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CONCERNING A PROBLEM OF ARENS ON REMOVABLE IDEALS IN BANACH ALGEBRAS

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Let A and B be commutative complex Banach algebras with unit element. We say that B is an *extension* or a *superalgebra* of A if there exists a topological isomorphism φ of A into B sending the unit of A onto the unit of B. We write in this case $A \subset B$ and call the map φ an *imbedding* of A into B. Two extensions consisting of the same algebra B and two different imbeddings are considered as different extensions. An ideal $I \subset A$ is called *removable* if there is an extension $B \supset A$ such that I, considered as a subset of B, is contained in no proper ideal of B. We say in this case that *the extension B removes the ideal I*. By the Kuratowski-Zorn lemma, it is easy to see that every non-removable ideal I of A is contained in a maximal non-removable ideal, i.e., in a non-removable ideal $J \subset A$ such that, for any ideal $J_1 \supset J$, either $J = J_1$ or J_1 is a removable ideal. We do not know whether every maximal non-removable ideal $I \subset A$ is a maximal ideal of A. (**P 895**)

A family $\{I_a\}$ of removable ideals of an algebra A is called *removable* if there exists an extension $B \supset A$ which removes all ideals in this family. The concept of a removable ideal and of a removable family of ideals is due to Arens [1] who posed the following problem:

(**RId**) Is it true that every family of removable ideals of a Banach algebra A is a removable family?

As we noticed in [6], this problem has a negative answer. It is an immediate consequence of the following result by Bollobás [3]:

There exists a commutative complex Banach algebra with unit element and with the property that there is a non-countable subset $S \subset A$ consisting entirely of elements which are not topological divisors of zero (and so, for each $s \in S$, there is an extension $B \supset A$ in which the element s has an inverse) such that, for every extension $B \supset A$, not all elements of S are invertible in B.

Denoting by (s) the principal ideal of A generated by s, we see that the family (s), $s \in S$, is a non-removable family of removable ideals.

In the same paper it is shown that, for any countable subset $S \subset A$ consisting of elements which are not topological divisors of zero, there is an extension $B \supset A$ in which all elements of S are invertible. Thus, a countable version of (**RId**) has no counter-example.

In this paper we discuss the finite version of (**RId**):

(**RId**$_f$) Is it true that any finite family of removable ideals is a removable family?

This problem, also posed by Arens in [1], is still open. We give here an alternative formulation of this problem in hope that it will serve as a step towards its solution. Our result is as follows:

THEOREM 1. *Let A be a commutative complex Banach algebra with unit element. Then the following conditions are equivalent.*

(1a) *Every finite family of removable ideals of A is a removable family.*

(1b) *Every family consisting of two removable ideals is a removable family.*

(1c) *Every maximal non-removable ideal is a prime ideal.*

We obtain this theorem as a corollary to a more general result which is of a purely algebraic character. Let $\mathfrak{b}$ be a class of commutative rings with unit elements together with a class of isomorphic mappings called *admissible imbeddings* between these rings. We say, for $A, B \in \mathfrak{b}$, that B is an *extension* of A if there is an admissible imbedding of A into B sending the unit of A onto the unit of B. The model we keep in mind is $\mathfrak{b}$ equal to the class of all complex Banach algebras, and admissible imbeddings being topological isomorphisms. Definitions of removable and non-removable ideals and of removable families of ideals in rings of the class $\mathfrak{b}$ are analogous to those in the case of Banach algebras and need no repetition.

The result we are going to prove is

THEOREM 2. *Let $\mathfrak{b}$ be a class of commutative rings with unit elements together with a class of admissible imbeddings. The following conditions imposed on a ring $A \in \mathfrak{b}$ are equivalent:*

(2a) *Every finite family of removable ideals is a removable family.*

(2b) *Every family consisting of two removable ideals of A is a removable family.*

(2c) *Every maximal non-removable ideal of A is a prime ideal.*

The following concepts will be useful in the proof. Let A be a commutative ring with unit element and let $\mathfrak{P}$ be the collection of all its prime ideals. For a subset $X \subset \mathfrak{P}$, we write

$$kX = \{\bigcap \mathfrak{p} \colon \mathfrak{p} \in X\}, \tag{1}$$

and, for a proper ideal $I \subset A$,

$$hI = \{\mathfrak{p} \in \mathfrak{P}: I \subset \mathfrak{p}\}. \tag{2}$$

Clearly,

$$I \subset khI \quad \text{for every ideal } I \subset A. \tag{3}$$

For $X \subset \mathfrak{P}$, its closure is defined by the formula $\bar{X} = hkX$, and it defines in $\mathfrak{P}$ a topology, called Zariski topology (cf. [2], Chapter I, Exercise 15), turning $\mathfrak{P}$ into a compact, in general, non-Hausdorff space. For our purposes it is essential that, for any ideal $I \subset A$, the set hI is a closed subset of $\mathfrak{P}$ and so, for any two ideals $I_1, I_2 \subset A$, the set $hI_1 \cup hI_2$ is also closed in $\mathfrak{P}$. It means that if $\mathfrak{p}_0 \in \mathfrak{P}$ and $\mathfrak{p}_0 \supset \{\bigcap \mathfrak{p}: \mathfrak{p} \in hI_1 \cup hI_2\}$, then either $\mathfrak{p}_0 \in hI_1$ or $\mathfrak{p}_0 \in hI_2$. We also make use of the formula

$$k(hI_1 \cup hI_2) = khI_1 \cap khI_2. \tag{4}$$

If A is a Banach algebra and we restrict the Zariski topology to the maximal ideal space, we obtain the usual hull-kernel topology (cf. [4]).

LEMMA 1. *Let I be an ideal in $A \in \mathfrak{b}$. Then I is a removable ideal if and only if khI is such one and an extension B removes the ideal I if and only if it removes the ideal khI.*

Proof. If B is an extension of A which removes I, then B removes also all ideals containing I, in particular, the ideal khI. If an extension $B \supset A$ removes khI, then it removes all prime ideals in hI. Thus I is contained in no prime ideal of B, since an intersection of such an ideal with A would be a prime ideal in hI which is not removed by the extension B. Thus I is contained in no maximal ideal of B, and so the extension B removes I.

LEMMA 2. *Let I_1 and I_2 be removable ideals in $A \in \mathfrak{b}$, and $I_i = khI_i$, $i = 1, 2$. Then $\{I_1, I_2\}$ is a removable family if and only if the ideal $I = I_1 \cap I_2$ is removable.*

Proof. If an extension $B \supset A$ removes I, then it, clearly, removes both I_1 and I_2. If an extension $B \supset A$ removes both I_1 and I_2, then it removes also all prime ideals in $hI_1 \cup hI_2$. But $hI_1 \cup hI_2$ is closed in $\mathfrak{P}$ in the Zariski topology, which means that if $\mathfrak{p} \supset k(hI_1 \cup hI_2)$, $\mathfrak{p} \in \mathfrak{P}$, then either $\mathfrak{p} \in hI_1$ or $\mathfrak{p} \in hI_2$. As in the proof of Lemma 1, it implies that $k(hI_1 \cup hI_2)$ is removed by the extension B. Applying formula (4), we see that

$$k(hI_1 \cup hI_2) = I_1 \cap I_2 = I,$$

and so the ideal I is also removed by the extension B.

Proof of Theorem 2.

(2a) $\Rightarrow$ (2b). Obvious.

(2b) $\Rightarrow$ (2a). Let $\{I_1, \ldots, I_n\}$ be an n-tuple of removable ideals. By Lemma 1, the ideals khI_i, $i = 1, 2, \ldots, n$, are also removable. We put $J_1 = khI_1$ and $J_k = J_{k-1} \cap khI_k$ for $k = 2, \ldots, n$. It can be easily seen that $J_k = khJ_k$, $k = 1, 2, \ldots, n$, so, by assumption (2b), by Lemma 2 and by an easy induction, we see that all ideals J_k, $k = 1, \ldots, n$, are removable. Thus there is an extension $B \supset A$ which removes the ideal J_n. Since $J_n \subset khI_i$, $i = 1, 2, \ldots, n$, the extension B removes also all ideals khI_i and so, by Lemma 1, all ideals I_i, $i = 1, \ldots, n$.

(2c) $\Rightarrow$ (2b). Let I_1 and I_2 be two removable ideals of A. Thus every prime ideal in $hI_1 \cup hI_2$ is also removable. Since $hI_1 \cup hI_2$ is a closed subset of $\mathfrak{P}$, every prime ideal containing $k(hI_1 \cup hI_2)$ is in $hI_1 \cup hI_2$, and so it is removable. This means that $k(hI_1 \cup hI_2)$ is a removable ideal itself, otherwise, by (2c), it would be contained in a non-removable prime ideal. Applying formula (4), we see that $khI_1 \cap khI_2$ is a removable ideal and there is an extension $B \supset A$ which removes both khI_1 and khI_2. By Lemma 1, the same extension removes also ideals I_1 and I_2, and so (2b) follows.

(2b) $\Rightarrow$ (2c). Suppose that I is a maximal non-removable ideal of A. If it is not a prime ideal, then there are elements $x, y \in A \setminus I$ with $xy \in I$. Put $I_1 = (x) + I$ and $I_2 = (y) + I$, where (x) and (y) denote the principal ideals defined by x and y, respectively. By the maximality of I, both I_1 and I_2 are removable ideals of A. Applying Lemma 1, we see that khI_1 and khI_2 are also removable ideals of A. Applying condition (2b) and Lemma 2, we see that the ideal $khI_1 \cap khI_2 = k(hI_1 \cup hI_2)$ is removable. But $hI_1 \cup hI_2 = hI$, since every prime ideal containing I must contain either x or y. This implies that khI is a removable ideal of A. But it is impossible, since, by the maximality of I and by Lemma 1, $I = khI$. The contradiction proves (2c).

Theorem 1 is an immediate corollary to Theorem 2.

Remarks. In the case where $\mathfrak{b}$ is the class of all commutative rings and admissible imbeddings are all isomorphisms into, it can be shown that non-removable ideals consist of joint divisors of zero (cf. [6], Proposition 4). In this case the maximal non-removable ideals are prime ideals (the proof of Proposition 2 given in paper [6] works also in this case). It follows that in this case a problem analogous to (**RId**$_f$) has a positive solution.

Theorem 1 raises hopes of an affirmative solution of problem (**RId**$_f$) of Arens. It seems that a maximal non-removable ideal of a Banach algebra is not only a prime ideal, but also that it is a maximal ideal of A. Such a result would follow immediately from a positive solution of a conjecture stating that an ideal I of a Banach algebra A is non-removable if and only if it consists of joint topological divisors of zero (cf. [6]), since, as proved by Słodkowski in [5], every maximal ideal consisting of joint topological divisors of zero is a maximal ideal of A. We hope that also the countable version of (**RId**) has a positive solution, since, by the above-mentioned result of Bollobás, every countable family of removable principal ideals is a removable family.

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