A note on holomorphic mappings with two fixed points

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Abstract. Let $X$ be a hyperbolic Riemann surface and let $a, b \in X, a \neq b$. We will prove that the set of all holomorphic mappings $f: X \rightarrow X$ with $f(a) = a, f(b) = b$ is a finite cyclic subgroup of the group of all holomorphic automorphisms of $X$.

Throughout the paper the following notation will be used:

$D = \{z \in \mathbb{C}: |z| < 1\}, \quad D_* = D \setminus \{0\},$

$P(r, R) = \{z \in \mathbb{C}: r < |z| < R\}, \quad 0 \leq r < R \leq +\infty;$

$X$ — a Riemann surface;

$\mathcal{G}(X; a, b)$ — the set of all holomorphic mappings $f: X \rightarrow X$ such that $f(a) = a, f(b) = b$ ($a, b \in X, a \neq b$);

$\text{Aut}(X)$ — the group of all holomorphic automorphisms of $X$;

$f^{(n)}$ — the $n$-th iterate of a mapping $f: X \rightarrow X$, i.e. $f^{(0)} = \text{id}_X$,

$f^{(n)} = f^{(n-1)} \circ f,$ $n \in \mathbb{N}.$

The main result is the following theorem.

Theorem 1. Let $X$ be a hyperbolic Riemann surface and let $a, b \in X, a \neq b$. Then $\mathcal{G}(X; a, b)$ is a finite cyclic subgroup of $\text{Aut}(X)$.

The proof will be based on the following elementary lemma.

Lemma 1. Let $\emptyset \neq A \subset D_*$ be a set with no accumulation points in $D$. Put $\mathcal{G}(A) := \{g \in \mathcal{G}(D, D): g(0) = 0, g(A) \subset A\}$. Then $\mathcal{G}(A)$ is a finite cyclic subgroup of the group $\text{Aut}_0(D)$ of all rotations of $D$.

Proof. Observe that

$\text{id}_D \in \mathcal{G}(A),$ \quad $g_2 \circ g_1 \in \mathcal{G}(A),$ \quad $g_1, g_2 \in \mathcal{G}(A).$

Put $\rho := \min \{\|w\|: w \in A\}, \quad B := \{w \in A: |w| = \rho\}.$ It is clear that $B$ is non-empty and finite. By the Schwarz lemma,

$|g(w)| \leq |w|, \quad w \in D, \quad g \in \mathcal{G}(A),$
which implies that
\[ (2) \quad \mathcal{G}(A) \subset \mathcal{G}(B). \]

Hence, in view of the definition of \( B \) (using again the Schwarz lemma), we conclude that \( \mathcal{G}(B) \subset \text{Aut}_0(D) \) and that \( \mathcal{G}(B) \) is finite. In consequence, in view of (1) and (2), for every \( g \in \mathcal{G}(A) \) there exists \( n = n(g) \in \mathbb{N} \) such that \( g^{-1} = g^{(n-1)}. \) This shows that \( \mathcal{G}(A) \) is a finite subgroup of \( \text{Aut}_0(D). \) In order to prove that \( \mathcal{G}(A) \) is also cyclic it is enough to observe that \( \mathcal{G}(A) \) is a subgroup of \( \{ \alpha \cdot \text{id}_D : \alpha \in \sqrt{1} \} \), where \( n = \prod_{g \in \mathcal{G}(A)} n(g). \) This completes the proof of the lemma.

\textbf{Proof of Theorem 1.} Let \( p : D \to X \) be a universal covering of \( X \) such that \( p(0) = a \) (cf. [2], Chapter 3, §27). Put \( A := p^{-1}(b). \) Note that \( A \) satisfies all the assumptions of Lemma 1. For \( f \in \mathcal{G}(X; a, b) \) let \( \tilde{f} : D \to D \) denote the lifting of \( f \) such that \( \tilde{f}(0) = 0. \) It is clear that \( \tilde{f} \in \mathcal{G}(A). \) In particular, in view of Lemma 1, for every \( f \in \mathcal{G}(X; a, b) \) there exists \( n = n(f) \in \mathbb{N} \) such that \( (f^{(n)})^{-1} = f^{(n)} = \text{id}_D. \) Consequently, \( f^{(n)} = \text{id}_X. \) This proves that \( \mathcal{G}(X; a, b) \) is a subgroup of \( \text{Aut}(X) \) (note that if \( X \) is a bounded domain in \( C \), then the inclusion \( \mathcal{G}(X; a, b) \subset \text{Aut}(X) \) follows, for instance, from Satz 29, §6 in [1]). The mapping
\[ \mathcal{G}(X; a, b) \ni f \mapsto \tilde{f} \in \mathcal{G}(A) \]
may be now regarded as a group monomorphism and therefore (again by Lemma 1) we conclude that \( \mathcal{G}(X; a, b) \) is a finite cyclic subgroup of \( \text{Aut}(X) \). The proof is finished.

\textbf{Theorem 1} will be illustrated by examples.

\textbf{Example 1.} If \( X \) is a simply connected hyperbolic Riemann surface, then \( \mathcal{G}(X; a, b) = \{ \text{id}_X \}, \ a, \ b \in X, \ a \neq b. \)

\textbf{Proof.} In this case \( p \) is biholomorphic and consequently \( \# A = 1. \)

\textbf{Example 2.} Let \( 0 < r < R < +\infty, \ X = P = P(r, R), \ a, \ b \in P, \ a \neq b. \) Then
\[ \mathcal{G}(P; a, b) = \begin{cases} \{ \text{id}_P \}, & z \to a^{2/z} \quad \text{iff} \ |a| = \sqrt{rR} \text{ and } b = -a, \\ \{ \text{id}_P \}, & \text{otherwise}. \end{cases} \]

\textbf{Proof.} By standard arguments one can reduce the proof to the case where \( R > 1, \ r = 1/R \) and \( 1/R < a < R. \) Define
\[ p(z) = \exp(F_1^{-1} \circ F_2^{-1}(z)), \quad z \in D, \]
where
\[ F_1(z) := \tan(\mu z), \quad -\log R < \text{Re} \ z < \log R, \quad \mu := \pi/(4\log R), \]
\[ F_2(z) := (z - \delta)/(1 - \delta z), \quad z \in D, \ \delta := \tan(\mu \log a). \]
It is easily seen that \( p : D \to X \) is a universal covering of \( P \) and that \( p(0) = a. \)
Let \( A := p^{-1}(b) \) and let \( B \) be as in the proof of Lemma 1. One can prove that \( \# B \leq 2 \). Hence, either \( \mathcal{G}(A) = \{ \text{id}_D \} \) (and so \( \mathcal{G}(P; a, b) = \{ \text{id}_P \} \)) or \( \mathcal{G}(A) = \{ \text{id}_D, -\text{id}_D \} \). In the second case we get \( \delta = 0 \) and \( \mathcal{G}(P; a, b) = \{ \text{id}_P, z \to 1/z \} \); consequently \( a = 1, b = -1 \).

**Example 3.** For every \( n \in \mathbb{N} \) there exist a hyperbolic Riemann surface \( X \) and points \( a, b \in X, a \neq b \), such that \( \# \mathcal{G}(X; a, b) = n \).

**Proof.** In view of Examples 1, 2, we may assume that \( n \geq 3 \). Let \( X := \mathbb{C} \setminus \sqrt{1}, a = 0, b = \infty \). Then, in virtue of the Picard theorem, \( \mathcal{G}(X; 0, \infty) \subset \mathcal{G}(\mathbb{C}, 0, \infty) \) and therefore, in view of Theorem 1, \( \mathcal{G}(X; 0, \infty) = \{ \text{a.id}_X; x \in \sqrt{1} \} \).

**Remark 1.** (a) If \( P = P(0, R), 0 < R < + \infty \), or \( P = P(r, + \infty), 0 < r < + \infty \), then \( \mathcal{G}(P; a, b) = \{ \text{id}_P \}, a, b \in P, a \neq b \).

(b) If \( P = P(0, + \infty) \) (\( P \) is a parabolic space) then \( \{ z \to z^{2k+1}, k \in \mathbb{Z} \} \subset \mathcal{G}(P; 1, -1) \) and so neither \( \mathcal{G}(P; 1, -1) \) is finite nor \( \mathcal{G}(P; 1, -1) \subset \text{Aut}(P) \).

**Remark 2.** Example 2 permits us to give an alternative method of the proof of the following well-known theorem on biholomorphisms between annuli (cf. [3], Theorem 14.22).

Let \( 0 < r_j < R_j < + \infty \), \( P_j := P(r_j, R_j), j = 1, 2 \). Then \( P_1, P_2 \) are biholomorphically equivalent iff \( r_1/R_1 = r_2/R_2 \). Moreover, every biholomorphism \( F: P_1 \to P_2 \) is either of the form \( z \to \alpha r_2/z \) or \( z \to \alpha r_1 R_2/z, z \in P_1, |\alpha| = 1 \).

**Proof.** We may assume that

\[ r_1/R_1 \leq r_2/R_2. \]

Fix a biholomorphism \( F: P_1 \to P_2 \) and let \( a := \sqrt{r_1 R_1}, b := -a \). The mapping

\[ G(P_1; a, b) \ni f \to F \circ f \circ F^{-1} \in G(P_2; F(a), F(b)) \]

is an isomorphism. Hence, by Example 2,

\[ |F(a)| = \sqrt{r_2 R_2}, \quad F(b) = -F(a). \]

Put

\[ f(z) := aF(z)/F(a), \quad z \in P_1. \]

Observe that, in view of (3), (4),

\[ f(P_1) = P(\sqrt{r_1 r_2 R_1/R_2}, \sqrt{r_1 R_1 R_2/r_2}) \subset P_1, \quad f(a) = a, f(b) = b. \]

Thus \( f \in \mathcal{G}(P_1; a, b) \) and therefore, by Example 2, \( f \in \text{Aut}(P_1) \) (in particular, \( f(P_1) = P_1 \) and so \( r_1/R_1 = r_2/R_2 \)) and either \( f = \text{id}_{P_1} \) (\( F(z) = \alpha r_2/z \)) or \( f(z) = a^2/z \) (\( F(z) = \alpha r_1 R_2/z \)).
References


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