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ALGORITHM 89

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A PROCEDURE REALIZING A FOURTH ORDER ONE-STEP METHOD FOR SOLVING A SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS OF THE FORM $y'' = f(x, y, y')$

1. Procedure declaration. The procedure *sodebis4* solves the initial value problem of the form

(1) $$y_k'' = f_k\big(x, y_1(x), y_2(x), \ldots, y_n(x), y_1'(x), y_2'(x), \ldots, y_n'(x)\big),$$

(2) $$y_k(x_0) = y_{0k},$$

(3) $$y_k'(x_0) = y_{0k} \quad (k = 1, 2, \ldots, n)$$

at the points $x_1, x_2, \ldots$

Data:

- *x* — the value of x_0 in (2) and (3);
- *x1* — the value of the argument for which we solve the problem;
- *eps* — the relative error, the given tolerance;
- *eta* — the number which is used instead of zero if the obtained solution is zero or near to zero; this number is used for evaluation of the relative error;
- *hmin* — the least admissible absolute value of the step length;
- *n* — the number of differential equations in (1)-(3);
- *y* [*1* : *n*] — the values of the right-hand sides of (2);
- *yp* [*1* : *n*] — the values of the right-hand sides of (3).

Results:

- *x* — the value of *x1*;
- *y* [*1* : *n*] — the values of the approximate solution $y_k(x1)$ $(k = 1, 2, \ldots, n)$;
- *yp* [*1* : *n*] — the values of the approximate solution $y_k'(x1)$ $(k = 1, 2, \ldots, n)$.

Additional parameters:

- *steph* — the label outside of the body of the procedure *sodebis4* to which a jump is made if the absolute value of the step length is smaller than *hmin*; after the jump, *x* is equal to the value of $\tilde{x}$ $(\tilde{x} < x1)$

for which the approximate solution has a relative error equal to the given one, and $y[1:n]$, $yp[1:n]$ contain the values of this approximate solution;

f — the identifier of the procedure which computes the values of the right-hand sides of (1) and puts them in $d[1:n]$, and which has the following heading:
procedure $f(x, n, y, yp, d)$; **value** x, n; **real** x; **integer** n; **array** y, yp, d;

2. Method used. We use a method of fourth order of the following form:

$$\eta p^2_{n+1/m} = \eta p^5_n + \frac{h}{m} f^{5,i}_n, \qquad \eta^3_{n+1/m} = \eta^5_n + \frac{h}{m}\eta p^5_n + \frac{h^2}{2m^2} f^{5,i}_n,$$

$$\eta p^3_{n+1/3} = \eta p^5_n + \frac{h}{3}\left(\frac{6-m}{6} f^{5,i}_n + \frac{m}{6} f^{3,2}_{n+1/m}\right),$$

$$\eta^4_{n+1/3} = \eta^5_n + \frac{h}{3}\eta p^5_n + \frac{h^2}{9}\left(\frac{9-m}{18} f^{5,i}_n + \frac{m}{18} f^{3,2}_{n+1/m}\right),$$

$$\eta p^4_{n+1/2} = \eta p^5_n + \frac{h}{8}\left(f^{5,i}_n + 3f^{4,3}_{n+1/3}\right),$$

$$\eta^4_{n+1/2} = \eta^5_n + \frac{h}{2}\eta p^5_n + \frac{h^2}{16}\left(f^{5,i}_n + f^{4,3}_{n+1/3}\right),$$

$$\eta p^4_{n+1} = \eta p^5_n + \frac{h}{2}\left(f^{5,i}_n - 3f^{4,3}_{n+1/3} + 4f^{4,4}_{n+1/2}\right), \qquad \eta^4_{n+1} = \eta^5_n + h\eta p^5_n + \frac{h^2}{2} f^{4,3}_{n+1/3},$$

$$\eta^5_{n+1} = \eta^5_n + h\eta p^5_n + \frac{h^2}{6}\left(f^{5,i}_n + 2f^{4,4}_{n+1/2}\right),$$

$$\eta p^5_{n+1} = \eta p^5_n + \frac{h}{6}\left(f^{5,i}_n + 4f^{4,4}_{n+1/2} + f^{5,4}_{n+1}\right),$$

where η^k_{n+a} is the approximate value of the solution of (1)-(3) at the point $x_n + ah$ with local error $O(h^k)$, ηp^k_{n+a} is the approximate value of the first derivative of the solution of (1)-(3) at the point $x_n + ah$ with local error $O(h^k)$ and $f^{i,j}_{n+a} = f(x_n + ah, \eta^i_{n+a}, \eta p^j_{n+a})$. The parameter m $(m > 0)$ was assumed to be equal to 4 and $i = 5$.

This method was given by Bobkov [2], p. 38-42. Twofold application of the method with step $h/2$ allows us to obtain the values with step h without evaluation of the function f.

In the algorithm, 9 evaluations of f in one step are made. The method of control of the step of integration described in [1] is used.

```
procedure sodebis4(x,x1,eps,eta,hmin,n,y,yp,steph,f);
 value x1,eps,eta,hmin,n;
 real x,x1,eps,eta,hmin;
 integer n;
 array y,yp;
 label steph;
 procedure f;
 begin
  real a1,b,b1,h,hh,hk,h1,h2,w,w1,w2,w3,w4;
  integer i;
  Boolean last;
  array d,d1,d2,d3,yp2,yp3,y2,y3[1:n];
  procedure stepbis4(x,d,y,ya,yp,yb);
   value x;
   real x;
   array d,y,ya,yp,yb;
   begin
    w:=hh×.25;
    w1:=hk×.03125;
    for i:=1 step 1 until n do
     begin
      w2:=d[i];
      w3:=yp[i];
      yb[i]:=w3+w×w2;
      ya[i]:=y[i]+w×w3+w1×w2
     end i;
    h1:=hh×.333333333333;
    h2:=hk×.006172777777;
    b:=2.0;
    a1:=5.0;
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b1:=4.0;
for w1:=hh×.11111111111,hh×.125 do
 begin
  f(x+w,n,ya,yb,d2);
  for i:=1 step 1 until n do
   begin
    w2:=d2[i];
    w3:=yp[i];
    w4:=d[i];
    yb[i]:=w3+w1×(w4+w2×b);
    ya[i]:=y[i]+h1×w3+h2×(a1×w4+b1×w2)
   end i;
  w:=h1;
  h1:=hh×.5;
  h2:=hk×.0625;
  b:=3.0;
  a1:=b1:=1.0
 end w1;
f(x+h1,n,ya,yb,d3);
h2:=hk×.166666666666;
for i:=1 step 1 until n do
 begin
  w2:=d[i];
  w3:=d3[i];
  w4:=yp[i];
  yb[i]:=w4+h1×(w2-3.0×d2[i]+4.0×w3);
  ya[i]:=y[i]+hh×w4+h2×(w2+w3+w3)
 end i;
f(x+hh,n,ya,yb,d2);
h1:=hh×.166666666666;
```

```
    for i:=1 step 1 until n do
     yb[i]:=yp[i]+h1×(d[i]+4.0×d3[i]+d2[i])
   end stepbis4;
  eps:=.033333333333/eps;
  h:=x1-x;
  last:=true;
  f(x,n,y,yp,d);
conth:
  hh:=h×.5;
  hk:=hh×hh;
  stepbis4(x,d,y,y2,yp,yp2);
  for i:=1 step 1 until n do
   d1[i]:=d2[i];
  stepbis4(x+hh,d1,y2,y3,yp2,yp3);
  w:=.0;
  h1:=h×.166666666666;
  h2:=h×h×.166666666666;
  for i:=1 step 1 until n do
   begin
    w2:=d[i];
    w1:=d1[i];
    w3:=yp[i];
    a1:=y3[i];
    w4:=yp3[i];
    b:=a1-y[i]-h×w3-h2×(w2+w1+w1);
    b1:=w4-w3-h1×(w2+4.0×w1+d2[i]);
    w1:=y3[i]:=a1+.066666666666×b;
    w2:=yp3[i]:=w4+.066666666666×b1;
    w1:=abs(w1);
    w2:=abs(w2);
```

```
    b1:=abs(b1);
    b:=abs(b);
    if w1<eta
     then w1:=eta;
    if w2<eta
     then w2:=eta;
    w1:=b/w1;
    w2:=b1/w2;
    if w1>w
     then w:=w1;
    if w2>w
     then w:=w2
   end i;
  w:=if w=.0 then eta else 1.25×(w×eps)↑.2;
  hh:=h/w;
  if w>1.25
   then
   begin
    if abs(hh)<hmin
     then go to steph;
    last:=false
   end w>1.25
   else
   begin
    x:=x+h;
    for i:=1 step 1 until n do
     begin
      y[i]:=y3[i];
      yp[i]:=yp3[i]
     end i;
```

```
          if last
           then go to endp;
          f(x,n,y,yp,d);
          w:=x1-x;
          if (w-hh)×h<0
           then
           begin
            hh:=w;
            last:=true
           end(w-hh)×h<0
         end w<1.25;
        h:=hh;
        go to conth;
      endp:
       end sodebis4
```

3. Certification. The procedure *sodebis4* has been verified on the Odra 1204 computer for many examples of the initial value problem. Some of them are presented here.

Examples.

(A)
$$\begin{cases} y_1'' = -y_2'/y_2^2, & y_1(0) = 1, \quad y_1'(0) = 1, \\ y_2'' = y_1'/y_1^2, & y_2(0) = 1, \quad y_2'(0) = -1 \end{cases}$$

with the exact solution $y_1 = e^x$, $y_2 = e^{-x}$.

(B)
$$\begin{cases} y_1'' = y_1, & y_1(0) = 1, \quad y_1'(0) = 1, \\ y_2'' = -y_2, & y_2(0) = 0, \quad y_2'(0) = -1 \end{cases}$$

with the exact solution $y_1 = e^x$, $y_2 = \sin x$.

(C)
$$\begin{cases} y_1'' = y_1/4, & y_1(0) = 1, \quad y_1(0) = -1/2, \\ y_2'' = (1+x^2)y_2, & y_2(0) = 1, \quad y_2(0) = 0 \end{cases}$$

with the exact solution $y_1 = e^{-x/2}$, $y_2 = e^{x^2/2}$.

The results, obtained for *eps* = *eta* and *hmin* = $_{10}-15$, are given below. As initial values, the exact solutions at the points .5, 1.0, 1.5 were used. The relative errors $(\eta - y)/y$ and the numbers of evaluations of the function f (denoted by $[f]$) are also given.

Results obtained for problem (A)

x	$eps = {}_{10}-3$	$[f]$	$eps = {}_{10}-6$	$[f]$	$eps = {}_{10}-9$	$[f]$
.5	$-1.0_{10}-6$	9	$-2.7_{10}-7$	26	$-1.1_{10}-9$	62
	$-1.2_{10}-6$		$-4.3_{10}-7$		$-2.8_{10}-9$	
1.0	$-1.0_{10}-6$	9	$-2.7_{10}-7$	26	$-1.1_{10}-9$	62
	$-1.2_{10}-6$		$-4.2_{10}-7$		$-2.9_{10}-9$	
1.5	$-1.0_{10}-6$	9	$-2.7_{10}-7$	26	$-1.1_{10}-9$	62
	$-1.2_{10}-6$		$-4.2_{10}-7$		$-2.7_{10}-9$	
10.0	(1)		$1.6_{10}-2$	224	$8.2_{10}-5$	827
			$-3.2_{10}-2$		$-1.6_{10}-4$	

Results obtained for problem (B)

x	$eps = {}_{10}-3$	$[f]$	$eps = {}_{10}-6$	$[f]$	$eps = {}_{10}-9$	$[f]$
.5	$1.5_{10}-7$	9	$2.6_{10}-8$	26	$1.4_{10}-10$	62
	$9.7_{10}-8$		$1.1_{10}-8$		$-4.5_{10}-11$	
1.0	$1.5_{10}-7$	9	$5.2_{10}-8$	26	$2.5_{10}-10$	62
	$-1.2_{10}-7$		$-4.2_{10}-8$		$-2.2_{10}-10$	
1.5	$1.5_{10}-7$	9	$1.9_{10}-8$	26	$1.8_{10}-10$	62
	$-2.2_{10}-7$		$-2.7_{10}-8$		$-2.9_{10}-10$	
10.0	$1.4_{10}-4$	70	$1.8_{10}-6$	291	$7.5_{10}-9$	1095
	$1.2_{10}-3$		$6.7_{10}-6$		$1.7_{10}-8$	

Results obtained for problem (C)

x	$eps = {}_{10}-3$	$[f]$	$eps = {}_{10}-6$	$[f]$	$eps = {}_{10}-9$	$[f]$
.5	$6.5_{10}-9$	9	$3.1_{10}-10$	35	$-3.7_{10}-11$	115
	$3.3_{10}-7$		$4.2_{10}-8$		$1.5_{10}-10$	
1.0	$6.5_{10}-9$	9	$2.8_{10}-10$	35	$-2.3_{10}-11$	98
	$-1.0_{10}-6$		$4.6_{10}-8$		$2.2_{10}-10$	
1.5	$6.5_{10}-9$	9	$1.3_{10}-10$	35	$6.1_{10}-11$	1107
	$-3.0_{10}-6$		$7.8_{10}-8$		$4.3_{10}-10$	
10.0	$1.2_{10}-3$	267	$1.5_{10}-6$	1195	$2.0_{10}-7$	4912
	$-1.9_{10}-4$		$1.5_{10}-5$		$6.8_{10}-8$	

(1) The determined step length was smaller than *hmin*.

References

[1] J. S. Chomicz, A. Olejniczak and M. Szyszkowicz, *A method for finding the step size of integration of a system of ordinary differential equations*, Zastos. Mat. 17 (1983), p. 645-654.

[2] V. I. Krylov, V. V. Bobkov and P. I. Monastyrnyĭ (В. И. Крылов, В. В. Бобков и П. И. Монастырный), *Вычислительные методы*, т. 2, Москва 1977.

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