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BICOLLARED n-MANIFOLD IN EUCLIDEAN SPACE

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Let M be a closed n-manifold and let Y be the Euclidean $(n+1)$-sphere. We assume that M is embedded in Y as a bicollared set [2]. Let R be a residual set for M as defined in [4].

THEOREM 1. *In Y there is a cellular set C [2] such that $C \cap M = R$.*

Proof. The construction of C is a modification of the method of constructing R in [4]. First, the bicollaredness of M is invoked to construct a nice bicollar on $M-R$ whose thickness goes uniformly to zero as it approaches R. The topological character of this bicollar is that of Euclidean $(n+1)$-space. Now, the construction in [4] can be followed with the exception that the construction of the open cell dense in Y is achieved by repeating the argument in both domains complementary to M *plus the collar on $M-R$*. Construction of the collar on either side of $M-R$ will allow a monotone union of open $(n+1)$-cells that contains $M-R$ at each stage. If U is the resulting dense cell, define C as $Y-U$. Then, certainly, C meets the requirements of the theorem.

COROLLARY 1. *If V is a neighborhood of R in M, there exists a closed $(n+1)$-cell Z in Y such that R lies in the interior of Z, $Z \cap M \subset V$, and the boundary of Z meets V in an $(n-1)$-sphere that is contained in V as a bicollared set.*

Proof. All of these properties follow from the facts that Y modulo C is a sphere again, that C does not meet $M-R$ and that, for $n \neq 2$, there are no pathological codimension one spheres in Y [3]. The case $n = 2$ can be got directly from the construction itself.

COROLLARY 2. *Each neighborhood of R in M nicely spans the boundary of an $(n+1)$-cell of Y.*

COROLLARY 3. *If M is not a sphere and $n \neq 3, 4$, then the set C must enter both domains complementary to M in Y.*

Proof. If C fails to meet one of the domains complementary to M in Y, then this domain is an open cell. The collaredness of M then implies that M is a homotopy n-sphere. Hence, by [6], M is a sphere unless n is 3 or 4.

THEOREM 2. *A necessary and sufficient condition that a closed n-manifold M embed in a bicollared manner in the $(n+1)$-sphere Y is that there exist a neighborhood V of R in M that embeds in Y in a bicollared manner and that there be a cellular set C in Y such that C meets V in R only.*

Proof. Certainly, if M embeds, then from the construction above-mentioned the neighborhood V exists.

On the other hand, if V exists, one can then shrink C to a point thus getting a disk in the quotient space corresponding to V. (This may require a special reselection of V.) The only case where the theorem does not now directly follow is for $n = 2$ [5]. The result follows for $n = 2$ from a capping theorem of Bing [1].

COROLLARY 4. *Let V be a Moebius band in the 3-sphere Y and C a compact set in Y that meets V in precisely its central curve. Then C is not cellular.*

Proof. By the above-mentioned argument, if C were cellular, then the projective plane would embed in the 3-sphere.

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