ANNALES
POLONICI MATHEMATICI
LIII.2 (1991)

A modern proof of the Maximum Principle

by ANDRZEJ W. TURSKI (Katowice)

Abstract. A new proof of the Weak Maximum Principle for parabolic partial differential equations of the second order is presented. The method of proof may also be used for nonclassical solutions (belonging to suitable Sobolev spaces).

1. Introduction. The Weak Maximum Principle is one of the most important theorems concerning elliptic and parabolic partial differential equations of the second order. All the classical proofs of the Principle make use of the pointwise properties of solutions (see [2]–[4]) and hence cannot cover the case of weak solutions in Sobolev spaces. [1] and [4] contain various proofs of the Principle in Sobolev spaces for particular cases (when the equation is in divergence form and the solution satisfies particular initial conditions). In the present paper we give a non-standard proof of the Weak Maximum Principle for the general form of linear parabolic equation of the second order. The method of proof may also be used for weak solutions.

2. Preliminaries. Consider the equation (where all the sums are taken from 1 to n)

$$u_t = \sum_{i,j}(a_{ij}(t, x)u_{x_j})_{x_i} + \sum_{j} b_j(t, x)u_{x_j} + c(t, x)u + f(t, x) \tag{1}$$

in $D = (0, T] \times \Omega$, where Ω is a bounded domain in $\boldsymbol{R}^n$ with suitably smooth boundary. We assume that

(U) a function u satisfying (1) is continuous in $\bar{D}$ and has partial derivatives u_t, u_{x_i}, $u_{x_i x_j}$ continuous in D.

Moreover, the following conditions on the coefficients are assumed:

(A) a_{ij} and $(a_{ij})_{x_i}$ are continuous in D and for every $(t, x) \in D$ and every $\xi \in \boldsymbol{R}^n$

$$\sum_{i,j} a_{ij}(t, x)\xi_i \xi_j \geqslant 0,$$

(C) $c(t, x) \leqslant h$ in D where h is a constant,

(D) f is globally bounded: $|f(t, x)| \leqslant M$ in D.

3. Main theorem. Under assumptions (U), (A), (C) and (D) we have the following:

THEOREM. *If b_j, $(b_j)_{x_j}$ are continuous in D for $j = 1, \ldots, n$ and u satisfies* (1), *then for every $t_0 \geqslant 0$ and $t \geqslant t_0$*

$$\|u(t,\cdot)\|_{L^\infty(\Omega)} \leqslant \{\max(\|u(t_0,\cdot)\|_{L^\infty(\Omega)}, m) + M(t_0)(t-t_0)\}\exp(h(t-t_0)), \tag{2}$$

where $M(t) = \sup_{z \geqslant t} \|f(z,\cdot)\|_{L^\infty(\Omega)}$ and $m = \sup_{(s,x)} |u(s,x)|$ for $(s,x) \in [t_0, T] \times \partial\Omega$.

Proof. Define $\Omega_l = \{x \in \Omega: \varrho(x, \partial\Omega) > 1/l\}$ and $D_l = (1/l, T] \times \Omega_l$, where l is an integer and $\varrho(x, \partial\Omega)$ denotes the distance from x to $\partial\Omega$. The functions a_{ij}, b_j, u and their derivatives $(a_{ij})_{x_i}$, $(b_j)_{x_j}$, u_{x_i}, $u_{x_i x_j}$ $(i, j = 1, \ldots, n)$ are continuous in $\bar{D}_l$. Note that if (2) is true for every Ω_l in place of Ω, then it is true for Ω itself, since for l tending to infinity we have

$$\|u(t,\cdot)\|_{L^\infty(\Omega_l)} \to \|u(t,\cdot)\|_{L^\infty(\Omega)}, \qquad \sup_{\substack{s \in [t_0, T] \\ x \in \partial\Omega_l}} |u(s,x)| \to \sup_{\substack{s \in [t_0, T] \\ x \in \partial\Omega}} |u(s,x)|.$$

Thus, with no loss of generality, we may additionally assume that

(E) a_{ij}, b_j, u and $(a_{ij})_{x_i}$, $(b_j)_{x_j}$, u_{x_i}, $u_{x_i x_j}$ are continuous in $\bar{D}$.

Next, observe that instead of (2) it is sufficient to prove

$$\|u(t,\cdot)\|_{L^\infty(\Omega)} \leqslant \max(\|u(t_0,\cdot)\|_{L^\infty(\Omega)}, m) + M(t_0)(t-t_0) \tag{3}$$

with (C) sharpened to $c(t,x) \leqslant 0$. Indeed, the transformation $v(t,x) = u(t,x)\exp(-ht)$ yields (1) for v with $\tilde{c}(t,x) = c(t,x) - h \leqslant 0$.

First, we show (3) under the additional assumption

$$c(t,x) \leqslant -h_1 < 0, \quad \text{where } h_1 \text{ is a constant.} \tag{4}$$

Multiplying (1) by u^{2^k-1} (the number $k \in N$, sufficiently large, is fixed until the limit passage after (14)) and integrating the result over Ω, we obtain

$$\int_\Omega u_t u^{2^k-1}\,dx = \int_\Omega \sum_{i,j} (a_{ij} u_{x_j})_{x_i} u^{2^k-1}\,dx + \int_\Omega \sum_j b_j u_{x_j} u^{2^k-1}\,dx + \int_\Omega c u^{2^k}\,dx + \int_\Omega f u^{2^k-1}\,dx. \tag{5}$$

The first and second terms on the right-hand side of (5) are integrated by parts and the fourth is estimated using the Hölder inequality with $p = 2^k(2^k-1)^{-1}$, $q = 2^k$:

$$\begin{aligned} 2^{-k}\frac{d}{dt}\int_\Omega u^{2^k}\,dx \leqslant{} & \int_{\partial\Omega} \sum_{i,j} a_{ij} u_{x_j} \cos(n, x_i) u^{2^k-1}\,ds \\ & - (2^k-1)\int_\Omega \sum_{i,j} a_{ij} u_{x_j} u_{x_i} u^{2^k-2}\,dx + 2^{-k}\int_{\partial\Omega} \sum_j b_j \cos(n, x_j) u^{2^k}\,ds \\ & - 2^{-k}\int_\Omega \sum_j (b_j)_{x_j} u^{2^k}\,dx + \int_\Omega c u^{2^k}\,dx + \Big(\int_\Omega f^{2^k}\,dx\Big)^{2^{-k}}\Big(\int_\Omega u^{2^k}\,dx\Big)^{1-2^{-k}}. \end{aligned} \tag{6}$$

Then using assumptions (A), (D), (E) and (4) we have

$$\frac{d}{dt}\int_\Omega u^{2^k}dx \leqslant 2^k m^{2^k-1}H_1 + m^{2^k}H_2 + \int_\Omega H_3(t, x)u^{2^k}dx \qquad (7)$$
$$+2^k M(t)|\Omega|^{2^{-k}}\Big(\int_\Omega u^{2^k}dx\Big)^{1-2^{-k}},$$

where H_1, $H_2 > 0$ are constants such that

$$H_1 \geqslant \sup_{t_0\leqslant t\leqslant T}\int_{\partial\Omega}\Big|\sum_{i,j} a_{ij}(t, x)u_{x_j}\cos(n, x_i)\Big|\,ds,$$

$$H_2 \geqslant \sup_{t_0\leqslant t\leqslant T}\int_{\partial\Omega}\Big|\sum_j b_j(t, x)\cos(n, x_j)\Big|\,ds,$$

$$H_3(t, x) = 2^k c(t, x) - \sum_j [b_j(t, x)]_{x_j}$$

and $|\Omega|$ is the Lebesgue measure of Ω.

Since $[b_j(t, x)]_{x_j}$ is bounded by (E) and $c(t, x) \leqslant -h_1 < 0$, $H_3(t, x)$ is negative for sufficiently large k. Set $y_k(t) := \|u(t, \cdot)\|^{2^k}_{L^{2^k}(\Omega)}$, $b_k(t) := M(t)|\Omega|^{2^{-k}}$, $\alpha := 2^{-k}$, $c_k := 2m^{2^k-1}(2^k H_1 + mH_2)$ and fix $a > 0$ such that $H_3(t, x) \leqslant -a$ for all $(t, x)\in D$. If $m = 0$ (or equivalently $c_k = 0$), then from (7) we obtain an ordinary differential inequality for y_k:

$$y_k'(t) \leqslant -ay_k(t) + \alpha^{-1}b_k(t)[y_k(t)]^{1-\alpha}. \qquad (8)$$

Its solution satisfies

$$y_k(t) \leqslant \Big\{[y_k(t_0)]^\alpha \exp(\alpha a t_0) + \int_{t_0}^t b_k(z)\exp(\alpha a z)\,dz\Big\}^{1/\alpha}\exp(-at). \qquad (9)$$

If $m > 0$ we obtain a strict ordinary differential inequality (compare the definition of c_k and (7)) for y_k:

$$y_k'(t) < c_k - ay_k(t) + \alpha^{-1}b_k(t)[y_k(t)]^{1-\alpha}. \qquad (10)$$

We will use the comparison technique to estimate the solution y_k of (10). Denoting the right-hand side of (10) by $\psi(t, y_k(t))$ and defining

$$y^*(t) = \Big\{[y_k(t_0) + c_k/a]^\alpha \exp\big(\alpha a(t+t_0)\big) + \int_{t_0}^t b_k(z)\exp(\alpha a z)\,dz\Big\}^{1/\alpha}\exp(-at)$$

it is easy to calculate that

$$y^*(t_0) = [y_k(t_0) + c_k/a]\exp(at_0) > y_k(t_0), \qquad (11)$$

$$(y^*)'(t) \geqslant c_k\exp(at_0) - ay^*(t) + \alpha^{-1}b_k(t)[y^*(t)]^{1-\alpha} \geqslant \psi\big(t, y^*(t)\big). \qquad (12)$$

From (10)–(12), and the comparison theorem ([5], p. 65, Th. 5) it follows that $y_k(t) < y^*(t)$ for every $t \in [t_0, T]$, hence

(13) $$y_k(t) < \{[y_k(t_0)+c_k/a]^\alpha \exp(\alpha a(t+t_0)) + \int_{t_0}^{t} b_k(z)\exp(\alpha a z)\,dz\}^{1/\alpha}\exp(-at).$$

Next we study (9) in the case $m = 0$ and (13) in the case $m > 0$. We only have to consider (13), since the proof is similar in both cases.

Since $M(t) = \sup_{z\geqslant t}\|f(z,\cdot)\|_{L^\infty(\Omega)}$ the function b_k $(= M(t)|\Omega|^{2^{-k}})$ is decreasing and we may write

$$[y_k(t)]^\alpha < \exp(\alpha a t_0)[y_k(t_0)+c_k/a]^\alpha + b_k(t_0)\exp(-\alpha a t)\frac{\exp(\alpha a t)-\exp(\alpha a t_0)}{\alpha a}$$

or using the previous notation

(14) $$\|u(t,\cdot)\|_{L^{2^k}(\Omega)} < \exp(2^{-k}at_0)[2\max(\|u(t_0,\cdot)\|^{2^k}_{L^{2^k}(\Omega)}, c_k/a)]^{2^{-k}} + b_k(t_0)\frac{1-\exp(-2^{-k}(t-t_0))}{2^{-k}a}.$$

For k tending to infinity we have $\|u(t,\cdot)\|_{L^{2^k}(\Omega)} \to \|u(t,\cdot)\|_{L^\infty(\Omega)}$ ([6], p. 34, Th. 1), $2^{2^{-k}} \to 1$,

$$(c_k/a)^{2^{-k}} = (a^{-1}2m^{2^k-1}(2^kH_1+mH_2))^{2^{-k}} \to m, \qquad b_k(t_0) \to M(t_0),$$
$$a^{-1}2^k[1-\exp(-2^{-k}a(t-t_0))] \to (t-t_0).$$

Hence as a consequence of (14) we obtain

$$\|u(t,\cdot)\|_{L^\infty(\Omega)} \leqslant \max(\|u(t_0,\cdot)\|_{L^\infty(\Omega)}, m) + M(t_0)(t-t_0),$$

which completes the proof under the strengthened assumption (4).

To cover the general case $c(t,x) \leqslant 0$, we take an arbitrary $h_1 > 0$, use the transformation $v(t,x) = u(t,x)\exp(-h_1 t)$ and by the previous procedure we obtain

$$\|v(t,\cdot)\|_{L^\infty(\Omega)} \leqslant \max(\|v(t_0,\cdot)\|_{L^\infty(\Omega)}, m_v) + M_v(t_0)(t-t_0),$$

where $m_v = \sup_{(z,x)}|\exp(-h_1 z)u(z,x)|$ for $(z,x) \in [t_0, T]\times\partial\Omega$ and $M_v(t_0) = \sup_{z\geqslant t_0}\|\exp(-h_1 z)f(z,\cdot)\|_{L^\infty(\Omega)}$. It is easy to verify that the last inequality yields for u

(15) $$\exp(-h_1 t)\|u(t,\cdot)\|_{L^\infty(\Omega)} \leqslant \max(\exp(-h_1 t_0)\|u(t_0,\cdot)\|_{L^\infty(\Omega)}, \exp(-h_1 t_0)m) + \exp(-h_1 t_0)M(t_0)(t-t_0).$$

Multiplying both sides of (15) by $\exp(h_1 t)$ we have

$$\|u(t,\cdot)\|_{L^\infty(\Omega)} \leqslant \{\max(\|u(t_0,\cdot)\|_{L^\infty(\Omega)}, m) + M(t_0)(t-t_0)\}\exp(h_1(t-t_0)).$$

Since $t \in [t_0, T]$ and the above inequality is true for every $h_1 > 0$, (3) follows. The proof is complete.

In the above Theorem we may omit the assumption that b_j and $(b_j)_{x_j}$ are continuous in D provided that the matrix $[a_{ij}(t, x)]_{i,j}$ is uniformly elliptic, i.e.:

(A1) there exists $\lambda > 0$ such that for every $(t, x) \in D$ and for every $\xi \in R^n$
$$\sum_{i,j} a_{ij}(t, x)\xi_i\xi_j \geqslant \lambda \sum_i \xi_i^2.$$

Under assumptions (U), (A), (A1), (C) and (D) we have

PROPOSITION. *If b_j for $j = 1, \ldots, n$ is locally bounded in D and u fulfils* (1), *then the estimate* (2) *holds.*

Proof. The proof is similar to that of the Theorem. We need only observe that b_j is globally bounded in D_l for every $l \in N$ (see the previous proof) and hence without loss of generality we may assume that

(E1) $(a_{ij})_{x_i}$ and $u_{x_ix_j}$ are continuous in $\bar{D}$ and b_j is globally bounded in D $(i, j = 1, \ldots, n)$.

Next we integrate by parts the first term on the right in (5) and estimate using condition (A1):

$$(16)\quad \int_\Omega \sum_{i,j} (a_{ij}u_{x_j})_{x_i} u^{2^k-1}dx = \int_{\partial\Omega} \sum_{i,j} a_{ij}u_{x_j}\cos(n, x_i)u^{2^k-1}ds$$
$$-(2^k-1)\int_\Omega \sum_{i,j} a_{ij}u_{x_j}u_{x_i}u^{2^k-2}dx$$
$$\leqslant m^{2^k-1}H_1-(2^k-1)\lambda\int_\Omega \sum_i (u_{x_i})^2u^{2^k-2}dx$$
$$= m^{2^k-1}H_1-2^{2-2k}(2^k-1)\lambda\int_\Omega \sum_i [(u^{2^{k-1}})_{x_i}]^2dx,$$

where H_1 is the same constant as in the proof of the Theorem.

The second term in (5) is estimated using the Hölder ($p = q = \frac{1}{2}$) and Cauchy ($ab \leqslant 2^{-1}\varepsilon a^2 + (2\varepsilon)^{-1}b^2$) inequalities:

$$(17)\quad \int_\Omega \sum_j b_ju_{x_j}u^{2^k-1}dx = 2^{1-k}\sum_j \int_\Omega (u^{2^{k-1}})_{x_j}b_ju^{2^{k-1}}dx$$
$$\leqslant 2^{1-k}\sum_j \Big\{\int_\Omega [(u^{2^{k-1}})_{x_j}]^2dx\Big\}^{1/2}\Big\{\int_\Omega b_j^2u^{2^k}dx\Big\}^{1/2}$$
$$\leqslant 2^{-k}\varepsilon\int_\Omega \sum_j [(u^{2^{k-1}})_{x_j}]^2dx + 2^{-k}\varepsilon^{-1}nB^2\int_\Omega u^{2^k}dx,$$

where $|b_j(t, x)| \leqslant B$ by (E1) for all $(t, x) \in \bar{D}$, $j = 1, \ldots, n$, and $\varepsilon > 0$ is arbitrary.

From (5), (16) and (17) we obtain

$$\frac{d}{dt}\int_\Omega u^{2^k}dx \leqslant 2^k m^{2^k-1}H_1 + \{\varepsilon-(2^k-1)2^{2-k}\lambda\}\int_\Omega \sum_i [(u^{2^{k-1}})_{x_i}]^2dx$$
$$+\int_\Omega (nB^2/\varepsilon + 2^kc(t, x))u^{2^k}dx + 2^kM(t)|\Omega|^{2^{-k}}\Big(\int_\Omega u^{2^k}dx\Big)^{1-2^{-k}}.$$

Choosing $\varepsilon > 0$ so small that $\varepsilon-(2^k-1)2^{2-k}\lambda < 0$ for all $k > 2$ and fixing $a > 0$ such that $nB^2/\varepsilon+2^k c(t, x) \leqslant -a$ for $k \geqslant k_0$, we obtain inequality (8) or (10) for $k \geqslant k_0$, which we may study in the same way as in the Theorem.

Remark. If $a_{ij} \equiv 0$ for all $i, j = 1, \ldots, n$, then our Theorem is applicable to linear partial differential equations of the first order. Under (C), (D) and assuming that $(b_j)_{x_j}$ is continuous in D (for $j = 1, \ldots, n$) we obtain the estimate (2) for solutions of the equation

$$u_t = \sum_j b_j(t, x)u_{x_j}+c(t, x)u+f(t, x), \quad (t, x) \in D.$$

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INSTITUTE OF MATHEMATICS
SILESIAN UNIVERSITY
Bankowa 14, 40-007 Katowice, Poland

Reçu par la Rédaction le 21.05.1989
Révisé le 01.10.1989