_1, \ldots, U_n$ of open subsets of N such that $N = U_1 + \ldots + U_n$, $\operatorname{diam} U_i < \varepsilon$ for $i = 1, \ldots, n$, and if $1 \leqslant i, j \leqslant n$, then $U_i \cap U_j \neq \emptyset$ if and only if $|i-j| \leqslant 1$. A finite collection of open sets satisfying the last two properties above is called an *ε-chain*; thus, N is snake-like if and only if, for every $\varepsilon > 0$, there is an ε-chain covering N. For the basic properties of snake-like continua, see [3]. We shall have more to say about snake-like continua in Chapter 3; at this point we mention several well-known facts about them.

(1) *A snake-like continuum is irreducible between a pair of its points.*

(2) *Every subcontinuum of a snake-like continuum is snake-like.*

(3) *A hereditarily decomposable, hereditarily unicoherent continuum containing no triod is snake-like.*

(4) *If N is a snake-like continuum, then there is a subcontinuum of the plane which is homeomorphic with N.*

Facts (3) and (4) are to be found in [3] and fact (1) can quickly be proved by combining several results in [3]. The proof of fact (3) given in [3] uses a decomposition theorem, and with the aid of our uniqueness theorem, Theorem 6 of Chapter 1, it can be seen that the decomposition given there is just the minimal admissible decomposition. To prove (2), let K be a subset of the snake-like continuum N. Given $\varepsilon > 0$, let $U_1, \ldots, U_n$ be an ε-chain of open subsets of N covering N. Let s be the least integer, i, such that $U_i \cap K \neq \emptyset$ and t the largest integer, i, such that $U_i \cap K \neq \emptyset$. Let $V_i = U_i \cap K$ for $i = s, \ldots, t$. If, for some j such that $s < j < t$, we have $V_j \cap V_{j+1} = \emptyset$, then $V_s + \ldots + V_j$ and $V_{j+1} + \ldots + V_t$ are disjoint open subsets of K whose union is K, i.e., K is not connected. Thus, if K is connected, then $V_s, \ldots, V_t$ is an ε-chain covering K.

It is convenient to combine these facts in the following simple result.

THEOREM 13. *A hereditarily decomposable continuum is hereditarily of type A' if and only if it is snake-like.*

Proof. The "if" part follows from facts (2) and (1); the "only if" part, from fact (3) and Theorem 6 of this chapter, which states that a continuum hereditarily of type A' is hereditarily unicoherent and contains no triod. ||

We now apply the preceding results to obtain an important example. The continuum of this example will be the inverse limit of a sequence $M_1 \xleftarrow{f_2} M_2 \xleftarrow{f_3} M_3 \leftarrow \ldots$ which satisfies the hypotheses of Theorems 11 and 12. In what follows, I denotes the unit interval and, for $n > 1$, I^n denotes the cartesian product of n unit intervals. For convenience we shall use x- and y-coordinates in I^2 and x-, y-, and z- coordinates in I^3 but for $n \geqslant 4$ we label coordinates with the subscripts $x_1, \ldots, x_n$.

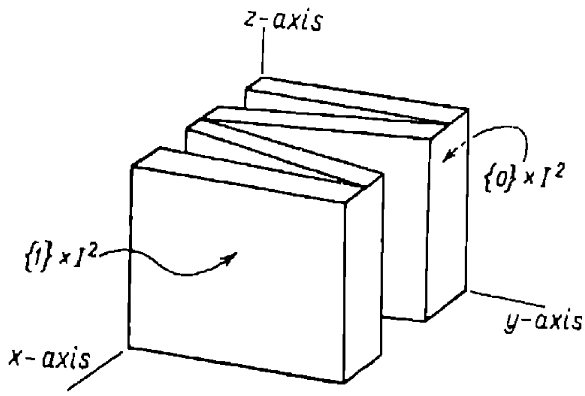

Fig. 9

The construction of the M_i's proceeds inductively. We begin by fixing a copy, M, of the accordionlike continuum of Chapter 1. It is convenient to assume that M is actually constructed as in Chapter 1; in particular, then, M is irreducible from $\{0\}\times I$ to $\{1\}\times I$, lies in $I\times I$ and each element of $\mathscr{D}(M)$ is either a line segment irreducible from a point with y-coordinate 1 to a point with y-coordinate 0, or the sum of two such line segments intersecting in a point with y-coordinate 1 or y-coordinate 0. We also fix a function $f \in \mathscr{F}$ such that $\mathscr{D}(f) = \mathscr{D}(M)$.

Now let $M_1 = I$ and let $M_2 = M$. Let $f_2 = f$ so that f_2 is a map of M_2 onto M_1. To get M_3 we shall replace each element of M_2 with a copy of M. To start with, notice that $M_2 \times I$ is a continuum in I^3 joining $\{0\} \times I^2$ to $\{1\} \times I^2$. $M_2 \times I$ is certainly not irreducible between these two sets because for any $z \in I$, $M \times \{z\}$ lies in $M_2 \times I$. We remove enough of $M_2 \times I$ to make it irreducible, doing this in a "uniform" fashion, as follows. Let $\tilde{M}$ be the image in I^3 of M under the rotation $(x, y, z) \to (z, x, y)$ of I^3. Then define $M_3 = (M_2 \times I) \cap (I \times \tilde{M})$.

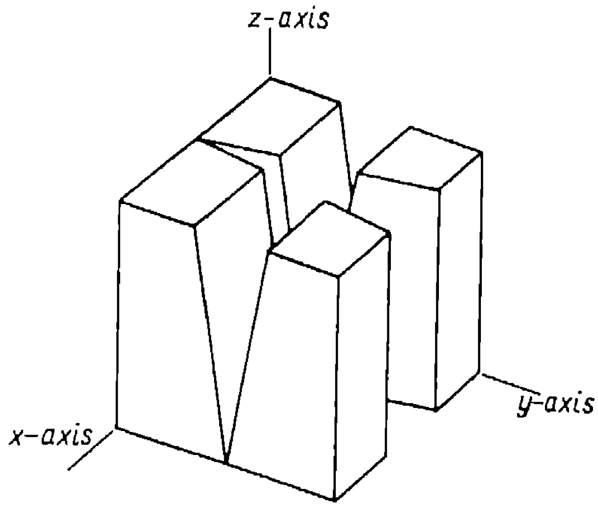

Fig. 10

It is easy to verify that M_3 is hereditarily decomposable, hereditarily unicoherent and contains no triod. Hence, M_3 is hereditarily decomposable and snake-like and, by Theorem 13, hereditarily of type A'. (As an alternative, one can proceed directly to prove M_3 snake-like by actually constructing an ε-chain cover for each $\varepsilon > 0$.)

It is easily seen that the elements of the first decomposition, $\mathscr{D}(M_3)$, of M_3 are copies of M, each perpendicular to the xy plane and projecting downwards onto an element of $\mathscr{D}(M_2)$. Each element of the second decomposition of M_3 is either a line segment irreducible from a point with z-coordinate 1 to a point with z-coordinate 0 or the union of two such segments intersecting in a point with z-coordinate 0 or z-coordinate 1. There is a homeomorphism, h, of the second decomposition space $(M_3)_2$ of M_3 onto M_2. If, as usual, q_2 denotes the quotient map of M_3 onto $(M_3)_2$, then $f_3 = h \circ q_2$ is a monotone, continuous mapping of M_3 onto M_2 such that if $x \in M_3$ then $f_3^{-1}[x]$ is a non-degenerate arc. Also condition (b) of Theorem 12 holds for the three term sequence: $M_1 \xleftarrow{f_2} M_2 \xleftarrow{f_3} M_3$. We can actually define f_3 explicitly. If f denotes the map we choose of M onto I such that $\mathscr{D}(f)$

$= \mathscr{D}(M)$, then $f_3(x, y, z) = (x, f(y, z), 0)$ is, for some homeomorphism h, the map given above. Roughly speaking, f_3 is projection of M_3 onto the xy plane except that instead of projecting directly downwards we project *via* f.

The last statement above indicates the general construction. To get M_n from $M_{n-1} \subset I^{n-1}$ we replace each of the arcs in the $(n-2)$-nd decomposition of M_{n-1} by a copy of M in I^n so that projection (*via* f) downwards into I^{n-1} gives the arc back again. Explicitly, having constructed $M_{n-1} \subset I^{n-1}$ $(n \geqslant 3)$, let $\tilde{M}$ be the image of $M \subset I^n$ under the rotation $(x_1, x_2, \ldots, x_{n-1}, x_n) \to (x_3, \ldots, x_n, x_1, x_2)$ and let $M_n = (M_{n-1} \times I) \cap (I^{n-2} \cap \tilde{M})$. Let $f_n(x_1, x_2, \ldots, x_{n-1}, x_n) = (x_1, x_2, \ldots, f(x_{n-1}, x_n), 0)$. The sequence $M_1 \xleftarrow{f_2} M_2 \xleftarrow{f_3} \ldots M_{n-1} \xleftarrow{f_n} M_n \leftarrow \ldots$ satisfies all hypotheses of Theorems 11 and 12; as usual, let M_∞ denote the inverse limit. We shall refer to M_∞ as the *inverse limit continuum*.

There are two properties of M_∞ of immediate interest. The first of these is evident from the construction, namely: *For each* $z \in M_\infty$, $\mathrm{O}(z) = \omega_0$. The second property is harder to state but might be roughly summarized as follows: *Every non-degenerate subcontinuum of* M_∞ *is almost homeomorphic with* M_∞. We shall break down the second statement into four more precise ones. First of all, from Theorem 12, the n-th decomposition of M_∞ is $\{p_n^{-1}[x] |\ x \in M_n\} = \{p_{n+1}^{-1}[f_{n+1}^{-1}[x]] |\ x \in M_n\}$. Fix x in M_n; then $D = f_{n+1}^{-1}[x]$ is an arc in M_{n+1}, and from the construction process, it is clear that $p_{n+1}^{-1}[D]$ is homeomorphic with M_∞. Thus: *For each* $n \geqslant 1$, *every element of the* n*-th decomposition of* M_∞ *is homeomorphic with* M_∞.

Let L be a subcontinuum of M_3 with non-void interior relative to M_3. Then $\overline{L^\circ}$ (both operations taken relative to M_3) is homeomorphic with M_3 and $L - \overline{L^\circ}$ is the sum of at most two connected sets whose closures in M_3 are homeomorphic to subcontinua of M_2. Each of the latter sets, if non-void, is either an arc or a copy of M_2 joining $\overline{L^\circ}$ along a common edge with (possibly) an arc attached to the other edge. This possibility is illustrated as in Fig. 11.

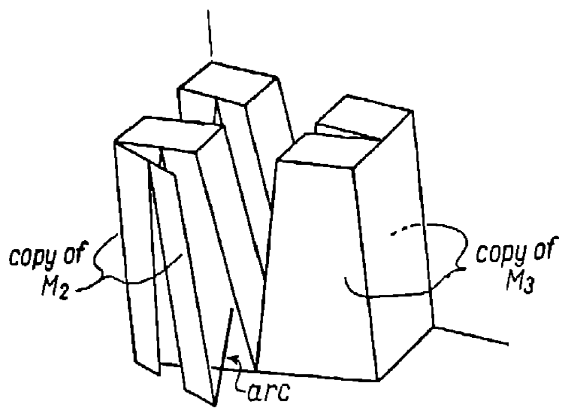

Fig. 11

In any case, the complete preimage, $p_3^{-1}[L]$, of L in M_3 is homeomorphic with M_∞, since this is true of each of the pieces. The same result holds for any $n \geqslant 1$: *If L is a subcontinuum of M_n $(n \geqslant 1)$, then $p_n^{-1}[L]$ is homeomorphic with M_∞.*

Let K be any subcontinuum of M_∞; then, for each n, $p_n(K)$ is a subcontinuum of M_n and $K = \bigcap_{n=1}^{\infty} p_n^{-1}[p_n(K)]$ (because K is closed). Combining this with the preceding remark we see that: *Every subcontinuum of M_∞ is the intersection of a nested sequence of subcontinua of M_∞, each homeomorphic to M_∞.*

The final result is a sort of dual to the preceding. Let K be a non-degenerate subcontinuum of M_∞ and i_0 the least integer i such that $p_i(K)$ is non-degenerate. In proving Theorem 11 we showed that if X_i and Y_i are the elements of $\mathscr{D}(p_i(K))$ between which $p_i(K)$ is irreducible $(i \geqslant i_0)$ then, for suitable labeling, K is irreducible from every point of $\bigcap_{i=i_0}^{\infty} p_i^{-1}[X_i] = A$ to every point of $\bigcap_{i=i_0}^{\infty} p_i^{-1}[Y_i] = B$ (and that A and B are non-void subsets of K). Now choose a function f from K onto I such that $\mathscr{D}(K) = \mathscr{D}(f)$ and, for $i \geqslant 3$, let $K_i = f^{-1}[[1/i, 1-1/i]]$; then each K_i misses $A+B$ and $\{K_i |\ i = 3, 4, \ldots\}$ is an increasing sequence of subcontinua of K whose union is $f^{-1}[(0, 1)]$. For $i \geqslant 3$, there is j_i such that $p_{j_i}(K_i)$ misses $X_i + Y_i$. By Theorem 11, $p_{j_i}^{-1}[p_{j_i}(K_i)] \subset K$ and we can choose the j_i so that $j_3 < j_4 < \ldots$ Since each $p_{j_i}^{-1}[p_{j_i}(K_i)]$ is homeomorphic with M_∞, we have proved the following result:

Let K be a non-degenerate subcontinuum of M_∞ and f a function such that $\mathscr{D}(K) = \mathscr{D}(f)$. Then there exists a countable increasing family of subcontinua of K, each of which is homeomorphic with M_∞, whose union is $f^{-1}[(0, 1)]$.

G. W. Henderson, in [6], proved that a decomposable continuum which is homeomorphic with each of its non-degenerate subcontinua is an arc. The previous example was an unsuccessful attempt at a counterexample (made before seeing Henderson's proof). In Theorem 2 of Chapter 3 we show that M_∞ contains subcontinua K such that the two elements of $\mathscr{D}(K)$ which do not separate K are degenerate. We also remark that L. K. Barrett, in [1], sketches the construction of a snake-like continuum which is very possibly homeomorphic to our inverse limit continuum. Our Theorems 11 and 12 set up machinery for investigating the decomposition properties of continua constructed by inverse limits. As no comparable machinery exists for construction by ε-chains, we cannot be sure that our example and Mrs. Barrett's are the same.

That our example is snake-like follows from our next theorem. (This theorem does not seem to be in the literature but is probably well known.)

THEOREM 14. *Let $\{M_i| \ i = 1, 2, \ldots\}$ be a countable collection of snake-like continua and, for $i \geqslant 2$, let f_i be a continuous map of M_i onto M_{i-1}; then the inverse limit, M_∞, of the M_i is snake-like.*

Proof. As usual, we use p_i for the projection of M_∞ onto M_i, also for $i \geqslant 1$, let d_i be a metric, for M_i. We assert that, for each $\varepsilon > 0$, there exist positive integers i and j such that if x and y are points of M_i and $d_i(x, y) < 1/j$, then the diameter of $p_i^{-1}[x]+p_i^{-1}[y]$ in M_∞ is less than ε.

Assume the assertion is false. Fix $i \geqslant 1$; then, for each $j \geqslant 1$, there exist points x_j and y_j of M_i such that $d_i(x_j, y_j) \leqslant 1/j$ and $\operatorname{diam}(p_i^{-1}[x_j]+ +p_i^{-1}[y_j]) \geqslant \varepsilon$. We may assume that x_j and y_j converge in M_i to a common point $z(i)$ of M_i. A simple calculation shows that $\operatorname{diam}(p_i^{-1}[z(i)]) \geqslant \varepsilon$ in M_∞.

Now for each i, let w_i be a point of $p_i^{-}\ [z(i)]$. We may assume the w_i converge to a point w in M_∞. (We do not exclude the possibility that $w_i = w$ for infinitely many i). Let $K = \limsup_i p_i^{-1}[z(i)]$; then $\operatorname{diam} K \geqslant \varepsilon$. Let $x \in K$ and let $\{x_i\}$ be a sequence converging to x such that $x_i \in p_i^{-1}[z(i)]$ for $i \geqslant 1$. For $1 \leqslant i \leqslant j$, $p_i(x_j) \in p_i(p_j^{-1}[z(i)]) = (f_{i+1} \circ \ldots \circ f_j)(z(j))$, and this is a point of M_i. Since w_j also lies in $p_j^{-1}[z(j)]$, we conclude that, if $1 \leqslant i \leqslant j$, $p_i(x_j) = p_i(w_j)$. Holding i fixed and using the fact that p_i is continuous, we see that, for each i, $p_i(x) = p_i(w)$. This holds for each $x \in K$, hence $p_i(K) = p_i(w)$ for $i \geqslant 1$. From this it follows that K is degenerate in M_∞, which contradicts $\operatorname{diam} K \geqslant \varepsilon$. Thus our original assertion holds.

Now let ε be a positive number; we may, by the assertion, choose positive integers i and j so that if U is a subset of M_i of diameter less than $1/j$, then $p_i^{-1}(U)$ has diameter less than ε in M_∞. Let $U_1, \ldots, U_n$ be a $1/j$-chain of open sets covering M_i. Then $p_i^{-1}(U_1), \ldots, p_i^{-1}(U_n)$ is a collection of open sets, each of diameter less than ε, and this collection covers M_∞. Also this is a chain, for $p_i^{-1}(U_j) \cap p_i^{-1}(U_k) \neq \emptyset$ in M_∞ if and only if $U_j \cap U_k \neq \emptyset$ in M_i.||

From Theorem 14 and the fact that every snake-like continuum is homeomorphic with a subcontinuum of the plane, we conclude that the inverse limit continuum is homeomorphic to a plane continuum.

We end this chapter with some results on embedding certain continua of type A' in other such continua. The 2-*sphere* is the one point compactification of the Euclidean plane. We shall use the following characterization of the 2-sphere, due to Kuratowski, [11]:

If S is a bicompact, metric, connected, locally connected space (i.e. a Peano space) such that every subcontinuum of S which does not separate S is unicoherent and such that no point of S separates S, then S is homeomorphic with the 2-sphere.

Conversely, the 2-sphere, S, is a Peano space with no separating points and a result of Janiszewski (see [12], pp. 353-354), states that any subcontinuum of S which does not separate S is unicoherent.

THEOREM 15. *Let M be a nowhere dense, closed and compact subset of the 2-sphere, S, and $\mathscr{D}$ an upper semicontinuous decomposition of M each of whose elements is a compact subcontinuum of S which does not separate S. Let $\mathscr{E}$ be the decomposition of S whose elements are the elements of $\mathscr{D}$ together with the family of sets $\{\{x\}|x\notin M\}$. Then the quotient space, T, of S relative to $\mathscr{E}$ is homeomorphic with S and, if q denotes the quotient map, then the restriction of q to $S-M$ is a homeomorphism of $S-M$ onto $T-q(M)$.*

Proof. First, notice that $\mathscr{E}$ is upper semi-continuous because M is closed and compact and $\mathscr{D}$ is upper semi-continuous. Since each element of $\mathscr{E}$ is closed and compact, the remark on quotient spaces (see the introduction) applies and T is metrizable, bicompact, connected and locally connected. The quotient map, q, is monotone because the elements of $\mathscr{E}$ are connected and q is closed.

T has no separating points because no element of $\mathscr{E}$ separates S.

Let K be a subcontinuum of T which does not separate T and suppose that the continuum $q^{-1}[K]$ separates S. Since S is locally connected, every component of $q^{-1}[K]$ is open; M contains no open subset of S, so there are points x and y of $S-M$ which lie in different components of $S-q^{-1}[K]$. Now $q(x) \neq q(y)$ and neither point lies in K, and since T is locally connected, there is a subcontinuum, L, of T such that $q(x)\in L$, $q(y)\in L$ and $L \cap K = \emptyset$. Then $q^{-1}[L]$ is a continuum in S joining x to y and missing $q^{-1}[K]$ which is impossible. Thus, if the subcontinuum K does not separate T, then $q^{-1}[K]$ does not separate S.

With K as above, suppose that P and Q are subcontinua of K with $K = P+Q$. Then $q^{-1}[P]$, $q^{-1}[Q]$ are subcontinua of $q^{-1}[K]$ and $q^{-1}[K] = q^{-1}[P]+q^{-1}[Q]$. Since $q^{-1}[K]$ does not separate S, Janiszewski's result implies that it is unicoherent and therefore $q^{-1}[P] \cap q^{-1}[Q]$ is connected. Since q is continuous $q(q^{-1}[P] \cap q^{-1}[Q]) = P \cap Q$ is connected. Thus K is unicoherent.

We have shown that S satisfies all hypotheses of Kuratowski's theorem and we conclude that T is homeomorphic with S.

The assertion about q follows immediately from the definition of $\mathscr{E}$. ||

COROLLARY. *Let S denote the 2-sphere and B an arc in S. Let M be a compact subcontinuum of S which is of type A and has an admissible decomposition, $\mathscr{D}$, no member of which separates S. Then there is a continuous monotone function, h, mapping S onto S such that $h(M) = B$, $\{h^{-1}[x]|\ x\in B\} = \mathscr{D}$, and the restriction of h to $S-M$ is a homeomorphism of $S-M$ onto $S-B$.*

Proof. Obviously a subcontinuum of S which contains an open subset of S is not irreducible between any pair of its points; hence M is nowhere dense in S. We may apply Theorem 15 to M and $\mathscr{D}$ and, using the notation of that theorem, we see that $q(M)$ is an arc and that there is a homeomorphism, ψ, of S onto T. Now $\psi(q(M))$ and B are arcs in S, hence, as is well known, there is a homeomorphism, φ, of S onto S such that $\varphi(\psi(q(M)) = B$. Let $h = \varphi\circ\psi\circ q$; then h has the desired properties.||

THEOREM 16. *Let M and N be continua hereditarily of type A' in the 2-sphere, S, and let D be a non-degenerate element of the α-th decomposition of M, where $\alpha < \mathrm{O}(M)$. Suppose that D satisfies the following condition: If L is a subcontinuum of M such that $L \cap D \neq \emptyset$ and $L-D \neq \emptyset$, then L is irreducible from some point of $L-D$ to every point of D (so, in particular, $D \subset L$).*

Then there is a continuum $M' \subset S$, hereditarily of type A', such that $N \subset M'$, N is an element of α-th decomposition of M', $M'-N$ is homeomorphic with $M-D$, and N satisfies the above condition relative to M'.

(We remark that the condition on D can be rephrased as follows: If L is a continuum in M such that $L\cap D \neq \emptyset$ and $L-D \neq \emptyset$, then D is contained in one of the two elements of $\mathscr{D}(L)$ which do not separate L.)

Proof. If a decomposable subcontinuum T of S separates S, then T is not unicoherent; thus no element of $\mathscr{D}(D)$ or $\mathscr{D}(N)$ separates S. Fix an arc, B, in S and apply the previous corollary to D and to N to obtain continuous monotone mappings g and h of S onto S such that: $g(D) = h(N) = B$; $\{g^{-1}[x]|\ x\in B\} = \mathscr{D}(D)$; $\{h^{-1}[x]|\ x\in B\} = \mathscr{D}(N)$, the restriction of g to $S-D$ is a homeomorphism of $S-D$ onto $S-B$, and the restriction of h to $S-N$ is a homeomorphism of $S-N$ onto $S-B$.

Let $M' = h^{-1}[g(M)]$; since h is monotone and g is continuous (and closed), M' is a subcontinuum of S containing N. We can also write $M' = h^{-1}[g(D)+g(M-D)] = h^{-1}[B-g(M-D)] = N+h^{-1}[g(M-D)]$. It is easy to see that $h^{-1}\circ g$ is a well defined function on $S-D$ and, in fact, a homeomorphism of $S-D$ onto $S-N$. It follows that $M-D$ is homeomorphic (*via* $h^{-1}\circ g$) with $M'-N$.

We now show that M' is hereditarily of type A'. Let K be a subcontinuum of M'; we first show that K is decomposable. This is trivial if $K\subset N$ or $K \subset M'-N$, so we assume $K\cap N \neq \emptyset$, $K-N \neq \emptyset$. The continuum $L = g^{-1}[h(K)]$ then satisfies $L\cap D \neq \emptyset$, $L-D \neq \emptyset$. Since M' is hereditarily of type A' and L and D are subcontinua of M, there exist proper subcontinua, P and Q, of L such that $L\cap D \subset Q-P$ and $P+Q = L$. Then $h^{-1}[g(P)]$ is a proper subcontinuum of K with non-void interior relative to K, and this implies that K is decomposable. In particular, M' is hereditarily decomposable.

With K and L as in the preceding paragraph, let us prove K is irreducible between a pair of its points. The condition on D implies that $D \subset L$ and that L is irreducible from a point, x, of $L-D$ to D. Now $K-N = h^{-1}[g(L-D)]$ is connected, contains the point $h^{-1}[g(x)]$ and its closure meets N. If Q were a proper subcontinuum of $\overline{K-N}$ joining $h^{-1}[g(x)]$ to N, then $K-N-Q \neq \emptyset$ and $g^{-1}[h(Q)]$ would be a subcontinuum of L joining x to D missing the non-void open subset $g^{-1}[h(K-N-$ $-Q)]$ of $L-D$. Since this is impossible, we conclude that $\overline{K-N}$ is irreducible from $h^{-1}[g(x)]$ to $\overline{K-N} \cap N$.

Suppose $\overline{K-N} \cap N = A+B$, separated and non-void. Let K' be irreducible in $\overline{K-N}$ from A to B. Then K' is a subcontinuum of M' such that $K' \cap N \neq \emptyset$ and $K'-N \neq \emptyset$ and applying the same argument to K' as we did to K, we conclude that $\overline{K'-N}$ is irreducible from a point of $K'-N$ to $\overline{K'-N} \cap N$, which is absurd. So, after all, $\overline{K-N} \cap N$ is connected. The facts that $\overline{L-D}$ contains D and $\{h^{-1}[x] | \ x \in B\} = \mathscr{D}(N)$ imply that $\overline{K-N} \cap N$ meets every element of $\mathscr{D}(N)$, and, therefore, $\overline{K-N} \cap N = N$, i.e., $N \subset \overline{K-N}$. Thus $\overline{K-N} = K$ and K is irreducible between a pair of its points.

We have proved that M' is hereditarily of type A' and, moreover, that N satisfies the same condition in M' as D does in M. It remains to show that D is an element of the α-th decomposition of M'. To this end, we shall prove the following statement by induction: If $\beta \leqslant \alpha$, then the collection of all sets of the form $h^{-1}[g(E)]$, where E is a member of the β-th decomposition of M, is the β-th decomposition of M'.

If $\beta = 1$, then $\{h^{-1}[g(E)] | \ E \in \mathscr{D}(M)\}$ is a decomposition, $\mathscr{E}$, of M' whose elements are continua, all except two of which, say $h^{-1}[g(E_0)]$ and $h^{-1}[g(E_1)]$, separate M'. Now D lies in one of E_0, E_1, say $D \subset E_0$, and, in this case, $N \subset h^{-1}[g(E_0)]$ (equality is not excluded). Now N has no interior relative to any subcontinuum of M' which contains it and this, together with the fact that $M-E_0$ is homeomorphic with $M'-$ $-h^{-1}[g(E_0)]$, implies that each member of $\mathscr{E}$ has void interior relative to M' and that $\mathscr{E}$ is upper semi-continuous. Thus $\mathscr{E}$ must be the minimal decomposition for M', and the statement is established for $\beta = 1$.

Suppose that $\beta \leqslant \alpha$ and that for every $\gamma < \beta$, the statement holds. If β is a non-limit ordinal, then the statement holds for that ordinal γ such that $\gamma+1 = \beta$. Let F be an element of the γ-th decomposition of M'; then $F = h^{-1}[g(E)]$ where E is in the γ-th decomposition of M. The argument used in the case $\beta = 1$ then applies and $\mathscr{D}(F) = \{h^{-1}[g(H)] | H \in \mathscr{D}(E)\}$. Letting E run through the γ-th decomposition of M, we see that the statement holds for $\gamma+1 = \beta$. If β is a limit ordinal, then a direct application of the definition of the β-th decomposition plus the fact that $h^{-1} \circ g$ is one-to-one on $M-D$, yields the desired result.‖

The mapping of M_β onto M'_β induced by the one-to-one correspondence between the β-th decomposition given above is easily seen to be bicontinuous. Hence we have the following:

COROLLARY TO THE PROOF. *If M, N, D and a are as in Theorem 16, then the continuum M' can also be chosen so that, for every $\beta \leqslant a$, M_β and M'_β are homeomorphic.*

In Theorem 16, the condition imposed on D is used mainly to prove that every non-degenerate subcontinuum of M' is irreducible between a pair of its points. We believe it should be possible to remove this condition.

We do not know whether there is a continuum hereditarily of type A' whose order is Ω. If M were such a continuum, then it would contain an uncountable family, $\mathscr{K}$, of subcontinua which could be indexed by the set $A = \{\alpha \in \Omega' | \ \alpha < \Omega\}$, say $\mathscr{K} = \{K_\alpha | \ \alpha \in A\}$, in such a way that, for each α in A, the order in M of every point of K_α is at least α. Hence one step towards constructing a continuum of order Ω would be the construction of a family $\mathscr{K}$ of continua each hereditarily of type A', such that for each countable ordinal, α, some member of $\mathscr{K}$ has order at least α. A promising, but so far unsuccessful, approach to this last problem is to attempt to construct $\mathscr{K}$ inductively, using the embedding Theorem 16 to get from each countable ordinal to its successor, and using the inverse limit Theorems 11 and 12 at the limit ordinals.

CHAPTER 3

In order to shorten some of the statements in this chapter, we shall introduce the following notation. Let M be of type A', $\mathscr{D}$ an admissible decomposition of M, and let f be a function such that $\mathscr{D} = \mathscr{D}(f)$. For each point s of I, let D_s denote the element $f^{-1}[s]$ of $\mathscr{D}$. Let $L_s = \overline{f^{-1}[[0, s)]}$ if $0 < s$ and let $L_0 = D_0$. Let $R_s = \overline{f^{-1}[[s, 1]]}$ if $s < 1$ and let $R_1 = D_1$. Thus, for each s in I, L_s and R_s are subcontinua of M. If $\mathscr{D} = \mathscr{D}(M)$, then $L_s + R_s = M$. (The capital letters L and R stand for left and right, respectively.)

DEFINITION 1. Let $\mathscr{D}$ be a decomposition of a metric space S, and D an element of $\mathscr{D}$. We say that $\mathscr{D}$ *is continuous at* D provided that, if $\{D_i\}$ is a sequence of elements in $\mathscr{D}$ and there exists a point $x_i \in D_i$ for $i = 1, 2, \ldots$, such that the sequence $\{x_i\}$ converges to a point of D, then $\limsup_{i} D_i = D$. We say that $\mathscr{D}$ is *continuous* provided it is continuous at each of its elements.

Notice that a continuous decompositions is automatically upper semi-continuous.

Let M denote the accordionlike continuum; then $\mathscr{D}(M)$ is not continuous at any of the $\vee$'s or $\wedge$'s, but is continuous at each of its other elements, i.e., at each straight line interval.

There is an example of a continuum of type A' whose minimal decomposition is continuous. By a *pseudo-arc* we mean any non-degenerate, snake-like, hereditarily indecomposable continuum. Any two of these are homeomorphic and, therefore, each non-degenerate subcontinuum of a pseudo-arc is a pseudo-arc. See [4] for further properties and historical remarks. In [4], R. H. Bing and F. B. Jones gave an example of what, in their terminology, would be called a continuous circle of pseudo-arcs embedded in the plane. The construction of this continuum and the derivation of its properties are too complicated to give here, however, a simple modification of that construction yields a continuous arc of pseudo-arcs. This is a continuum, M, with the following properties.

(a) M is snake-like and a subset of the plane.

(b) M is of type A'.

(c) Every element of $\mathscr{D}(M)$ is a pseudo-arc of diameter at least 1.

(d) $\mathscr{D}(M)$ is continuous.

Our first theorem says that if M is of type A, then any admissible decomposition is continuous almost everywhere.

THEOREM 1. *Let M be of type A, $\mathcal{D}$ an admissible decomposition and f a function such that $\mathcal{D} = \mathcal{D}(f)$. The set of points r in I such that $\mathcal{D}$ is continuous at $f^{-1}[r]$ is a dense G_δ set, G, in I.*

Proof. For each positive integer i, let X_i be the set of points x in M such that if U is an open subset of M containing x and $\operatorname{diam} U \leqslant 1/i$, then $f(U)$ is not a neighborhood of $f(x)$ in I.

We show that X_i is closed in M. Let $\{x_j\}$ be a sequence in X_i converging to $x \in M$ and let U be an open subset of M containing x with $\operatorname{diam} U \leqslant 1/i$. Let N be a positive integer such that, for $j \geqslant N$, $x_j \in U$. Then, for $j \geqslant N$, $f(U)$ is not a neighborhood of $f(x_j)$ in I, and there is a point y_j in I such that $|x_j - y_j| \leqslant 1/j$ and $y_j \notin f(U)$. The sequence $\{y_j | j \geqslant N\}$ lies in $I - f(U)$ and converges to x, so $f(U)$ is not a neighborhood of $f(x)$. Thus X_i contains all of its limit points in M and is closed. Since f is a closed map, $f(X_i)$ is closed in I.

We next show that $f(X_i)$ is nowhere dense in I. Assume, to the contrary, that $f(X_i)$ contains a non-degenerate interval, $[r, s]$, in I. Let $U_1, \ldots, U_n$ be a cover of X_i with open sets of diameter not greater than $1/4i$. Then $f(\overline{U}_1) + \ldots + f(\overline{U}_n)$ contains $f(X_i)$ and this implies that, for some k, $f(\overline{U}_k)$ contains an open subset of $[r, s]$. We may reorder so that $f(\overline{U}_1)$ contains an open subset of $[r, s]$; hence there exist r_0, s_0 in I such that $r \leqslant r_0 < s_0 \leqslant s$ and $[r_0, s_0] \subset f(\overline{U}_1)$. Let r_1 and s_1 be points of I such that $r_0 < r_1 < s_1 < s_0$; then we assert that $U_1 \cap f^{-1}[[r_1, s_1]] \cap X_i = \emptyset$. To see this, let $\tilde{U}_1 = \{x \in M | \operatorname{dist}(x, U_1) < 1/4i\}$. Then $\tilde{U}_1$ is open and $\operatorname{diam} \tilde{U}_1 \leqslant 3/4i$ and $\overline{U}_1 \subset \tilde{U}_1$. Let $x \in U_1 \cap X_i$, then $f(\tilde{U}_1)$ is not a neighborhood of $f(x)$. Since $[r_1, s_1] \subset (r_0, s_0) \subset f(\overline{U}_1) \subset f(\tilde{U}_1)$, this means that $f(x)$ cannot lie in $[r_1, s_1]$, i.e., $x \notin f^{-1}[[r_1, s_1]]$. Let $V_1 = U_1 - f^{-1}[[r_1, s_1]]$; then $V_1, U_2, \ldots, U_n$ is a cover of X_i with open sets of diameter not greater than $1/4i$ and $f(V_1) \cap [r_1, s_1] = \emptyset$.

Now $[r_1, s_1] \subset f(X_i)$ and since $\overline{f(V_1)} \cap [r_1, s_1] = \emptyset$, there is $k \geqslant 2$ such that $f(\overline{U}_k)$ contains an open subset of $[r_1, s_1]$. Reordering if necessary, we may assume $k = 2$ and choose points r_2, s_2 in I such that $r_1 \leqslant r_2 < s_2 \leqslant s_1$ and $[r_2, s_2] \subset \overline{f(\overline{U}_2)}$. Let r_3, s_3 be points of I such that $r_2 < r_3 < s_3 < s_2$; then, as before, $U_2 \cap f^{-1}[[r_3, s_3]] \cap X_i = \emptyset$. Let $V_2 = U_2 f^{-1}[[r_3, s_3]]$; then $V_1, V_2, U_3, \ldots, U_n$ is a cover of X_i with open sets of diameter not greater than $1/4i$. Also $[r_3, s_3] \subset f(X_i)$, but $[f(V_1) + f(V_2)] \cap [r_3, s_3] = \emptyset$.

Continuing for $n-2$ more steps, we obtain points $r_{2n-1} < s_{2n-1}$ and open sets $V_1, \ldots, V_n$ such that $[r_{2n-1}, s_{2n-1}] \subset f(X_i) \subset \overline{f(V_1) + \ldots + f(V_n)}$ but $[f(V_1) + \ldots + f(V_n)] \cap [r_{2n-1}, s_{2n-1}] = \emptyset$. This contradiction shows that $f(X_i)$ has void interior in I as asserted.

Let $G = I - \bigcup_{i=1}^{\infty} f(X_i)$; then G is a dense G_δ in I. It is easy to see that $\mathscr{D}$ is not continuous at $D_r \in \mathscr{D}$ if and only if D_r contains a point which lies in some X_i, i.e., if and only if $r \in \bigcup_{i=1}^{\infty} f(X_i)$. Thus $r \in G$ if and only if $\mathscr{D}$ is continuous at D_r.

COROLLARY. *Assume the hypotheses and notation of Theorem* 1; *then for every* $r \in G$, $D_r = R_r \cap L_r$.

Proof. Suppose that $0 < r < 1$ and $\mathscr{D}$ is continuous at r. Let $a_1, a_2, \ldots$ be a sequence in $[0, r)$ converging to r. Then, for each i, $D_{a_i} \subset f^{-1}[[0, r)]$; hence, $D_r = \limsup_{i}{}_M D_{a_i} \subset L_r$. Similarly $D_r \subset R_r$, so $D_r \subset R_r \cap L_r$. The reverse containment always holds. The argument for $r = 0$ or 1 is obvious from the above.‖

Before stating our next major result, we need some preparatory remarks concerning ε-chains. In Chapter 2 we defined the term "ε-chain cover" in order to explain the term snake-like. We now extend this concept. Let S be a metric space and ε a positive number; an *ε-chain in S* is a finite ordered collection, $U_1, \ldots, U_n$, of open subsets of S such that the diameter of each U_i is less than ε and $U_i \cap U_j \neq \emptyset$ if and only if $|i-j| \leqslant 1$. By a *chain in S* we mean a collection of open sets, $U_1, \ldots, U_n$, satisfying the last condition above. There is some useful terminology commonly used in discussing ε-chains. Let $U_1, \ldots, U_n$ be an ε-chain in the metric space S. The U_i are called the *links of the chain* and U_1 and U_n are called the *first* and *last link* respectively. If $1 \leqslant s \leqslant t \leqslant n$, then $U_s, \ldots, U_t$ is called a *subchain* of $U_1, \ldots, U_n$ and if $s = 1$ ($t = n$), then $U_s, \ldots, U_t$ is called an *initial* (a *terminal*) subchain. We shall always use script letters to denote chains. If $\mathscr{C}$ is a chain, say $\mathscr{C} = \{U_1, \ldots, U_n\}$, then by *mesh* $\mathscr{C}$ we mean the number $\min\{\operatorname{diam} U_i |\ i = 1, \ldots, n\}$, and by $\tilde{\mathscr{C}}$ we mean the chain obtained by reversing the order of the links of $\mathscr{C}$, i.e., $\tilde{\mathscr{C}} = \{V_1, \ldots, V_n\}$ where $V_i = U_{n-i+1}$ for $i = 1, \ldots, n$.

DEFINITION 2. Let S be a bicompact metric space and $\mathscr{C} = \{U_1, \ldots, U_n\}$ an ε-chain in S. Then $\mathscr{C}$ is called a *strong ε-chain in S* provided $\overline{U_i} \cap \overline{U_j} \neq \emptyset$ if and only if $|i-j| \leqslant 1$.

Remark. *If $\mathscr{C} = \{U_1, \ldots, U_n\}$ is an ε-chain cover of the bicompact metric space, S, then there is a strong ε-chain cover, $\mathscr{D} = \{V_1, \ldots, V_n\}$, of S such that $V_i \subset U_i$ for $i = 1, \ldots, n$.*

Proof. Suppose that, for $1 \leqslant i < j \leqslant n$, $\overline{U_i} \cap \overline{U_i} \neq \emptyset$; now $\overline{U_i} \cap \overline{U_j} \subset \partial U_i \cap \overline{U_j} \subset U_{i+1} \cap \overline{U_j}$ and this is non-void only if $U_{i+1} \cap U_j \neq \emptyset$, in which case, $j-(i+1) \leqslant 1$, i.e., $j = i \leqslant 2$. Thus, $\overline{U_i} \cap \overline{U_j} = \emptyset$ if $|i-j| \geqslant 3$.

We define the V_i successively by shrinking the U_i where necessary and we will assume $n \geqslant 4$ so that we can illustrate the process. Let F

$= \overline{U_1} \cap \overline{U_3}$; then $F \subset U_2$. If $F = \emptyset$, then we define $V_1 = U_1$ and $W_3 = U_3$. If $F \neq \emptyset$, then there is a positive number, ε, such that the closure, $\overline{N_\varepsilon(F)}$, in S of the ε neighborhood of F lies in U_2. (This fact follows easily from the compactness of F.) We shrink U_1 and U_3 away from each other as follows. Let $V_1 = U_1 - \overline{N_\varepsilon(F)}$ and $W_3 = U_3 - \overline{N_\varepsilon(F)}$. In either case, $F = \emptyset$ or $F \neq \emptyset$, consider the collection, $V_1, U_2, W_3, U_4, \ldots, U_n$, of open subsets of S we have defined. Since $\overline{N_\varepsilon(F)} \subset U_2$, this is an ε-chain cover of S whose j-th link lies in U_j for $j = 1, \ldots, n$. Moreover $\overline{V_1} \cap \overline{W_3} = \emptyset$ and by our first remark $\overline{V_1} \cap \overline{U_i} = \emptyset$ for $i \geqslant 4$.

At the next step we let $F = \overline{U_2} \cap \overline{U_4}$. If $F = \emptyset$, let $V_2 = U_2$ and $W_4 = U_4$. If $F \neq \emptyset$, then let ε be a positive number such that $\overline{N_\varepsilon(F)} \subset W_3$ and define $V_2 = U_2 - \overline{N_\varepsilon(F)}$ and $W_4 = U_4 - \overline{N_\varepsilon(F)}$. Then $V_1, V_2, W_3, W_4, \ldots, U_n$ is an ε-chain cover and $\overline{V_2} \cap \overline{W_4} = \emptyset$.

Since we already know that $\overline{W_3} \cap \overline{V_1} = \emptyset$, the next step will only involve shrinking W_3 away from U_5 to get V_3 and W_5. This proccess terminates in n steps with the desired strong ε-chain.||

DEFINITION 3. Let $\mathscr{C}$ and $\mathscr{D}$ be chains in a space S; we say that $\mathscr{D}$ *is deeply embedded in* $\mathscr{C}$ provided the closure of each link of $\mathscr{D}$ is contained in some link of $\mathscr{C}$. If U is a subset of S, then $\mathscr{D}$ *is deeply embedded in* U means that $\mathscr{D}$ is deeply embedded in the one link chain $\mathscr{C} = \{U\}$. We say that $\mathscr{D}$ *loops in* $\mathscr{C}$ provided $\mathscr{D}$ is deeply embedded in $\mathscr{C}$ and both end links of $\mathscr{D}$ lie in the first link of $\mathscr{C}$ while some link of $\mathscr{D}$ lies in the last link of $\mathscr{C}$.

We now prove two facts. The first one, which we will call a remark, is probably well known but we include a proof for the sake of completeness. The second is a lemma which we use in proving the major theorem of this chapter.

Recall that if S is a metric space with metric d, and A, B are subsets of S, then $\operatorname{dist}(A, B) = \inf\{d(x, y) \mid x \in A, y \in B\}$.

Remark on nested chains. *Let M be a continuum and $\{\mathscr{C}_i \mid i = 1, 2, \ldots\}$ a collection of chains in M such that* mesh $\mathscr{C}_i \leqslant 1/i$, *for $i = 1, 2, \ldots$ Suppose that $\mathscr{C}_{i+1}$ is deeply embedded in $\mathscr{C}_i$, for $i = 1, 2, \ldots$; then $\bigcap_{i=1}^{\infty} (\bigcup \{C \mid C \in \mathscr{C}_i\})$ is a snake-like subcontinuum, K, of M. If, in addition, the closures of first and last links of $\mathscr{C}_{i+1}$ are contained, respectively, in the first and last links of $\mathscr{C}_i$, for $i = 1, 2, \ldots$, then K is irredducible from the intersection of the first links to the intersection of the last links.*

Proof. Let $G_i = \bigcup \{C \mid C \in \mathscr{C}_i\}$ for $i = 1, 2, \ldots$; then G_i is open in M and $G_1 \supset \overline{G}_2 \supset G_2 \supset \overline{G}_3 \supset G_3 \supset \ldots$ and the intersection, K, of the G_i is a non-void closed subset of M. If K is not connected, then we write

$K = A+B$, non-void and separated. By compactness, there is a positive integer, i, such that $3/i < \operatorname{dist}(A, B)$. No link of $\mathscr{C}_i$ meets both of A and B and each of A, B meets some link of $\mathscr{C}_i$, hence there is a link C of $\mathscr{C}_i$ such that $C \cap (A+B) = \emptyset$. It follows that there is an increasing sequence of integers, $i < i_1 < i_2 < \ldots$, and a sequence of links, C_j, where $C_j \in \mathscr{C}_{i_j}$, such that $C \supset \bar{C}_1 \supset C_1 \supset \bar{C}_2 \supset C_2 \supset \ldots$ But then $\bigcap_{j=1}^{\infty} C_j$ is a non-void closed subset of K which lies in C. This contradiction shows that K is connected. K is snake-like since, for each i, the sets $\{C \cap K \mid C$ is a link of $\mathscr{C}_i$ such that $C \cap K \neq \emptyset\}$ form a $1/i$-chain cover of K. For each i, let F_i be the first link of $\mathscr{C}_i$ and L_i the last link of $\mathscr{C}_i$ and suppose that $F_1 \supset \bar{F}_2 \supset F_2 \supset \bar{F}_3 \supset \ldots$ and $L_1 \supset \bar{L}_2 \supset L_2 \supset \bar{L}_3 \supset \ldots$ Let $\{x\} = \bigcap_{i=1}^{\infty} F_i$ and $\{y\} = \bigcap_{i=1}^{\infty} L_i$ and suppose that L is a subcontinuum of K joining x to y. Then for each i, the $F_i \cap L \neq \emptyset \neq L_i \cap L$; hence every link of $\mathscr{C}_i$ meets L. If L were proper in K, then some link of some $\mathscr{C}_i$ would miss L, hence $L = K$. Thus, K is irreducible from x to y.||

Lemma. *Let M be a snake-like continuum and, for $i = 1, 2, \ldots$, let $\mathscr{C}_i$ be a strong $1/i$-chain covering M. Let $\mathscr{B}_1$ be a subchain of $\mathscr{C}_1$ with at least three links and let F and L denote the first and last links of $\mathscr{B}_1$, respectively. For some integer, n, either $\mathscr{C}_n$ or $\tilde{\mathscr{C}}_n$ has a subchain, $\mathscr{B}_n$, with an initial subchain, $\mathscr{I}_n$, and a terminal subchain, $\mathscr{T}_n$, such that:*

(a) *$\mathscr{B}_n$ is deeply embedded in $\mathscr{B}_1$; and,*

(b) *$\mathscr{I}_n$ and $\mathscr{T}_n$ have at least three links and are deeply embedded in F and, L, respectively.*

Proof. Write $\mathscr{C}_1 = U_1, \ldots, U_s, \ldots, U_t, \ldots, U_n$ where $\mathscr{B}_1 = U_s, \ldots, U_t$. Let $W_1 = (U_1 + \ldots + U_s) - \bar{U}_{s+1}$ and $W_2 = (U_t + \ldots + U_n) - \bar{U}_{t-1}$. If no component of $M-(W_1+W_2)$ joins $\partial W_1 = U_s \cap \partial U_{s+1}$ to $\partial W_2 = U_t \cap \partial U_{t-1}$, then we may write $M-(W_1+W_2) = A+B$, separated, where $\partial W_1 \subset A$ and $\partial W_2 \subset B$. But then M would be the union of the non-void, separated subset W_1+A and W_2+B, which is impossible. So, some component, K, of $M-(W_1+W_2)$ joins ∂W_1 to ∂W_2. This means that K is a subcontinuum of M joining U_s to U_t and lying in $U_s + \ldots + U_t$. Let x and y be points of $K \cap U_s$ and $K \cap U_t$, respectively, and pick a positive number ε such that $\operatorname{dist}(x, M-U_s)$ and $\operatorname{dist}(y, M-U_t)$ are each greater than 4ε. We may also take ε small enough so that, for each $z \in K$, $\{y \in M \mid d(z, y) < 2\varepsilon\}$ lies in $U_s + \ldots + U_t$. Finally, we may require that ε be less than one-half the Lebesgue number of the cover $U_s, \ldots, U_t$ of K.

Choose a positive integer, n, such that $1/n < \varepsilon$. The collection of all links of $\mathscr{C}_n$ which meet K is a subchain, $\mathscr{A}_n$, of $\mathscr{C}_n$. If L is a link of $\mathscr{A}_n$, then diam $\bar{L} \leqslant 1/n < 2\varepsilon$. The last two restrictions on ε imply that $\bar{L}$ lies in some link of $\mathscr{B}_1$. Now $\mathscr{A}_n$ contains at least one subchain one of whose endlinks contains x and the other y. Reversing the order of $\mathscr{C}_n$ (and hence of $\mathscr{A}_n$) if necessary, let $\mathscr{B}_n$ be a subchain with first link containing x and last link containing y. Since $4/n < \operatorname{dist}(x, M-U_s)$, there is an initial subchain, $\mathscr{I}_n$, of $\mathscr{B}_n$ having at least three links such that $\mathscr{I}_n$ is deeply embedded in $U_s = F$. Similarly, $\mathscr{B}_n$ has a terminal subchain, $\mathscr{T}_n$, with at least three links, deeply embedded in $U_t = L$.||

The following theorem generalizes the results of Henderson, [6], and Barrett, [1]. For completeness, we shall include a proof which, except for an application of the preceding lemma, is essentially that of Henderson.

THEOREM 2. *Let M be a snake-like continuum and suppose that, for some positive number, ε, every non-degenerate subcontinuum of M having diameter less than ε is decomposable. If U and V are disjoint open subsets of M, then there is a subcontinuum, K, of M irreducible from a point, x, in U to a point, y, in V such that the complement in K of the x-composant of K is $\{y\}$, i.e., $\{z \in K \mid K$ is irreducible from x to $z\} = \{y\}$.*

Proof. Let Γ denote a sequence, $\mathscr{C}_1, \mathscr{C}_2, \ldots$, of strong chain covers of M such that, for each i, mesh $\mathscr{C}_i < \min(\varepsilon/2, 1/i)$.

There is an integer $j_1 \geqslant 1$ such that $\mathscr{C}_{j_1}$ or $\tilde{\mathscr{C}}_{j_1}$ has a subchain, $\mathscr{B}_1$, with at least three links whose first link, F_1, lies in U and whose last link, L_1, lies in V.

We assert that there is an integer $j_2 > j_1$ and a subchain $\mathscr{B}_2$ of $\mathscr{C}_{j_2}$ or $\tilde{\mathscr{C}}_{j_2}$ with a terminal subchain $\mathscr{T}_2$ such that the following hold:

(a) $\mathscr{B}_2$ is deeply embedded in $\mathscr{B}_1$ and the closure of its first link lies in F_1.

(b) $\mathscr{T}_2$ has at least three links and is deeply embedded in L_1.

(c) If $\mathscr{C} \in \Gamma$ and $\mathscr{B}$ is any subchain of $\mathscr{C}$ or $\tilde{\mathscr{C}}$ which is deeply embedded in $\mathscr{B}_2$ and the closure of the first link of $\mathscr{B}$ lies in the first link of $\mathscr{B}_2$, then no terminal subchain, $\mathscr{T}$, of $\mathscr{B}$ loops (see Definition 3) in $\mathscr{T}_2$.

Assume the assertion is false. By the lemma, there exist j_2, $\mathscr{B}_2$ and $\mathscr{T}_2$ satisfying conditions (a) and (b). Thus condition (c) must fail and there is an integer j_3 such that $\mathscr{C}_{j_3}$ or $\tilde{\mathscr{C}}_{j_3}$ has a subchain $\mathscr{B}_3$ with a terminal subchain $\mathscr{T}_3$ such that $\mathscr{B}_3$ is deeply embedded in $\mathscr{B}_2$, the closure of the first link of $\mathscr{B}_3$ lies in the first link of $\mathscr{B}_2$, and $\mathscr{T}_3$ loops in $\mathscr{T}_2$. Note that we must have $j_3 > j_2 > j_1$. Since $\mathscr{B}_3$ and $\mathscr{T}_3$ satisfy (a) and (b), it fails to satisfy (c). The above argument then yields j_4, $\mathscr{B}_4$, $\mathscr{T}_4$, such that $\mathscr{B}_4$ is deeply embedded in $\mathscr{B}_3$, the closure of the first link of $\mathscr{B}_4$ lies in that of $\mathscr{B}_3$ and $\mathscr{T}_4$ loops in $\mathscr{T}_3$.

Let us consider the sequence of chains $\mathscr{T}_2, \mathscr{T}_3, \ldots$ obtained by continuing the above process. For each $i \geqslant 2$, $\mathscr{T}_{i+1}$, is deeply embedded in, and loops in, $\mathscr{T}_i$. By our remark on nested chains, the set $T = \bigcap_{i=2}^{\infty} (U\{T|$

$T \in \mathscr{T}_i\}$), is a subcontinuum of M. It is proved in [1] and in [6] that, because each $\mathscr{T}_{i+1}$ loops in $\mathscr{T}_i$, the continuum T is indecomposable. On the other hand, $\mathscr{C}_1$ is deeply embedded in L_1, the last link of $\mathscr{B}_1$, and $\operatorname{diam} L_1 < \varepsilon/2$. This implies that $\operatorname{diam} T < \varepsilon/2$ and, by hypothesis, T is decomposable. The contradiction proves the assertion.

The assertion now applies, with $\mathscr{B}_1$ and $\mathscr{T}_1$ replaced by $\mathscr{B}_2$ and $\mathscr{T}_2$, to yield $\mathscr{B}_3$ and $\mathscr{T}_3$ with properties (a), (b) and (c) (relative to $\mathscr{B}_2$ and $\mathscr{C}_2$). In fact, proceeding inductively, we obtain a sequence, $j_1 < j_2 < \ldots$, of integers and, for each i, a subchain $\mathscr{B}_i$ of $\mathscr{C}_{j_i}$ or $\tilde{\mathscr{C}}_{j_i}$ with a terminal subchain $\mathscr{T}_i$ (where $\mathscr{T}_1 = L_1$) such that if F_i and L_i denote the first and last links of $\mathscr{B}_i$, then:

(A) For $i \geqslant 1$, $\mathscr{B}_{i+1}$ is deeply embedded in $\mathscr{B}_i$ and $\overline{F}_{i+1} \subset F_i$.

(B) For $i \geqslant 1$, $\mathscr{T}_{i+1}$ has at least three links and is deeply embedded in L_i. (Notice that, in particular, $\overline{L}_{i+1} \subset L_i$ for $i \geqslant 2$.)

(C) For $i \geqslant 2$, if $\mathscr{C} \in \Gamma$ and $\mathscr{B}$ is any subchain of $\mathscr{C}$ or $\tilde{\mathscr{C}}$ which is deeply embedded in $\mathscr{B}_i$ and the closure of the first link of $\mathscr{B}$ lies in the first link of $\mathscr{B}_i$, then no terminal subchain of $\mathscr{B}$ loops in $\mathscr{T}_i$.

Let $K = \bigcap_{i=2}^{\infty} (\bigcup \{B \mid B \in \mathscr{B}_i\})$; by our remark on nested chains K is a subcontinuum of M, irreducible from x to y where $x = \bigcap_{i=1}^{\infty} F_i$ and $y = \bigcap_{i=1}^{\infty} L_i$. Since $x \in F_1 \subset U$ and $y \in L_1 \subset V$, all that remains to be proved is that $\{z \mid K$ is irreducible from x to $z\}$, which we shall denote by Y, contains exactly the one element y.

Assume, to the contrary, that w is a point of Y distinct from y. Let U be an open subset of M such that $y \in U$ and $w \notin \overline{U}$. The component C of $K \cap U$ which contains y has a limit point in ∂U. Let H denote $\overline{C}$, then H is a subcontinuum of K containing y but not w. If H contains a point, z, of $K-Y$, then let L be a subcontinuum of K joining x to z. Now L misses Y, hence $H+L$ is a subcontinuum of M joining x to y missing w which contradicts the definition of Y. So, after all, $H \cap (K-Y) = \emptyset$, i.e., H is a non-degenerate subcontinuum of K lying in Y.

Let z be a point of H distinct from y and let N be an integer such that $1/N$ is less than one-half the distance from y to z. In particular, L_N contains y but not z and therefore, $\bigcup \{T \mid T \in \mathscr{T}_{N+1}\}$ contains y but not z. Let w be a point of $K-Y$ which also lies in L_{N+1} and let R be an integer such that $R \geqslant N+1$ and $1/R$ is less than one-half the distance from w to H and less than one-half the Lebesgue number of the cover $\mathscr{B}_{N+1}$ of K.

Let U_s and U_t be links of $\mathscr{B}_R$ containing w and z, respectively. Then every link of $\mathscr{B}_R$ between U_t and the last link, L_R, of $\mathscr{B}_R$ meets H and therefore misses w. Hence U_s precedes U_t in $\mathscr{B}_R$.

Now $\mathscr{B}_R$ is strongly embedded in $\mathscr{B}_{N+1}$ and w lies in the last link L_{N+1} of $\mathscr{B}_{N+1}$. By choice of R, the subchain of $\mathscr{B}_R$ consisting of those links preceding U_s has a link whose closure lies in the first link of $\mathscr{T}_{N+1}$ and we let U_p denote the last link preceding U_s with this property. Then the subchain $U_p, \ldots, U_s$ of $\mathscr{B}_R$ is deeply embedded in $\mathscr{T}_{N+1}$. Similarly, the terminal subchain $U_s, \ldots, U_t, \ldots, L_R$ of $\mathscr{B}_R$ must contain a link whose closure lies in the first link of $\mathscr{T}_{N+1}$, and if U_q denotes the first link following U_s with this property, then $U_s, \ldots, U_q$ is deeply embedded in $\mathscr{T}_{N+1}$.

Let $\mathscr{B}$ be the initial subchain, $F_R, \ldots, U_p, \ldots, U_s, \ldots, U_q$, of $\mathscr{B}_R$ and let $\mathscr{T} = U_p, \ldots, U_s, \ldots, U_q$. Then $\mathscr{B}$ is a subchain of $\mathscr{C}_{j_R}$ or $\tilde{\mathscr{C}}_{j_R}$ which is deeply embedded in $\mathscr{B}_{N+1}$ and the closure of the first link of $\mathscr{B}$ lies in F_{N+1}. By construction, the terminal subchain, $\mathscr{T}$, of $\mathscr{B}$ loops in $\mathscr{T}_{N+1}$. This violates condition (c) and we have obtained a contradiction to the assumption that $Y \neq \{y\}$. ||

It is convenient to shorten the hypothesis of Theorem 2 to the statement "M does not contain small non-degenerate indecomposable subcontinua."

We point out that, although we shall only apply Theorem 2 to our decomposition theory, one may apply this result to investigate any snake-like continuum which does not contain small non-degenerate indecomposable subcontinua.

We now give several applications of Theorem 2.

THEOREM 3 (Henderson). *Let M be a non-degenerate continuum which is homeomorphic with each of its non-degenerate subcontinua. If M is decomposable, then M is an arc.*

Proof. Certainly M is hereditarily of type A'. By Theorem 2, there is a non-degenerate subcontinuum, K, of M such that one of the two elements of $\mathscr{D}(K)$ which do not separate K is degenerate. Thus, one of the two elements of $\mathscr{D}(M)$ which do not separate M is degenerate and this in turn implies that every element of $\mathscr{D}(M)$ is degenerate, i.e., M is an arc. ||

THEOREM 4. *Let M be a snake-like continuum of type A which does not contain small non-degenerate indecomposable subcontinua, and let f be a function such that $\mathscr{D}(f) = \mathscr{D}(M)$. With the notation at the beginning of this chapter, either there is r in I such that $D_r \not\subset L_r$ or $G = \{r \in I \mid D_r$ is degenerate$\}$ is a dense G_δ set in I.*

Proof. If G is not a dense G_δ in I, then, for some positive integer, n, the set $\{r \in I \mid \operatorname{diam} D_r \geqslant 1/n\}$ has non-void interior. Pick s and t in I such that $s < t$ and, for each r in (s, t), $\operatorname{diam} D_r \geqslant 1/n$. Let N denote the continuum $\overline{f^{-1}[(s, t)]}$; Theorem 2 applies to N and there is a subcontinuum, K, of N irreducible from a point x of $f^{-1}[(s, (s+t)/3)]$ to a point y of $f^{-1}[(2(s+t)/3, t)]$ and the complement in K of the x-composant of K is $\{y\}$.

Write $f(K) = [s_1, t_1]$ where $s \leqslant s_1 < t_1 \leqslant t$ and suppose that, for each $r \in [s, t]$, $D_n \subset L_r$. Then, in particular, $D_{t_1} \subset K$ and, of course, $y \in D_{t_1}$. Suppose that L is any subcontinuum of M joining x to D_{t_1}. Then, by Theorem 5 of Chapter 1, L contains $\overline{f^{-1}[(s_1, t_1)]}$; therefore, $x \in D_{t_1} \subset L$ and, by irreducibility, $L = K$. This means that K is irreducible from x to every point of D_{t_1}, i.e., D_{t_1} is contained in the complement of the x-composant of K. Since $\operatorname{diam} D_{t_1} \geqslant 1/n$, we have a contradiction and we conclude that, for some $r \in [s, t]$, $D_r \not\subset L_r$.||

We remark that the some result holds with L_r replaced by R_r.

THEOREM 5. *Let M, f and G be as in the preceding theorem and denote by L and R, respectively, the sets $\bigcup \{D_r - L_r | r \in I\} + f^{-1}[G]$, $\bigcup \{D_r - R_r | r \in I\} + f^{-1}[G]$. Each of L and R is dense in M.*

Proof. Let U be an open subset of M; we show that $U \cap L \neq \emptyset$. If there exists $r \in I$ such that $U \subset D_r$, then $U \cap D_r^0 = \emptyset$ and $D_r^0 \subset D_r - L_r$ so $U \cap L \neq \emptyset$. Suppose then that U meets at least two elements of $\mathscr{D}(M)$. Using Theorem 2, one can easily construct a subcontinuum K of M irreducible from D_0 to some point of U such that the complement of the proper composant of K which meets D_0 is $\{y\}$. Let $f(K) = [0, s]$, where $0 < s$; then $y \in D_s$ and K contains L_s (by definition of L_s). We now consider two cases.

Case 1. Suppose $K = L_s$; since L_s is irreducible from D_0 to $D_s \cap L_s$ we must have $\{y\} = D_s \cap L_s$. If $\{y\} = D_s$, then $\{y\} \in f^{-1}[G] \subset L$, and we are done. If $\{y\} \neq D_s$, then since U contains y, U meets $D_s - \{y\} = D_s - L_s \subset L$, q.e.d.

Case 2. Suppose $K \neq L_s$; then $y \notin L_s$ (by irreducibility of K), hence $y \in D_s - L_s \subset L$ and we are done in this case.

Dually, U contains a point of R.||

In [4], Jones obtains the continuous circle of pseudoarcs by starting with a pair of accordionlike continua intersecting in the end elements of their decomposition and then replacing each $\vee$, $\wedge$, and straight line interval by a pseudoarc. It is pointed out there that, by modifying the construction, one can replace each $\vee$ and $\wedge$ with an indecomposable continuum, retaining the straight line intervals, so that the collection of continua thus obtained will be continuous. Theorem 4 applies to this intermediate example to show that the continua used to replace the $\vee$ and $\wedge$ must not only be indecomposable, but, for each positive integer, i, a dense collection of these continua must contain non-degenerate indecomposable subcontinua of diameter not more than $1/i$.

DEFINITION 4. Let M be of type A and let f be such that $\mathscr{D}(M) = \mathscr{D}(f)$. For $r \in I$, we say that *$\mathscr{D}(M)$ is continuous from the left at D_r* $(= f^{-1}[r])$ provided that if $\{r_i\}$ is a sequence in I converging from the left to r, then $\limsup_i D_{r_i} = D_r$. A similar definition holds for "continuous from the right."

Notice that $\mathscr{D}(M)$ is continuous if and only if it is continuous from the left and from the right at each of its elements. Also, if $\mathscr{D}(M)$ is continuous from the left (right) at D_r, then $D_r \subset L_r$ ($D_r \subset R_r$), but the converse is not true.

THEOREM 6. *Let M be a snake-like continuum of type A which does not contain small non-degenerate indecomposable subcontinua and let f be a function such that $\mathscr{D}(M) = \mathscr{D}(f)$. Suppose, also, that $\mathscr{D}(M)$ is continuous from the left at each of its elements; then, for $0 < r \leqslant 1$, D_r is degenerate.*

Proof. Continuity from the left implies that for each $r \in I$, $D_r - L_r = \emptyset$. By Theorem 5, this implies that $G = \{r \in I \mid D_r \text{ is degenerate}\}$ is dense in I. If $r > 0$, let $\{r_i\}$ be a sequence in G converging from the left to r. Then each D_{r_i} is a singleton and, by left continuity, D_r is a singleton.||

COROLLARY. *Let M and f be as in Theorem 6. Then M is an arc if and only if D_0 is degenerate.*

At this juncture we should mention a result of Eldon Dyer. In [5] he proves the following result: *No irreducible continuum has a continuous decomposition whose elements are non-degenerate indecomposition continua.* In our teminology this can be phrased as follows: *If M is of type A and $\mathscr{D}(M)$ is continuous then some element (possibly degenerate) of $\mathscr{D}(M)$ is indecomposable.* In one sense this is stronger than our results because M is not assumed to be snake-like. On the other hand, if we add the hypothesis that M is snake-like, then we obtain much stronger conclusions, e.g., either M is an arc or else M contains arbitrarily small non-degenerate indecomposable subcontinua. Indeed, stronger conclusions are obtained with much weaker forms of continuity.

We now introduce a notion of sidedness for certain subcontinua of the 2-sphere, S. We begin by defining this notion for an arc in S.

DEFINITION 5. Let A be an arc in S and let B and C be subcontinua of S each of which meets A. We say that *B and C meet A on the same side* provided there is a simple closed curve C in S such that A is contained in C and $(B-A)+(C-A)$ lies in the same component of $S-C$. (Note: $S-C$ has exactly two components; these are frequently called the *complementary domains* of C.)

Fix an arc A in S and define a relation on the collection of continua in S which meet A as follows: B and C are related provided they meet A on the same side. As a consequence of our definition, this relation is not transitive. It is possible to modify Definition 5 so as to make the relation transitive, but this introduces complications in our results which, we feel, are not justified.

Let K be a continuum of type A in S and $\mathscr{D}$ an admissible decomposition of K no element of which separates S and extend $\mathscr{D}$ to the decomposition $\mathscr{E}$ of S whose elements are those of $\mathscr{D}$ together with the collection of sets $\{x\}$, where $x \in S - K$. If T denotes the quotient space of S modulo $\mathscr{E}$, then, by Theorem 15 of Chapter 2, there is a homeomorphism, h, of T onto S. If q denotes the quotient map of S onto T, then $h \circ q$ is a continuous mapping of S onto S and $h(q(K))$ is an arc.

DEFINITION 6. Assuming the hypotheses and notation of the preceding paragraph, let B and C be continua in S each of which meets K. We say that *B and C meet K on the same side relative to $\mathscr{D}$* provided $h(q(B))$ and $h(q(C))$ meet $h(q(K))$ on the same side in the sense of Definition 5.

We must prove that Definition 6 is independent of the choice of the homeomorphism h. Let h and g be any two homemorphism of T onto S and suppose that $(hq(B))$ and $h(q(C))$ meet $h(q(K))$ on the same side. Let C be a simple closed curve in S such that $h(q(K)) \subset C$ and $[h(q(B)) - -h(q(K))] + [h(q(C)) - h(q(K))]$, which we will denote by Q, lies in one complementary domain of C. Then $g \circ h^{-1}$ is a homeomorphism of S onto S taking $h(q(K))$ onto $g(q(K))$ and taking Q onto $[g(q(B)) - g(q(K)) + +[g(q(C) - g(q(K))]$. Moreover, being a homeomorphism, $g \circ h^{-1}$ maps C onto a simple closed curve, C', in S and maps the two complementary domains of C onto those of C'. Therefore, $g(q(B))$ and $g(q(C))$ meet $g(q(K))$ on the same side in the sense of Definition 5, q.e.d.

Notice that Definition 6 does depend on the particular admissible decomposition of K chosen. As a simple illustration (in the plane rather than S) let K denote the unit interval, B the interval $\{(x, y) | x = 1/2, 0 \leqslant y \leqslant 1\}$ and C the reflection of B in the x-axis. If $\mathscr{D} = \mathscr{D}(K)$, then B and C do not meet K on the same side relative to $\mathscr{D}$. However, if $\mathscr{D}$ consists of the subinterval $\{(x, y) | \ 0 \leqslant x \leqslant 1/2, y = 0\}$ of K together with $\{\{(x, y)\} | \ 1/2 < x \leqslant 1, y = 0\}$, then B and C meet K on the same side relative to $\mathscr{D}$.

Until further notice we will work under the following hypotheses and notation. All continua lie in S; M is a continuum of type A and K is a subcontinuum of M of type A meeting at least two elements of $\mathscr{D}(M)$, and $\mathscr{D}$ is the admissible decomposition of K whose elements are $\{D \cap K | \ D \in \mathscr{D}(M), D \cap K \neq \emptyset\}$. We assume that no element of $\mathscr{D}$ seperates S. Extend $\mathscr{D}$ to a decomposition of S as usual and let q be the induced quotient map and h a homeomorphism of the quotient space onto S (see the paragraph preceding Definition 6). Let $\tilde{K} = \mathrm{cl}_M(\mathrm{int}_M(K))$; then $\tilde{K}$ is a subcontinuum of K and $h(q(\tilde{K})) = h(q(K))$.

Remark. *If C is a simple closed curve in S containig $h(q(K))$ and W is a complementary domain of C, then there is a sequence $B_1, B_2, \ldots,$of arcs in $q^{-1}[h^{-1}[W]]$ such that $\limsup_i B_i$ contains $\tilde{K}$; in particular $\tilde{K} \subset \partial_S(q^{-1}[h^{-1}[W]])$.*

Proof. Let $A_1, A_2, \ldots,$ be a sequence of arcs in W such that $\limsup_i A_i = h(q(K))$, and let $C_i = q^{-1}[h^{-1}[A_i]]$ for $i = 1, 2, \ldots$. Since

$h \circ q$ is a homeomorphism on $S-K$, the C_i are arcs in $q^{-1}[h^{-1}[W]]$ and $\limsup_i C_i$ lies in K.

It is not hard to show that there is a subsequence, $B_1, B_2, \ldots$ of the C_i such that, if B_i has endpoints a_i and b_i, then $\{a_i\}$ and $\{b_i\}$ converge in S to points a and b in $\partial_S\big[q^{-1}[h^{-1}[W]]\big]$ such that $h(q(a))$ and $h(q(b))$ are the endpoints of $h(q(K))$.

By Theorem 8 of Chapter 1, $\tilde{K}$ is irreducible in K from a to b. Since $\limsup_i B_i$ is a subcontinuum of K containing a and b, $K \subset \limsup_i B_i$.

THEOREM 7. *Assume the standing hypotheses on M, K and $\tilde{K}$, let C be a simple closed curve containing $h(q(K))$, and let W be a complementary domain of C. Suppose that U and V are disjoint open sets in S such that $U+V = S$, U and V have common boundary, say, $\partial_S(U) = \partial_S(V) = L$ and suppose, also, that $\tilde{K}$ meets both U and V. Then $h(q(L))$ meets W.*

Proof. Since $h\circ q$ is a closed and continuous map,

$$\overline{h(q(U))} \cap \overline{h(q(V))} = h(q(\overline{U})) \cap h(q(\overline{V}))$$
$$= h(q(U+L)) \cap h(q(V+L)) \subset h(q(K))+h(q(L)).$$

Now by the remark, $\tilde{K} \subset \partial_S(q^{-1}[h^{-1}[W]])$, and since K contains no open subset of S and $\tilde{K}$ meets U, there is a point, x, of $q^{-1}[h^{-1}[W]] \cap (U-K)$. Then $h(q(x))$ lies in $W \cap \operatorname{int}_S\big(h(q(U))\big)$. Similarly, $W \cap \operatorname{int}_S\big(h(q(V))\big) \neq \emptyset$.

Now assume the theorem false. Then W misses $h(q(L))$. Since W also misses $h(q(K))$, it follows from our initial computation that W misses $\overline{h(q(U))} \cap \overline{h(q(V))}$. But this contradicts the fact that W is connected and meets the interior of each of $h(q(U))$ and $h(q(V))$. ||

We shall use Theorem 7 to prove Theorem 8 which, in a restricted form, says that if B and C are continua which meet K on the same side, then there is an arc, J, meeting K on the other side. Before proving Theorem 8 we establish a useful result.

LEMMA. *Let C be a simple closed curve in S, let A be an arc lying in C with endpoints a and b and let W be a complementary domain of C. Suppose that L is a locally connected continuum in S with the property that if J is an arc in L which meets A and $J-A \subset W$, then $J \cap A \subset \{a, b\}$. Then S contains a simple closed curve, C', containing A such that one complementary domain of C' lies in W and misses L.*

(What this lemma says is that if L satisfies the given conditions, then "inside" C there is another simple closed curve C' which bounds L away from $A-\{a, b\}$.)

Proof. We may assume that C is the boundary of the unit square, $I\times I$, in the plane, $A = I\times\{0\}$, $a = (0, 0)$, $b = (1, 0)$, and $W = \{(x, y)|$

$0 < x < 1, 0 < y < 1\}$. For $n \geqslant 3$, let $a_n = (1/n, 0)$ and $b_n = (1-1/n, 0)$. We assert that, if $n \geqslant 3$, there is a positive number d_n such that $L \cap \{(x, y)| a_n < x < b_n, 0 < y < d_n\} = \emptyset$. If, for some n, this were false, then $L \cap W$ would contain a sequence $\{z_i\}$ converging to a point z of the form $z = (c, 0)$ where $a_n \leqslant c \leqslant b_n$. Let U be an open subset of S such that $z \in U$ and $\operatorname{diam} U < 2/n$. Since $z \in L$ and L is locally connected, $U \cap L$ would contain an arc joining some z_i to z. This arc would contain a subarc, J, meeting A such that $J - A \subset W$. Since this is impossible, the assertion is proved.

We also require that the d_i be monotone decreasing to 0. The simple closed curve C' is then the set $A+A_3+A_4+\ldots$, where $A_3 = \{(x, y)| a_3 \leqslant x \leqslant b_3, y = d_3\}+\{(x, y)| x = a_3$ or $x = b_3$, and $d_4 \leqslant y \leqslant d_3\}$ and, for $i \geqslant 4$, $A_i = \{(x, y)| a_i \leqslant x \leqslant a_{i-1}$ or $b_{i-1} \leqslant x \leqslant b_i$, and $y = d_i\}+\{(x, y)| x = a_i$ or $x = b_i$, and $d_{i+1} \leqslant y \leqslant d_i\}$.

THEOREM 8. *Assume all the hypotheses of Theorem 7 on M, K, U, V, L and, in addition, assume that L is locally connected. Suppose that B and C are continua in S meeting K on the same side relative to $\mathcal{D}$. Then there is an arc, J, lying in L, with endpoints x and y, such that $J \cap (K+B+C) = J \cap K = x$ or y.*

Proof. Let D be a simple closed curve in S with complementary domains W_1 and W_2 such that $h(q(K)) \subset D$ and $[h(q(B))-h(q(K))]+ +[h(q(C))-h(q(K))]$ lies in W_1. By Theorem 7, $h(q(L))$ meets W_2. Since $h(q(L))$ is locally connected, it contains an arc, I, such that $I \cap h(q(K)) \neq \emptyset$, $I-h(q(K)) \subset W_2$ and $I \cap h(q(K))$ is neither endpoint of $h(q(K))$. For if this fails, then we may apply the lemma to get a simple closed curve, D', containing $h(q(K))$ such that one complementary domain of D' lies in W_2 and misses $h(q(L))$. This violates Theorem 7, so there is such an arc I in $h(q(L))$. By taking a subarc of I if necessary, we may also require that $I \cap h(q(K))$ be an endpoint, p, of I.

Denote by J the closure of $h^{-1}[q^{-1}[I-p]]$. Then J is an arc lying in L. Also $J \cap K$ is an endpoint, x, of J. Since $I-p \subset W_2$, $J-x = h^{-1}[q^{-1}[I-p]]$ misses $(B-K)+(C-K)$. Hence $J \cap (K+B+C) = J \cap K = x.\|$

We are now ready to prove our final structure theorem.

THEOREM 9. *Let M be a plane continuum of type A and write $\mathcal{D}(M) = \{D_r| r \in I\}$. Suppose that each element of $\mathcal{D}(M)$ is either degenerate or of type A and that, for each non-degenerate D_r, no element of $\mathcal{D}(D_r)$ separates the plane. Let R denote the set of $r \in I$ such that D_r contains a subcontinuum K_r of type A meeting at least two elements of $\mathcal{D}(D_r)$ such that R_r and L_r meet K_r on the same side relative to the decomposition $\{D \cap K_r| D \in \mathcal{D}(D_r), D \cap K_r \neq \emptyset\}$ of K; then R is at most countable.*

Proof. Let R_1 denote the set of $r \in R$ such that K_r meets at least two distinct horizontal lines in the plane. Notice that if $r \in R_1$, then $\tilde{K}_r$

$= \overline{\mathrm{int}_{D_r}(K_r)}$ also meets at least two distinct horizontal lines in the plane. Assume that R_1 is uncountable; then there is a horizontal line L, with upper and lower half-planes U and V, such that for uncountable many r in R_1, $\tilde{K}_r$ meets U and V; let R' denote the set of such r.

Let $\leqslant$ be the usual (left to right) ordering on L. By Theorem 8 there is, for each r in R', an arc J_r in L, $J_r = [x_r, y_r]$ where $x_r < y_r$ in L, such that $J_r \cap (K_r + R_r + L_r) = x_r$ or y_r, i.e., $J_r \cap M = x_r$ of y_r. If for $r \neq s$, we have $J_r \cap M = x_r$ and $J_r \cap M = x_s$, then $J_r \cap J_s = \emptyset$ (otherwise one of these would contain two points of L). Since L contains at most countably many disjoint arcs, the set $P = \{r \in R' |\ J_r \cap M = x_r\}$ is at most countable. Similary $Q = \{r \in R' | J_r \cap M = y_r\}$ is at most countable. Since $R' = P + Q$, R' is at most countable and we have a contradiction.

So R_1 is countable; the complement in R of R_1 is just the set $\{r \in R|\ K_r$ lies in a horizontal line$\}$. Applying the same argument as above, only with L replaced with a suitable vertical line, L', in the plane, we see that $R - R_1$ is countable. Thus R is countable.||

The conclusion of this theorem could be roughly summarized by the statement "All but countably many of the non-degenerate elements of $\mathscr{D}(M)$ are deeply embedded from both sides in M". If M is the accordionlike continuum, then the set R is precisely the set of r in I such that D_r is a $\vee$ or $\wedge$. Theorem 9 says that, in the above sense, any plane continuum of type A (which has reasonably nice decomposition elements) enjoys this property.

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