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Smoothness properties of solutions of mixed boundary value problems for elliptic equations in sectionally smooth n-dimensional domains

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Abstract. We study here the smoothness of solutions of mixed boundary value problem for elliptic equations in sectionally smooth n-dimensional domains.

1. Introduction. Main result. In a bounded domain $\Omega \subset \boldsymbol{R}^n$, $n \geqslant 2$, we consider here the boundary value problems for the elliptic equation

$$Lu \equiv a_{ij}(x)u_{ij}+a_i(x)u_i+a(x)u = f(x), \tag{1.1}$$

where $x = (x_1, \ldots, x_n)$; $n \geqslant 2$, $u_i = \partial u/\partial x_i$, $u_{ij} = \partial^2 u/\partial x_i \partial x_j$ and we use the summation convention. General boundary value problems for (1.1), in a domain Ω with a smooth boundary, have been thoroughly investigated, cf. [1], [2]. A known result is the following. Let the boundary $\partial\Omega$ of Ω be of class $C^{m+2+\alpha}$ and let the coefficients of (1.1) be of class $C^{m+\alpha}(\bar{\Omega})$. Suppose that u satisfies (1.1) in Ω and on $\partial\Omega$ satisfies the boundary condition

$$\eta_1 u+\eta_2 u_\nu = \eta_1 \varphi+\eta_2 \psi, \tag{1.2}$$

where η_i is constant, $i = 1, 2$ and u_ν is the outward normal derivative of u. If $\varphi \in C^{m+2+\alpha}(\partial\Omega)$ and $\psi \in C^{m+1+\alpha}(\partial\Omega)$, then $u \in C^{m+2+\alpha}(\bar{\Omega})$. If $\partial\Omega$ is sectionally smooth, this conclusion may not be true. The reason is that, it is not possible in this case to smooth the boundary by means of a smooth transformation. Moreover, from the simplest examples, it is apparent that when the boundary contains edges (angular points if $n = 2$), the solution of (1.1)–(1.2) may not be of class $C^1(\bar{\Omega})$ even for infinitely differentiable coefficients in (1.1) and boundary data. In weighted Sobolev spaces, Kondrat'ev considered in [5]–[7] different boundary value problems for (1.1) when $\partial\Omega$ is sectionally smooth.

We consider here the problem in the space $C^{m+\alpha}$. See also [3], [4], [9], [11], [12] and the references given in [4].

Consider the simply connected bounded domain $\Omega \subset \boldsymbol{R}^n$, $n \geqslant 2$. We assume that the boundary Γ of Ω consists of $(n-1)$-dimensional surfaces $\Gamma_1, \ldots, \Gamma_k$ belonging to $C^{m+2+\alpha}$, $m \geqslant 0$ an integer and $0 < \alpha < 1$. We assume that the surface Γ_i may intersect only with Γ_{i-1} and Γ_{i+1} across $(n-2)$-dimensional manifolds S_{i-1} and S_i, $i = 1, \ldots, k$; $\Gamma_{k+1} = \Gamma_1$ and $\Gamma_0 = \Gamma_k$. Consider a bounded solution of the mixed boundary value problem

$$Lu = f, \quad \text{cf. (1.1)}, \tag{1.3}$$

$$\eta_i u + (1-\eta_i) u_\nu = 0 \quad \text{on } \Gamma_i, \tag{1.4}$$

where η_i is either 0 or 1 and $\eta_i + \eta_{i+1} \neq 0$. It is known from [1] that, if a_{ij}, a_i, a and f in (1.3) are of class $C^{m+\alpha}(\bar{\Omega})$, then

$$u \in C^{m+2+\alpha}(\Omega_1) \cap C^0(\bar{\Omega}), \tag{1.5}$$

where Ω_1 is any compact subregion of $\bar{\Omega}$ with positive distance from $S = \bigcup_i S_i$. To investigate the smoothness of u near the edges, consider any point $P \in S_i$. Let $R_i(P)$ and $R_{i+1}(P)$ be the two planes which touch Γ_i and Γ_{i+1} at P making an angle $\gamma_i(P)$. We transform the equation

$$a_{ij}(P) u_{ij} = 0 \tag{1.6}$$

to canonical form. This equation is an equation with constant coefficients since the point P is fixed. After the transformation, the planes $R_i(P)$ and $R_{i+1}(P)$ will be transformed to other planes with angle $\omega_i(P)$ between them. It is clear that $\omega_i(P)$ does not depend upon the transformation used to transform (1.6) to canonical form. We now state our main result.

THEOREM 1. *Let a_{ij}, a_i, a and f in* (1.3) *be of class $C^{m+\alpha}(\bar{\Omega})$. If for any P and i; $\omega_i(P) < (\eta_i + \eta_{i+1})\pi/2(m+2+\alpha)$, then the solution of* (1.3)–(1.4) *belongs to $C^{m+2+\alpha}(\bar{\Omega})$.*

From (1.5) it follows that, to prove the theorem, it suffices to show that, for any point $P \in \bigcup S_i$, $u \in C^{m+2+\alpha}(N_{P,q})$, where $N_{P,q}$ is the intersection of $\bar{\Omega}$ with a sphere of radius $q > 0$ and center at P. This local result will follow from considering the sector case in the next section.

2. The sector case. Consider the domain G given by

$$G = \{x \mid 0 < r < \infty,\ 0 < \theta < \omega,\ |x_i| < \infty,\ i > 2\},$$

where $r = \sqrt{x_1^2 + x_2^2}$, $\theta = \arctan(x_2/x_1)$ and $\omega = \omega(0)$ is constant. We define Γ_1, Γ_2 and S as follows

$$\Gamma_1 = \{x \mid x_2 = 0\}, \quad \Gamma_2 = \{x \mid x_2 = x_1 \tan \omega\}, \quad S = \Gamma_1 \cap \Gamma_2.$$

Assume that the bounded function u satisfies in G the elliptic equation

$$Lu = f, \tag{2.1}$$

where $a_{ij}(0) = \delta_{ij}$, $i, j = 1, 2$. We also assume that u satisfies the boundary conditions

(2.2a) $$u = 0 \quad \text{on } \Gamma_1,$$

(2.2b) $$\mathscr{B}u = \eta u + (1-\eta)u_v = 0 \quad \text{on } \Gamma_2.$$

Here η is either 0 or 1. Let $P \in S$ be any point and let $\omega(P)$ be the angle obtained after transforming the principal part of (2.1) at P to canonical form. We now state a theorem from which the result of Theorem 1 follows.

THEOREM 2. *Let a_{ij}, a_i, a and f in* (2.1) *be of class $C^{m+\alpha}(\bar{G})$. If for any $P \in S$ we have*
$$\omega(P) < (\eta+1)\pi/2(m+2+\alpha),$$
then $u \in C^{m+2+\alpha}(N_{0,q})$, for some constant $q > 0$.

We shall prove this result assuming that an additional condition (Condition I below) is satisfied. In the next sections we shall show that Condition I is not a restriction.

CONDITION I. The right-hand side f of (2.1) and the solution u of (2.1), (2.2) satisfy in G_q; $G_q = \{x|\ x \in \bar{G},\ r \leqslant q,\ |x_i| \leqslant q,\ i > 2\}$, the conditions

(2.3) $$D^k f(x)|_{x_1 = x_2 = 0} = 0, \quad k \leqslant m,$$

and

(2.4) $$|D^k u(x)| \leqslant M_1 r^{m+2-k+\alpha}, \quad k \leqslant m+2.$$

Here $D^k g(x)$ is any partial derivative of $g(x)$ of order k.

THEOREM 2′. *If Condition* I *is satisfied, then under the assumptions of Theorem* 2 *we have*
$$u \in C^{m+2+\alpha}(G_q).$$

Proof. The result follows if we can prove that

(2.5) $$\frac{|D^{m+2}u(Q) - D^{m+2}u(R)|}{\overline{QR}^{\alpha}} \leqslant H,$$

with some constant $H > 0$ and for any two different points Q and R in G_q. Let the distances of Q and R from S be r_1 and r_2 respectively. Without restriction we assume that $0 \leqslant r_2 \leqslant r_1$. If $r_2 \leqslant r_1/2$, then in this case $\overline{QR} \geqslant r_1/2$, and from (2.4) (with $k = m+2$) we obtain (2.5). We now consider the case where $r_2 > r_1/2$. Let $Q = (x_1^0, \ldots, x_n^0)$. Consider the region Ω_Q defined by

$$\Omega_Q = \{x|\ x \in G_q,\ r_1/2 \leqslant r \leqslant r_1,\ |x_i - x_i^0| \leqslant r_1/2,\ i > 2\}.$$

We perform the transformation

(2.6)
$$x_i = \frac{2r_1}{q} x_i', \qquad i = 1,2,$$
$$x_i - x_i^0 = \frac{2r_1}{q}(x_i' - x_i^0), \quad i > 2.$$

The region Ω_Q will be transformed by (2.6) to $\tilde{\Omega}_Q$;

$$\tilde{\Omega}_Q = \{q/4 \leqslant r' \leqslant q/2,\ |x_i' - x_i^0| \leqslant q/4,\ i > 2\}.$$

Here $r'^2 = x_1'^2 + x_2'^2$. Let the functions u, a_{ij}, a_i, a and f be transformed by (2.6) to U, A_{ij}, A_i, A and F. In $\tilde{\tilde{\Omega}}_Q$:

$$\tilde{\tilde{\Omega}}_Q = \{q/8 \leqslant r' \leqslant q,\ |x_i' - x_i^0| \leqslant q/4,\ i > 2\},$$

the function U satisfies the elliptic equation

(2.7) $$A_{ij}(x')U_{ij} + (2r_1/q)A_i(x')U_i + (2r_1/q)^2 A(x')U = (2r_1/q)^2 F(x'),$$

where $x' = (x_1', \ldots, x_n')$. U also satisfies the boundary conditions

(2.8a) $$U = 0 \quad \text{on } \Gamma_1,$$

(2.8b) $$\eta U + (1-\eta)U_\nu = 0 \quad \text{on } \Gamma_2.$$

The images $\tilde{Q}$ and $\tilde{R}$ of Q and R have distances r_1' and r_2' from S, where $r_1' = q/2$ and $r_2' = qr_2/2r_1 > q/4$. In $\tilde{\Omega}_Q$ and $\tilde{\tilde{\Omega}}_Q$ the Schauder inequality for the solution U of (2.7) and (2.8) yields

(2.9) $$\|U\|_{m+2+\alpha}^{\tilde{\Omega}_Q} \leqslant b\left(\|U\|_0^{\tilde{\tilde{\Omega}}_Q} + (2r_1/q)^2 \|F\|_{m+\alpha}^{\tilde{\tilde{\Omega}}_Q}\right),$$

cf. [1]. From (2.4) (with $k = 0$) it follows that

$$\|U\|_0^{\tilde{\tilde{\Omega}}_Q} \leqslant \tilde{M} r_1^{m+2+\alpha}.$$

Also from the definition of F and (2.3) it follows that

$$\|F\|_{m+\alpha}^{\tilde{\tilde{\Omega}}_Q} \leqslant \tilde{\tilde{M}} r_1^{m+\alpha}.$$

Thus (2.9) gives

(2.10) $$\|U\|_{m+2+\alpha}^{\tilde{\Omega}_Q} \leqslant M_0 r_1^{m+2+\alpha}.$$

We note that

(2.11) $$D_1^k U = (2r_1/q)^k D^k u, \quad k = 0, 1, \ldots, m+2,$$

where D_1^k is the derivative corresponding to D^k. We also note that

(2.12) $$H_\alpha^{\tilde{\Omega}_Q}(D_1^{m+2} U) = (2r_1/q)^{m+2+\alpha} H_\alpha^{\Omega_Q}(D^{m+2} u),$$

where $H_\alpha^{\tilde{\Omega}}(g)$ is the Hölder coefficient of g, with index α in the region $\tilde{\Omega}$.

Thus from (2.10)–(2.12) we obtain in Ω_Q

(2.13) $$|D^k u| \leqslant C_1 r_1^{m+2-k+\alpha}$$

and

(2.14) $$H_\alpha^{\Omega_Q}(D^{m+2} u) \leqslant C_2.$$

We now prove (2.5) in the case $r_2 > r_1/2$. Besides the points Q and R, consider the point Q_1 lying on the normal from R to S and having distance r_1 from S. If $\overline{QQ_1} \leqslant r_1/2$, then $R \in \Omega_Q$, where $u \in C^{m+2+\alpha}$, cf. (2.13), (2.14). If $\overline{QQ_1} > r_1/2$, then $\overline{QR} \geqslant \overline{QQ_1} > r_1/2$ and also $\overline{QR} \geqslant \overline{Q_1R}$. Thus

$$\frac{|D^{m+2}u(Q)-D^{m+2}u(R)|}{\overline{QR}^\alpha} \leqslant \frac{|D^{m+2}u(Q)-D^{m+2}u(Q_1)|}{\overline{QQ_1^\alpha}}+$$
$$+\frac{|D^{m+2}u(Q_1)-D^{m+2}u(R)|}{\overline{Q_1R^\alpha}} \leqslant \frac{2C_1 r_1^\alpha}{(r_1/2)^\alpha}+C_2 \leqslant C_3,$$

since in this case $R \in \Omega_{Q_1}$, where $u \in C^{m+2+\alpha}(\Omega_{Q_1})$. This completes the proof of Theorem 2′.

In the next section we shall prove that there exists a function $v \in C^{m+2+\alpha}(\bar{G})$, such that the function $g = f - Lv$ will satisfy (2.3) of Condition I.

3. Two lemmas. In this section we shall prove two lemmas. The first one will be used in proving the second one.

LEMMA 1. *Let $F(x_3, \ldots, x_n) \in C^{p+\alpha}$ be defined on $x_1 = x_2 = 0$. There exists an extension $F^*(x) \in C^{p+\alpha}(R^n)$ of $F(x_3, \ldots, x_n)$ which satisfies*

(3.1) $$x_1^{k-k_1} x_2^{k_1} F^*(x) \in C^{p+k+\alpha}(R^n), \quad k_1 \leqslant k \text{ and } k \geqslant 0.$$

Proof. Consider the averaging kernal $K(t) = K(t_3, \ldots, t_n) \in C^\infty$ which has the following properties

(a) $K(t) \geqslant 0$ for $|t_i| < \infty$, $i = 3, \ldots, n$,

(b) $K(t) = 0$ if $|t_i| \geqslant 1$, $i = 3, \ldots, n$,

(c) $\int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} K(t)\,dt_3 \ldots dt_n = 1$.

We define the function F^* as follows

(3.2) $$F^*(x) = r^{2-n} \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} K\left(\frac{x_3-\tau_3}{r}, \ldots, \frac{x_n-\tau_n}{r}\right) F(\tau_3, \ldots, \tau_n)\,d\tau_3 \ldots d\tau_n.$$

Here $r = \sqrt{x_1^2+x_2^2}$. We first show that F and F^* coincide when $x_1 = x_2 = 0$.

By a change of variables and in virtue of property (b), (3.2) takes the from

$$F^*(x) = \int_{-1}^{1} \dots \int_{-1}^{1} K(t) F(x_3 - rt_3, \dots, x_n - rt_n)\, dt_3 \dots dt_n.$$

Using the Mean Value Theorem and property (c) of K we get

$$F^*(x) = F(x_3 - r\tilde{t}_3, \dots, x_n - r\tilde{t}_n); \qquad |\tilde{t}_i| \leqslant 1.$$

As $r \to 0$ we obtain

$$F^*(0, 0, x_3, \dots, x_n) = F(x_3, \dots, x_n).$$

To illustrate the proof of (3.1) we only consider the case $p = 0$, $k_1 = 0$ and $k = 1$. By a straightforward calculation it can be shown that $F^* \in C^\alpha$. To show that $x_1 F^*(x) \in C^{1+\alpha}$ it suffices to prove that $x_1 \dfrac{\partial F^*}{\partial x_i} \in C^\alpha$, $i = 1, \dots, n$. We first prove that

$$\left| x_1 \frac{\partial F^*(x)}{\partial x_i} \right| \leqslant A r^\alpha, \quad i = 1, \dots, n. \tag{3.3}$$

Indeed, for $i = 1, 2$ we find from (3.2) that

$$x_1 \frac{\partial F^*}{\partial x_i} = -\frac{x_1 x_i}{r^2} \int_{-1}^{1} F(x_3 - rt_3, \dots, x_n - rt_n) \sum_{p=3}^{n} \frac{\partial}{\partial t_p}(t_p K)\, dt_3 \dots dt_n.$$

Using property (b) of K we can write

$$x_1 \frac{\partial F^*}{\partial x_i} = -\frac{x_1 x_i}{r^2} \int_{-1}^{1} [F(x_3 - rt_3, \dots, x_n - rt_n) - F(x_3, \dots, x_n)] \times$$

$$\times \sum_{p=3}^{n} \frac{\partial}{\partial t_p}(t_p K)\, dt_3 \dots dt_n.$$

From $F(x_3, \dots, x_n) \in C^\alpha$ inequality (3.3) may be obtained. Similarly if $i > 2$ we can obtain (3.3). Using (3.3) it can be shown that $x_1 \dfrac{\partial F^*}{\partial x_i} \in C^\alpha$. We omit the details.

In the next lemma we show that relation (2.3) of Condition I is not a restriction.

LEMMA 2. *There exists a function* $v(x) \in C^{m+2+\alpha}(\bar{G})$ *satisfying*

$$v = 0 \quad on\ \Gamma_1, \tag{3.4a}$$

$$\mathscr{B} v = \eta v + (1 - \eta) v_\nu = 0 \quad on\ \Gamma_2, \tag{3.4b}$$

and is such that the function $w = u - v$ satisfies in G the equation $Lw = g$ with g satisfying

$$D^k g(x)|_{x_1 = x_2 = 0} = 0, \quad k \leqslant M,$$

cf. (2.3).

Proof. We prove the lemma by induction. Consider the function

$$v_1(x) = F_{1,0}(x_3, \ldots, x_n) x_1 x_2 + F_{0,1}(x_3, \ldots, x_n) x_2^2. \tag{3.5}$$

This function vanishes on Γ_1 and we shall show that coefficients $F_{ij}(x_3, \ldots, x_n)$, exist such that

$$\mathscr{B} v_1 = 0 \quad \text{on } \Gamma_2$$

and

$$\sum_{i,j=1}^{2} a_{ij}(0, 0, x_3, \ldots, x_n) v_{1,ij} = f(0, 0, x_3, \ldots, x_n),$$

cf. (2.1) and (3.4b). Thus $A_2 = \{F_{1,0} F_{0,1}\}$ satisfies an algebraic equation of the form

$$M_2 A_2 = E_2, \tag{3.6}$$

where here and in the rest of this proof M_p, A_p and E_p denote matrices of order $p \times p$, $p \times 1$ and $p \times 1$ respectively. The entries of these matrices are functions of the variables $x_3, \ldots, x_n$ of class $C^{m-p+2+\alpha}$. If E_2 in (3.6) is identically zero we take $A_2 = 0$. If $E_2 \not\equiv 0$, then A_2 will be uniquely determined if M_2 is non-singular, we show this actually is the case. Suppose contrarily that det M_2 vanishes at $(x_3^0, \ldots, x_n^0)$. Let $A_2^0 = \{F_{1,0}^0 F_{0,1}^0\}$ be any *non-trivial* solution of $M_2 A_2 = 0$. Using a linear transformation $x \mapsto y$, we transform the equation

$$\sum_{i,j=1}^{2} a_{ij}(0, 0, x_3^0, \ldots, x_n^0) \frac{\partial^2 u}{\partial x_i \partial x_j} = 0 \tag{3.7}$$

to canonical form. The domain G will be transformed to another one bounded by the hyperplanes $y_2 = 0$ and $y_2 = y_1 \tan \omega(P_0)$, where $P_0 = (0, 0, x_3^0, \ldots, x_n^0)$. The function

$$v_1^0(x) = F_{1,0}^0(x_3, \ldots, x_n) x_1 x_2 + F_{0,1}^0(x_3, \ldots, x_n) x_2^2$$

will be transformed to a function $w(y) = \varrho^2 h(\varphi)$, where $\varrho^2 = y_1^2 + y_2^2$, $\varphi = \arctan(y_2/y_1)$ and the coefficients in $h(\varphi)$ are functions of $y_3, \ldots, y_n$. The function $h(y)$ is harmonic (in y_1 and y_2), vanishes on $y_2 = 0$ and on $y_2 = y_1 \tan \omega(P_0)$ satisfies the condition

$$\eta h + (1 - \eta) \frac{\partial h}{\partial \varphi} = 0.$$

Thus $h(\varphi)$ satisfies

$$h''+4h = 0, \quad h(0) = 0, \quad \eta h(\varphi)+(1-\eta)h'(\varphi) = 0 \quad \text{on } \varphi = \omega(P_0).$$

Here η is either 0 or 1. This problem has a non-trivial solution if and only if

$$\omega(P_0) = \frac{2k+\eta-1}{4}\pi, \quad k = 1, 2, \ldots,$$

which is not satisfied since

$$\omega(P_0) < (\eta+1)\pi/2(m+2+\alpha) < (\eta+1)\pi/4.$$

Thus $h \equiv 0$ and consequently $v_1^0 \equiv 0$. This contradicts $\det M_2 = 0$. Thus $\det M_2$ does not vanish and a solution A_2 of (3.6) exists with entries $F_{ij}(x_3, \ldots, x_n) \in C^{m+\alpha}$. Using Lemma 1, we now construct the functions $F_{ij}^*(x) \in C^{m+\alpha}$, where $x_1^{2-p} x_2^p F_{ij}^*(x) \in C^{m+2+\alpha}$, $p = 1, 2$. Then instead of (3.5) we take

$$v_1 = x_1 x_2 F_{1,0}^*(x) + x_2^2 F_{0,1}^*(x).$$

v_1 now satisfies

$$v_1 = 0 \quad \text{on } \Gamma_1, \quad \mathcal{B}v_1 = 0 \quad \text{on } \Gamma_2 \quad \text{and} \quad Lv_1 = f(0, 0, x_3, \ldots, x_n)+o(r).$$

We now proceed to prove the lemma by induction. Consider the functions

$$(3.8) \qquad \begin{aligned} v_{p-1} &= \sum_{i+j \leq p-1} x_1^i x_2^{j+1} F_{ij}^*(x), \quad 2 \leq p \leq m+1, \\ V_p &= \sum_{i+j=p} F_{ij}(x_3, \ldots, x_n) x_1^i x_2^{j+1}. \end{aligned}$$

Assume that $v_{p-1}(x) \in C^{m+2+\alpha}$ has already been found such that

$$\mathcal{B}v_{p-1} = 0 \quad \text{on } x_2 = x_1 \tan \omega$$

and

$$(3.9) \qquad Lv_{p-1} = \sum_{i+j \leq p-2} \frac{x_1^i x_2^j}{i!\,j!} \frac{\partial^{i+j} f(x)}{\partial x_1^i \partial x_2^j}\bigg|_{x_1 = x_2 = 0} + P_{p-1}(x_1, x_2) + o(r^{p-1}),$$

where $P_{p-1}(x_1, x_2)$ is a homogeneous polynomial in x_1 and x_2 of degree $p-1$. We now show that the coefficients F_{ij} in (3.8) may be found such that V_p will satisfy

$$(3.10) \qquad \mathcal{B}V_p = 0 \quad \text{on } x_2 = x_1 \tan \omega$$

and

$$(3.11) \qquad LV_p = \sum_{i+j=p-1} \frac{x_1^i x_2^j}{i!\,j!} \frac{\partial^{i+j} f}{\partial x_1^i \partial x_2^j}\bigg|_{x_1 = x_2 = 0} - P_{p-1}(x_1, x_2) + o(r^{p-1}),$$

where P_{p-1} is taken from (3.9). Comparing the coefficients of $x_1^i x_2^j$ on both sides of (3.10) and (3.11) we obtain a system of algebraic equations

$$M_{p+1} A_{p+1} = E_{p+1}. \tag{3.12}$$

If $E_{p+1} \equiv 0$ we take $A_{p+1} = 0$. If $E_{p+1} \not\equiv 0$ we shall prove as before that in this case det M_{p+1} does not vanish anywhere. Suppose that det $M_{p+1} = 0$ at $(x_3^0, \ldots, x_n^0)$ and consider the function

$$V_p^0 = \sum_{i+j=p} F_{ij}^0(x_3, \ldots, x_n) x_1^i x_2^{j+1},$$

where $A_{p+1}^0 = \{F_{ij}^0\}$ is a non-trivial solution of $M_{p+1} A_{p+1} = 0$. As before, by a linear transformation $x \mapsto y$ we transform (3.7) to canonical form. The function $V_p^0(x)$ will be transformed to a function $h(y)$, which is harmonic as a function of y_1 and y_2 and satisfies

$$h(y) = 0 \quad \text{on } y_2 = 0,$$
$$\eta h(y) + (1-\eta) h_\nu(y) = 0 \quad \text{on } y_2 = y_1 \tan \omega(P_0).$$

By the principal of symmetry, $h(y)$ vanishes on all the hyperplanes of the form

$$y_2 = y_1 \tan \frac{2k\omega(P_0)}{\eta+1}; \quad k = 0, 1, \ldots$$

If $\pi/\omega(P_0)$ is irrational, then the number of these hyperplanes is infinite. If $\pi(\eta+1)/2\omega(P_0) = N/D$, a reduced fraction; $D \geqslant 1$, then the number of these hyperplanes is N, and $N \geqslant N/D = \pi(\eta+1)/2\omega(P_0) > m+2$. This contradicts the fact that h is a homogeneous polynomial of order $p+1$ with $p \leqslant m+1$. Thus det $M_{p+1} \neq 0$ and $F_{ij} \in C^{m-p+1+\alpha}$ are uniquely defined from (3.10) and (3.11). Again using Lemma 1 we construct $F_{ij}^*(x)$; $x_1^{p-j} x_2^{j+1} F_{ij}^*(x) \in C^{m+2+\alpha}$. In (3.8) we then replace F_{ij} by F_{ij}^* to get $V_p \in C^{m+2+\alpha}$. Finally the function

$$v = v_m + V_{m+1},$$

will satisfy the required conditions of Lemma 2. This proves the lemma.

4. Bounds for the solution and its derivatives. In Section 3 we have shown that the function

$$w = u - v, \tag{4.1}$$

with $v \in C^{m+2+\alpha}(\bar{G})$, satisfies the equation

$$Lw = g \tag{4.2}$$

and the boundary conditions

(4.3a) $$w = 0 \quad \text{on } \Gamma_1,$$

(4.3b) $$\mathscr{B}w = 0 \quad \text{on } \Gamma_2,$$

with $g(x)$ satisfying in $\bar{G}$

(4.4) $$D^k g(x)|_{x_1=x_2=0} = 0, \quad k \leqslant m.$$

In this section we shall prove that the solution w of (4.2), (4.3) and its derivatives satisfy in G_q the bounds

(4.5) $$|D^k w(x)| \leqslant Mr^{m+2-k+\alpha}, \quad k \leqslant m+2.$$

If this step is established, we can apply Theorem 2′ to conclude that $w \in C^{m+2+\alpha}(G_q)$, which together with (4.1) gives the result of Theorem 2. Note that $N_{0,q} \subset G_q$.

LEMMA 3. *The solution w of* (4.2), (4.3) *satisfies in G_{2q} the inequality*

(4.6) $$|w(x)| \leqslant Mr^{m+2+\alpha},$$

where $M > 0$ and $q > 0$ are constants.

Proof. Without loss of generality we can assume that w vanishes outside $N_{0,2q}$. This condition may be relaxed as it was done in [6]. We consider first the case $\eta = 0$. The modifications in the proof for the case $\eta = 1$ will be given in the end of this proof. Consider the function

$$U(x) = -Mr^{\beta} \cos \lambda(\omega - \theta),$$

where $\lambda = (\pi - 2\delta)/2\omega$ and $\delta > 0$ is such that $\lambda > m+2+\alpha = \beta$. We write LU as follows

$$LU = \Delta U + L_1 U,$$

where $\Delta = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2$. Now

$$\Delta U = M(\lambda^2 - \beta^2) r^{m+\alpha} \cos \lambda(\omega - \theta),$$

while $L_1 U = o(r^{m+\alpha})$ as $r \to 0$. Thus for any $\varepsilon > 0$, we can find $q > 0$ sufficiently small that in G_{2q} we have

(4.7) $$LU \geqslant M[(\lambda^2 - \beta^2) \cos \lambda(\omega - \theta) - \varepsilon] r^{m+\alpha}.$$

Since $g \in C^{m+\alpha}(\bar{G})$, then from (4.4) it follows that

$$|g(x)| \leqslant Kr^{m+\alpha}, \quad x \in \bar{G}.$$

Now for $0 \leqslant \theta \leqslant \omega$ we have $\cos \lambda(\omega - \theta) \geqslant \sin \delta$, thus taking $\varepsilon < (\lambda^2 - \beta^2) \sin \delta$, then $M > K/[(\lambda^2 - \beta^2) \sin \delta - \varepsilon]$ we obtain from (4.7) that $LU \geqslant g$ in G_{2q}. That is

(4.8) $$L(w - U) \leqslant 0 \quad \text{in } G_{2q}.$$

On Γ_2 we have $U_\nu = \left.\frac{1}{r}\frac{\partial U}{\partial \theta}\right|_{\theta=\omega} = 0$. Thus $w-U$ cannot attain its minimum on Γ_2. We now show that $w-U$ can be made non-negative on the remaining parts of the boundary of G_{2q}. On Γ_1 we have

$$w-U = Mr^{\beta}\sin\delta \geqslant 0.$$

On $r = 2q$ or $|x_i| = 2q$, $i > 2$ we have $w = 0$ and $w-U$ is also non-negative on these portions of the boundary. To apply the maximum principal we take q sufficiently small, cf. [8]. Thus in G_{2q} we have $w-U \geqslant 0$. That is

$$w \geqslant -Mr^{m+2+\alpha}\cos\lambda(\omega-\theta) \geqslant -Mr^{m+2+\alpha}. \tag{4.9}$$

Similarly we can show that in G_{2q}

$$w \leqslant Mr^{m+2+\alpha}, \tag{4.10}$$

provided that M is taken sufficiently large and q sufficiently small. This gives the proof in the case $\eta = 0$. For $\eta = 1$ we may take the barrier function

$$\tilde{U} = -Mr^{\tilde{\beta}}\cos\tilde{\lambda}\left(\frac{\omega}{2}-\varphi\right),$$

where $\tilde{\lambda} = (\pi-2\tilde{\delta})/\omega > m+2+\alpha = \tilde{\beta}$.

Then (4.7) (with U replaced by $\tilde{U}$) can be similarly obtained. $w-\tilde{U}$ may be made non-negative on the boundary of G_{2q} by taking M sufficiently large. This leads to (4.9). Similarly (4.10) can be proved. This completes the proof of Lemma 3.

Using Lemma 3 we now find bounds for the derivatives of w in G_q.

LEMMA 4. *The solution w of (4.2), (4.3) satisfies in G_q the bounds*

$$|D^k w(x)| \leqslant M_0 r^{m+2-k+\alpha}, \quad k \leqslant m+2. \tag{4.11}$$

Proof. In G_{2q} consider the regions

$$\Omega_p = \{x|\ 2^{-p-1}q \leqslant r \leqslant 2^{-p}q,\ |x_i| \leqslant 2^{-p}q,\ i > 2\},$$

$$\Omega'_p = \Omega_{p-1}\cup\Omega_p\cup\Omega_{p+1}.$$

The transformation

$$x_i = 2^{-p}y_i, \quad i = 1, 2, \ldots, n,$$

transforms Ω_p and Ω'_p onto Ω_0 and Ω'_0 respectively. In Ω'_0 the function $w^0(y) = w(2^{-p}y)$, $y = y_1, \ldots, y_n$ satisfies the elliptic equation

$$B_{ij}(y)w^0_{ij} + 2^{-p}B_i(y)w^0_i + 2^{-2p}B(y)w^0 = 2^{-2p}g^0(y) \tag{4.12}$$

as well as the boundary conditions

(4.13a) $$w^0 = 0 \quad \text{on } \Gamma_1,$$

(4.13b) $$\mathscr{B}w^0 = 0 \quad \text{on } \Gamma_2.$$

In Ω_0 and Ω'_0 we apply the Schauder inequality to obtain

(4.14) $$\|w^0\|_{m+2+\alpha}^{\Omega_0} \leqslant b(\|w^0\|_0^{\Omega'_0} + 2^{-2p}\|g^0\|_{m+\alpha}^{\Omega'_0}).$$

As it was done in the proof of Theorem 2′ we can show that

$$\|w^0\|_0^{\Omega'_0} \leqslant C_1 \cdot 2^{-p(m+2+\alpha)}$$

and

$$\|g^0\|_{m+\alpha}^{\Omega'_0} \leqslant C_2 \cdot 2^{-p(m+\alpha)}.$$

Thus

(4.15) $$\|w^0\|_{m+2+\alpha}^{\Omega_0} \leqslant b_1 \cdot 2^{-p(m+2+\alpha)}.$$

Now if D_0^k is the partial derivative corresponding to D^k, then

(4.16) $$D_0^k w_0(y) = 2^{-pk} D^k w(x),$$

and from (4.15), (4.16) and

$$|D_0^k w_0(y)| \leqslant \|w_0\|_{m+2+\alpha}^{\Omega_0}, \quad k = 0, 1, \ldots, m+2,$$

we obtain

$$|D^k w(x)| \leqslant b_2 \cdot 2^{-p(m+2-k+\alpha)} \leqslant M r^{m+2-k+\alpha}.$$

This completes the proof of Lemma 4.

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