COLLOQUIUM MATHEMATICUM

LI DÉDIÉ À M. STANISŁAW HARTMAN 1987

SOME EMBEDDING THEOREMS FOR CLASSES OF SEQUENCES

BY

J. MUSIELAK (POZNAŃ)

1. In [3] there were obtained necessary and sufficient conditions for functions x belonging to an Orlicz class $L^{\varphi}(E, \mathscr{E}, \mu)$ in order that the function $g(\xi) = \int_E \varphi(\xi, |x(t)|)\,d\mu$ defined for indices $\xi \in \Xi$, where $(\Xi, \mathscr{X}, m)$ is a measure space, be integrable over subsets $Z \in \mathscr{X}$ of finite, positive measure m, in the case of a finite, atomless measure μ. This was extended in [4] to infinite, atomless μ, and in [5] there was considered a more general case with two families of indices (see also [1], § 19) again in case of μ finite and atomless; in [2], these considerations were extended to general (convex or concave) functionals in place of integrals.

The aim of this note is to obtain analogous results in case of spaces of sequences, i.e. a purely atomic, infinite measure μ. Let $(\Xi, \mathscr{X}, m)$ and $(H, \mathscr{Y}, n)$ be two measure spaces, where $\mathscr{X}$ resp. $\mathscr{Y}$ are σ-algebras of subsets of Ξ resp. H. Let $\mathscr{Z}$ and $\mathscr{Z}^*$ be two fixed non-empty families from $\mathscr{X}$ and $\mathscr{Y}$, respectively, such that $0 < m(Z) < \infty$ and $0 < n(Z^*) < \infty$ for $Z \in \mathscr{Z}$, $Z^* \in \mathscr{Z}^*$. We consider two functions $\varphi\colon \Xi \times R_+ \to R_+$ and $\psi\colon H \times R_+ \to R_+$, $R_+ = \langle 0, \infty)$, such that

1° $\varphi(\cdot, u)$ is $\mathscr{X}$-measurable with respect to $\xi \in \Xi$ and $\psi(\cdot, u)$ is $\mathscr{Y}$-measurable with respect to $\eta \in H$, for every $u \geqslant 0$,

2° $\varphi(\xi, \cdot)$ and $\psi(\eta, \cdot)$ are φ-functions (see e.g. [1], p. 4) for m-a.e. $\xi \in \Xi$ resp. n-a.e. $\eta \in H$,

3° $\varphi(\cdot, u)$ is m-integrable over Z and $\psi(\cdot, u)$ is n-integrable over Z^* for all $Z \in \mathscr{Z}$, $Z^* \in \mathscr{Z}^*$.

In the sequel, we shall need the following condition:

$$\int_{Z^*} \psi(\eta, u)\,dn \leqslant c \int_{Z} \varphi(\xi, u)\,dm \quad \text{for} \quad 0 \leqslant u \leqslant u_0. \tag{1}$$

Finally, let X be the space of all real-valued (or complex-valued) sequences convergent to 0, and let (p_j) be a sequence of positive numbers. We define two functions

$$g_{\varphi}(\xi) = \sum_{j=1}^{\infty} p_j \varphi(\xi, |t_j|) \quad \text{and} \quad g_{\psi}(\eta) = \sum_{j=1}^{\infty} p_j \psi(\eta, |t_j|)$$

for $x = (t_j) \in X$. Moreover, we shall write $L(Z) = L^1(Z, \mathscr{X}, m)$ resp. $L(Z^*) = L^1(Z^*, \mathscr{Y}, n)$ for the spaces of all m-integrable functions over $Z \in \mathscr{Z}$ resp. n-integrable functions over $Z^* \in \mathscr{Z}^*$.

2. The following direct theorem on connection between spaces $L(Z)$ and $L(Z^*)$ holds:

THEOREM 1. *Let* $0 < p_j < \infty$ *for* $j = 1, 2, \ldots$ *Then*:

1) *If*

(a) *there exist sets* $Z \in \mathscr{Z}$, $Z^* \in \mathscr{Z}^*$ *and constants* $c, u_0 > 0$ *such that* (1) *holds*,

then

(a′) *if* $g_\varphi \in L(Z)$ *for every* $Z \in \mathscr{Z}$, *then there exists* $Z^* \in \mathscr{Z}^*$ *such that* $g_\psi \in L(Z^*)$.

2) *If*

(b) *for every* $Z \in \mathscr{Z}$ *and* $Z^* \in \mathscr{Z}^*$ *there exist* $c, u_0 > 0$ *such that* (1) *holds*,

then

(b′) *if there exists* $Z \in \mathscr{Z}$ *such that* $g_\varphi \in L(Z)$, *then* $g_\psi \in L(Z^*)$ *for every* $Z^* \in \mathscr{Z}^*$.

3) *If*

(c) *for every* $Z^* \in \mathscr{Z}^*$ *there exist* $Z \in \mathscr{Z}$, $c, u_0 > 0$ *such that* (1) *holds*,

then

(c′) *if* $g_\varphi \in L(Z)$ *for every* $Z \in \mathscr{Z}$, *then* $g_\psi \in L(Z^*)$ *for every* $Z^* \in \mathscr{Z}^*$.

4) *If*

(d) *for every* $Z \in \mathscr{Z}$ *there exist* $Z^* \in \mathscr{Z}^*$ *and* $c, u_0 > 0$ *such that* (1) *holds*,

then

(d′) *if there exists* $Z \in \mathscr{Z}$ *such that* $g_\varphi \in L(Z)$, *then there exists* $Z^* \in \mathscr{Z}^*$ *such that* $g_\psi \in L(Z^*)$.

Proof. Let us fix the sets $Z \in \mathscr{Z}$, $Z^* \in \mathscr{Z}^*$ and numbers $c, u_0 > 0$ and let us suppose that (1) is satisfied. Then, taking any $x \in X$ and denoting $T = \{j\colon |t_j| \leqslant u_0\}$, the complement T' of T is a finite set and we obtain

$$\int_{Z^*} g_\psi(\eta)\,dn \leqslant c \int_{Z} g_\varphi(\xi)\,dm + \sum_{j \in T'} p_j \int_{Z^*} \psi(\eta, |t_j|)\,dn, \tag{2}$$

where the second term on the right-hand side of this inequality is finite, due to assumption 3°. In order to prove 1), let us suppose that $g_\varphi \in L(Z)$ for all $Z \in \mathscr{Z}$ and let us take sets $Z \in \mathscr{Z}$, $Z^* \in \mathscr{Z}^*$ specified by (a). Then the right-hand side of the inequality (2) is finite, and so $\int_{Z^*} g_\psi(\eta)\,dn < \infty$, i.e. $g_\psi \in L(Z^*)$. Parts 2)–4) of the theorem are proved analogously, applying inequality (2).

3. Now we are going to prove, under some additional assumptions, a converse theorem. Namely, the family $\mathscr{Z}$ is called σ-*absorbed* if there exists a nondecreasing sequence (Z_i) of sets $Z_i \in \mathscr{Z}$ such that for every $Z \in \mathscr{Z}$ there exists an index k for which $Z \subset Z_k$, and the family $\mathscr{Z}^*$ is called *σ-absorbing*, if there exists a sequence (Z_i^*) of sets $Z_i^* \in \mathscr{Z}^*$ such that for every $Z^* \in \mathscr{Z}^*$ there is an index k for which $Z_i^* \subset Z^*$ for all $i \geqslant k$ (see [5], 1.2 or [1], 19.2). The following theorem holds:

THEOREM 2. *Let us suppose that there are $a, b > 0$ such that $a \leqslant p_j \leqslant b$ for $j = 1, 2, \ldots$ Then*:

1) *if $\mathscr{Z}$ is σ-absorbed and $\mathscr{Z}^*$ is σ-absorbing, then* (a') *implies* (a),
2) (b') *implies* (b),
3) *if $\mathscr{Z}$ is σ-absorbed, then* (c') *implies* (c),
4) *if $\mathscr{Z}^*$ is σ-absorbing, then* (d') *implies* (d).

Proof. In order to prove 1) let us suppose that (a) does not hold. Then, taking the sets Z_i and Z_i^* from the definitions of σ-absorbicity, we may find a sequence (u_i) decreasing to zero such that

$$\int_{Z_i^*} \psi(\eta, u_i)\,dn > 2^i \int_{Z_i} \varphi(\xi, u_i)\,dm, \qquad i = 1, 2, \ldots \tag{3}$$

Moreover, $\int_{Z_i} \varphi(\xi, u_j)\,dm \to 0$ as $j \to \infty$, by the Lebesgue dominated convergence theorem and by 3°; hence we may choose u_i so small that $\int_{Z_i} \varphi(\xi, u_i)\,dm < 2^{-i-1}$ for $i = 1, 2, \ldots$ Then

$$2^{-i}\Big(\int_{Z_i} \varphi(\xi, u_i)\,dm\Big)^{-1} - 2^{-i-1}\Big(\int_{Z_i} \varphi(\xi, u_i)\,dm\Big)^{-1} > 1, \qquad i = 1, 2, \ldots$$

and so we may find positive integers N_i, $i = 1, 2, \ldots$, such that

$$2^{-i-1}\Big(\int_{Z_i} \varphi(\xi, u_i)\,dm\Big)^{-1} \leqslant N_i < 2^{-i}\Big(\int_{Z_i} \varphi(\xi, u_i)\,dm\Big)^{-1}, \qquad i = 1, 2, \ldots \tag{4}$$

We take a sequence (E_i) of pairwise disjoint sets of positive integers such that the number of elements of E_i is equal to N_i. Now, let us take $t_j = u_i$ if $j \in E_i$, $i = 1, 2, \ldots$, $t_j = 0$ for remaining j's, and let us define $x = (t_j)$. Of course, $x \in X$. We shall see that $g_\varphi \in L(Z)$ and $g_\psi \notin L(Z^*)$ for all $Z \in \mathscr{Z}$, $Z^* \in \mathscr{Z}^*$, whence (a') does not hold. Indeed, let $Z \in \mathscr{Z}$ be arbitrary and let k be such that $Z \subset Z_k \subset Z_i$ for $i \geqslant k$. Then

$$\begin{aligned}\int_Z g_\varphi(\xi)\,dm &\leqslant b \sum_{i=1}^{\infty} N_i \int_{Z_k} \varphi(\xi, u_i)\,dm \\ &\leqslant b \sum_{i=1}^{k-1} N_i \int_{Z_k} \varphi(\xi, u_i)\,dm + b \sum_{i=k}^{\infty} N_i \int_{Z_i} \varphi(\xi, u_i)\,dm \\ &< b \sum_{i=1}^{k-1} N_i \int_{Z_k} \varphi(\xi, u_i)\,dm + \frac{1}{2^{k-1}} < \infty,\end{aligned}$$

by (4) and 3°. Moreover, taking any $Z^* \in \mathscr{Z}^*$ and k such that $Z_i^* \subset Z^*$ for

$i \geqslant k$, and applying (3) and (4), we obtain

$$\int_{Z^*} g_\psi(\eta)\,dn \geqslant a \sum_{i=k}^{\infty} N_i \int_{Z_i^*} \psi(\eta, u_i)\,dn \geqslant a \sum_{i=k}^{\infty} N_i 2^i \int_{Z_i} \varphi(\xi, u_i)\,dm = \infty.$$

Now, in order to prove 2) we observe that supposing (b) to be not true, we get the inequality (3) with some fixed $Z \in \mathscr{Z}$, $Z^* \in \mathscr{Z}^*$ in place of Z_i and Z_i^*. We choose N_i by means of (4), where Z_i is replaced by Z and we define (E_i) and $x = (t_j)$ as in case 1). Then

$$\int_{Z} g_\varphi(\xi)\,dm \leqslant b \sum_{i=1}^{\infty} N_i \int_{Z} \varphi(\xi, u_i)\,dm < b < \infty,$$

but

$$\int_{Z^*} g_\psi(\eta)\,dn \geqslant a \sum_{i=1}^{\infty} N_i 2^i \int_{Z} \varphi(\xi, u_i)\,dm = \infty,$$

a contradiction to (b′). Proofs of 3) and 4) are performed in an analogous manner, with N_i defined as in 1) in case of 3), and N_i defined as in 2) in case of 4).

4. A very special case of the above theorems is obtained supposing Ξ and H to be one-element sets, the measures m, n being normalized to 1, $\mathscr{Z} = \{\Xi\}$, $\mathscr{Z}^* = \{H\}$. Then, denoting by l_p^φ and l_p^ψ the Orlicz classes of sequences, generated by modulars $\sum_{j=1}^{\infty} p_j \varphi(|t_j|)$ and $\sum_{j=1}^{\infty} p_j \psi(|t_j|)$, respectively, Theorems 1, 2 give the known result that supposing $0 < a \leqslant p_j \leqslant b < \infty$, $j = 1, 2, \ldots$, there holds $l_p^\varphi \subset l_p^\psi$, if and only if, there exist constants $c, u_0 > 0$ such that $\psi(u) \leqslant c\varphi(u)$ for $0 \leqslant u \leqslant u_0$.

Another application is obtained taking as $\Xi = H$ the set of all positive integers, the measures m, n of a single element being 1, $\mathscr{Z}$ and $\mathscr{Z}^*$ constituted by sets $\{1\}, \{1, 2\}, \{1, 2, 3\}, \ldots$ Then e.g. application of part 3) of our theorems yields the following

Corollary. *Let $0 < a \leqslant p_j \leqslant b < \infty$, $j = 1, 2, \ldots$ and let (φ_j), (ψ_j) be two sequences of φ-functions. There holds*

$$\bigcap_{i=1}^{\infty} l_p^{\varphi_i} \subset \bigcap_{i=1}^{\infty} l_p^{\psi_i},$$

if and only if, for every index l there are an index k and numbers $c, u_0 > 0$ such that

$$\sum_{j=1}^{l} \psi_j(u) \leqslant c \sum_{j=1}^{k} \varphi_j(u) \quad \textit{for all } 0 \leqslant u \leqslant u_0$$

(for the case of a single function ψ in place of a sequence (ψ_j), see [6], Th. 1 (1)).

The reader may easily conclude other statements of the same type.

5. Let us still remark that analogous results may be obtained also in case of a space X of measurable, finite a.e. functions $x = x(t)$ defined over a measure space $(E, \mathscr{E}, \mu)$ with an infinite, atomless measure μ, with g_φ and g_ψ defined as

$$g_\varphi(\xi) = \int_E \varphi(\xi, |x(t)|)\,d\mu \quad \text{and} \quad g_\psi(\eta) = \int_E \psi(\eta, |x(t)|)\,d\mu.$$

Condition (1) has to be replaced by an analogous one, replacing $0 \leqslant u \leqslant u_0$ by all $u \geqslant 0$. One has then to omit u_0 in the formulation of the theorems.

REFERENCES

[1] J. Musielak, *Orlicz spaces and modular spaces*, Lecture Notes in Mathematics 1034, Springer-Verlag, Berlin–Heidelberg–New York–Tokyo 1983, p. 1–222.

[2] –, *Some embedding theorems for modular classes*, p. 165–172 in: *Functional Analysis and Approximation, Proceedings of Conference Oberwolfach, 30. July – 6. August 1983*, Birkhäuser Verl., 1984.

[3] – and A. Waszak, *On some families of functions integrable with a parameter*, Publ. Elektr. Fak. Univ. u Beogradu 480 (1974), p. 127–137.

[4] –, –, *Some remarks on functions integrable with a parameter*, Functiones et Approximatio Commentarii Mathematici 3 (1976), p. 175–183.

[5] –, –, *Some remarks on families of Orlicz classes*, Commentationes Mathematicae 19 (1977), p. 297–303.

[6] –, –, *On Orlicz sequence spaces with application to two-modular spaces*, Functiones et Approximatio Commentarii Mathematici 11 (1981), p. 127–132.

Reçu par la Rédaction le 27. 02. 1984