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On homogeneous rotation invariant distributions and the Laplace operator

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Abstract. This note deals with some special classes of distributions distinguished by their behaviour with respect to some linear transformations. For example, Theorem 4 gives a characterization of rotation invariant distributions $u \in D'(E^n \setminus \{0\})$, where E^n denotes the n-dimensional Euclidean space. This result is more complete and, moreover, is obtained in an essentially simpler way than the analogous characterizations (with proofs based on the concept of the Haar measure on locally compact groups) given by other authors. Also some applications are added. For example, Theorem 4 mentioned above allows one to find a fundamental solution of the Laplace operator Δ_n for $n \geqslant 2$ in a completely natural way.

We begin with some general remarks which serve as an introduction to this paper as well as to two other papers which follow it.

As it is well known, with the help of the Fourier transformation one can obtain fundamental solutions of the linear operators with constant coefficients. A partial differential equation is transformed into an easily solvable ordinary equation and it remains to "re-translate" its solution applying the inverse Fourier transform. The difficulties which arise consist in the computation of the Fourier transformations involved, which is sometimes very troublesome. Fortunately, we can avoid such computations taking into account some properties of the fundamental solution we are going to find. These are mostly the property of homogenity and invariance with respect to some linear transformations. For example, in the study of the Laplace operator, since it is rotation invariant, orthogonal transformations appear. Similarly, it is natural and convenient to exploit Lorentz transformations in the study of the wave operator.

Genarally, the problem of characterization of homogeneous distributions and distributions invariant with respect to some affine transformations proves to be an important one.

A characterization of homogeneous rotation invariant distributions is given in Section 23 of [1], which, although elementary in its form, was obtained in not an elementary way. The proof presented there refers to the theory of the Haar measure on locally compact groups. A similar approach to finding a fundamental solution of the Laplace operator Δ_n for $n > 3$, is also applied by Treves in his recent book [5].

In Theorem 4 of Section 3 of this paper we give a characterization of rotation invariant distributions. We prove it in a completely elementary way. Then, in Section 4, this theorem is applied to determining, up to a constant factor the general form of homogeneous rotation invariant distributions defined outside the origin. We end the paper with Section 5 containing an application of these results to the Laplace operator.

A first theorem on the characterization of distributions invariant with respect to the Lorentz transformations is due to Methée [2]. He also has applied this theorem to finding all Lorentz-invariant fundamental solutions of the wave operator. This method of determining a fundamental solution of the wave operator did not pierce into text-books owing to its difficulty and complexity. Treves, who presented the above mentioned application of the Methée theorem (omitting the proof of the theorem itself) in his book [4], did not place it in the text-book [5]. The difficulties of this problem are serious and it seems rather impossible to give a complete, elementary and short proof. But such a proof can be clear and natural, as it will be shown in the following papers.

In paper [8] a comprehensive and elementary proof of the Methée theorem is presented. Moreover, this paper gives another characterization of Lorentz invariant distributions, namely one stated in differential terms. Also, as far as only basic facts from the theory of distributions are being used in these proofs, they may by suitable(1) for introductory text-books on partial differential equations, which as a rule discuss the wave operator (which is of course Lorentz invariant).

In Section 1 of the present paper we collect the principal notation and definitions used troughout this paper and two others which follow.

We also give some auxiliary theorems which will be useful in the next sections.

In Section 2 we characterize homogeneous distributions in the one-dimensional Euclidean space, adding as an application and an immediate consequence a proof of a theorem due to Zieleźny.

The contents of the remaining Sections 3–5 has already been exposed.

1. Notation, definitions and auxiliairy theorems. The variable in the n-dimensional real Euclidean space E^n will be denoted by $x = (x_1, \ldots, x_n)$. We denote $|x| = (x_1^2 + \ldots + x_n^2)^{1/2}$. If $x, y \in E^n$, we write $x \cdot y = x_1 y_1 + \ldots$ $\ldots + x_n y_n$.

(1) I have included some of the results of this paper, which are basic for finding a fundamental solution of the wave operator, in the appendix to my book [3] as soon as they were obtained.

Half spaces are defined and denoted as follows:

$$E^1_+ = (0, +\infty), \quad E^{n+1}_+ = \{(t, x):\ t > 0, x \in E^n\} \quad (n \in N),$$
$$\overline{E^1_+} = \langle 0, +\infty), \quad \overline{E^{n+1}_+} = \{(t, x):\ t \geqslant 0, x \in E^n\} \quad (n \in N).$$

Half spaces E^1_-, E^{n+1}_-, $\overline{E^1_-}$, $\overline{E^{n+1}_-}$ are defined analogously. Here and in the sequel N denotes the set of positive integers, N_0 is the set of non-negative integers, and N_0^n denotes the product of n copies of the set N_0. For $a \in N_0^n$, $a = (a_1, \ldots, a_n)$, we set $|a| = a_1 + \ldots + a_n$.

We shall be concerned with complex valued functions defined in an open set $\Omega \subset E^n$. We denote by $C^k(\Omega)$, $k \in N_0 \cup \{+\infty\}$, the set of functions defined in Ω and continuous together with their derivatives up to order k including k if $k < +\infty$, or with the derivatives of any order if $k = +\infty$. The symbol $C_0^k(\Omega)$ denotes the set of functions of class $C^k(\Omega)$, whose supports are compact subsets of Ω ([2]).

As usual, $S(E^n)$ stands for the space of C^∞-functions rapidly decaying at infinity. $D'(\Omega)$ denotes the space of distributions in Ω, and $S'(E^n)$ the space of tempered distributions in E^n. By $L_1^{\text{loc}}(\Omega)$ we denote the class of locally integrable functions defined on Ω.

Definition 1. Let $x = f(y)$ be a one-to-one mapping of $\tilde{\Omega}$ into Ω ([3]) of class C^∞ with non-vanishing Jacobian $J = Dx/Dy$. A substitution $x = f(y)$ in a distribution $u \in D'(\Omega)$ is denoted by $u \circ f$ and defined by

$$(1) \qquad (u \circ f)[\varphi] = u\left[(\varphi \circ f^{-1}) \frac{1}{|J|}\right] \quad \text{for } \varphi \in C_0^\infty(\tilde{\Omega}).$$

It is easy to see that $v = u \circ f \in D'(\tilde{\Omega})$.

Remark 1. The substitution $v \circ f^{-1}$ is also well defined and it is equal to the restriction of u to $f(\tilde{\Omega})$.

Definition 2. A distribution u is said to be *invariant* with respect to the transformation f if $u \circ f = u$.

We shall consider some specific cases of a linear transformation f:

$$(2) \qquad f(y) = Ay \quad \text{for } y \in E^n \ (\operatorname{Det} A \neq 0).$$

(a) Inversion ($A = -I$).

(b) Rotation ($\operatorname{Det} A = 1$, $A^{-1} = A^t$, where A^t denotes the transposed matrix to A).

(c) Homothety ($A = rI$ with $r > 0$).

([2]) Every function $\varphi \in C_0^k(\Omega)$ can be identified with a function from $C^k(\tilde{\Omega})$ with compact support in Ω, $\Omega \subset \tilde{\Omega}$.

([3]) If $\varphi \in C_0^\infty(\tilde{\Omega})$, then the function $\varphi \circ f^{-1} \in C_0^\infty(f(\tilde{\Omega}))$ can be regarded as a function from $C_0^\infty(\Omega)$ (see footnoe ([2])). If $f(\Omega) \subset \Omega$, we say that Ω is *invariant* with respect to the transformation f.

Remark 2. Let Ω be an open set symmetric with respect to the origin. Let $u \in D'(\Omega)$ and set

$$\check{u}[\varphi] = u[\check{\varphi}] \quad \text{for } \varphi \in C_0^\infty(\Omega), \quad \check{\varphi}(x) = \varphi(-x) \quad \text{for } x \in \Omega.$$

Distributions invariant with respect to the inversion are precisely the even distributions, i.e., such that $\check{u} = u$.

DEFINITION 3. Let Ω be a region invariant with respect to homotheties f_r, $f_r(y) = ry$ for $y \in \Omega$ $(r > 0)$. A distribution $u \in D'(\Omega)$ is called *homogeneous of order m* iff for every homothety f_r we have

$$u \circ f_r = r^m u \quad (r > 0). \tag{3}$$

We begin with an easy to prove

PROPOSITION 1. *Let $f_t(x) = tx$ for $x \in E^n$ $(t \in E^1)$ and let k be an integer. If k is even, then the distribution $u \in D'(E^n \setminus \{0\})$ is an even distribution homogeneous of order k if and only if it satisfies the equation*

$$u \circ f_t = t^k u \quad \text{for } t \in E^1 \setminus \{0\}. \tag{4}$$

If k is odd, the distribution $u \in D'(E^n \setminus \{0\})$ is even and homogeneous of order k if and only if it satisfies the equation

$$u \circ f_t = \begin{cases} t^k u & \text{for } t > 0, \\ -t^k u & \text{for } t < 0. \end{cases}$$

THEOREM 1 ([4]). *Let $\Omega \subset E^n$ be a region invariant with respect to homotheties. A distribution $u \in D'(\Omega)$ is homogeneous of order m iff it satisfies the Euler equation*

$$mu = \sum_{j=1}^{n} x_j \frac{\partial u}{\partial x_j}. \tag{5}$$

PROPOSITION 2. *Every two distributions on E^n homogeneous of order $p > -n$ and coinciding outside the origin are equal.*

Proof. Any distribution w with support at the origin is equal to a finite sum $\sum_{|\alpha| \leqslant k} a_\alpha D^\alpha \delta$, where δ denotes the Dirac distribution at the origin. The distributions $D^\alpha \delta$ are homogeneous of order $-n-|\alpha| \leqslant -n$, hence the assertion follows.

Remark 3. Let $u, v \in D'(E^n)$ be two distributions homogeneous of order p, coinciding outside the origin. Suppose $p = -n-k$ with $k \in N_0$. Then there exist constants a_α such that $u - v = \sum_{|\alpha|=k} a_\alpha D^\alpha \delta$.

([4]) See [1], p. 111.

To conclude this section let us list some properties of the Fourier transforms F, F^{-1} of the substitution, which can be easily derived([5]) from the definitions given above.

PROPOSITION 3. *Let f be a non-singular linear mapping* (2). *Let $\tilde{f} = (f^{-1})^t$ denote the mapping given by the transposed matrix to A^{-1}. Then*

$$(6)\qquad F(u\circ f) = \frac{1}{|\operatorname{Det} A|} Fu\circ\tilde{f} \quad \text{for } u \in S'(E^n).$$

The analogous equality holds for F^{-1}.

Applying Definition 2 and that of rotation, we derive from (6) the following

COROLLARY 1. *The Fourier transformations F, F^{-1} commute with rotations and consequently preserve invariance with respect to rotations.*

From Definition 3 and Proposition 3 follows

COROLLARY 2. *The Fourier transforms F and F^{-1} of a distribution $u \in S'(E^n)$ homogeneous of order m are distributions homogeneous of order $-m-n$.*

2. Homogeneous distributions in E^1.

THEOREM 2. *If $v \in D'(E^1_+)$ is homogeneous of order k, it is necessarily a function of the form Cr^k for $r > 0$. If $v \in D'(E^1\setminus\{0\})$ is homogeneous of order k, then it is a function $C_1 r^k$ for $r > 0$ and $C_2 r^k$ for $r < 0$. In particular, if $v \in D'(E^1\setminus\{0\})$ is homogeneous of order 0, then outside the origin $v = C_1 Y + C_2$, where Y denotes the Heaviside function.*

The proof follows from the Euler equation $kv = rv'$, which has only classical solutions on $E^1\setminus\{0\}$.

COROLLARY 3. *Suppose that a distribution $u \in D'(E^1)$ satisfies the equation*

$$(7)\qquad u\circ f_t = u \quad \text{for } t \in E^1\setminus\{0\} \quad (f_t(x) = tx \quad \text{for } x \in E^1).$$

Then u is a constant distribution.

Proof. By Theorem 2, outside the origin we have $u = C + \tilde{C}Y$; hence $\tilde{C} = 0$ by Proposition 1. Therefore Proposition 2 yields $u = C$ on E^1.

THEOREM 3 (Zieleźny [7]). *Fix $x^* \in E^1$ and $v \in D'(E^1)$. Let*

$$h_\varepsilon(x) = x^* + \varepsilon x \quad \text{for } x \in E^1 \ (\varepsilon \in E^1).$$

If there exists the limit $u = \lim_{\varepsilon\to 0} v\circ h_\varepsilon$, it is necessarily a constant distribution: $u[\varphi] = c\int_{-\infty}^{+\infty} \varphi(t)\,dt$ for $\varphi \in C_0^\infty(E^1)$ ([6]).

([5]) See for example [3], Section 20.7.

([6]) Following the definition given by S. Łojasiewicz, c is called the *value of the distribution u at the point x^*.*

Proof. Following Zieleźny, we first prove that the limit distribution u satisfies equation (7). Indeed, we have:

$$u \circ f_t = \lim_{\varepsilon \to 0} \big((v \circ h_\varepsilon) \circ f_t\big) = \lim_{\varepsilon \to 0} v \circ h_{\varepsilon t} = u.$$

By Corollary 3, u is a constant distribution.

3. Rotation invariant distributions.

Definition 4. Let $g \in C^0(E^n \setminus \{0\})$ and let ω_n denote the surface measure of the unit sphere in E^n. The symbol g_s will denote a function defined in $E^n \setminus \{0\}$ by

$$(8) \qquad g_s(x) = \frac{1}{\omega_n |x|^{n-1}} \int_{|\eta|=|x|} g(\eta) d\sigma_\eta = \frac{1}{\omega_n} \int_{|\xi|=1} g(|x|\xi) d\sigma_\xi \text{ (7)}.$$

If $g \in C^0(E^n)$, then setting $g_s(0) = g(0)$ we extend g_s to a continuous function in E^n.

Let us note the following, easy to prove,

Lemma 1. *The assignment $\varphi \mapsto \varphi_s$ for $\varphi \in C_0^\infty(E^n \setminus \{0\})$ is a continuous mapping of the space $D(E^n \setminus \{0\})$ into itself.*

Definition 5. For any distribution $u \in D'(E^n \setminus \{0\})$ we denote by u_s the distribution defined by

$$u_s[\varphi] = u[\varphi_s] \quad \text{for } \varphi \in D(E^n \setminus \{0\}).$$

The correctness of this definition is provided by Lemma 1. In the case when u is a continuous function in $E^n \setminus \{0\}$, Definition 5 coincides with Definition 4.

Lemma 2. *If a distribution u from $D'(E^n)$ and a $C_0^\infty(E^n)$-function v are rotation invariant, then the convolution $u * v$ is a rotation invariant C^∞-function.*

Proof. Because $v \in C_0^\infty(E^n)$, the convolution $u * v$ exists and it is a C^∞-function given by

$$(9) \qquad (u * v)(x) = u[v(x - \xi)] \quad \text{for } x \in E^n.$$

Since the function v is rotation invariant, there exists a function h such that $v(y) = h(|y|)$ for $y \in E^n$. Let $f(y) = Ay$ be a rotation in E^n. Then from (9) we see that

$$\big(((u * v)) \circ f\big)(y) = u[h(|Ay - \xi|)] = u[h(|y - A^{-1}\xi|)]$$
$$= u[h(|y - \xi|)] \quad \text{for } y \in E^n;$$

(7) Here $d\sigma_\eta$ is the element of the surface area of the sphere $\{\eta: |\eta| = |x|\}$; the meaning of $d\sigma_\xi$ is analogous.

the last equality follows from the assumption that u is invariant with respect to rotations. Applying once more formula (9), we obtain $((u*v)\circ f)(y) = (u*v)(y)$ for $y \in E^n$.

LEMMA 3. *If $u \in D'(E^n)$ is a rotation invariant distribution, then $u = u_s$ in $E^n \setminus \{0\}$, where $u_s \in D'(E^n \setminus \{0\})$ denotes the distribution defined by Definition 5.*

Proof. Let ψ be a rotation invariant $C_0^\infty(E^n)$-function such that $\int_{E^n} \psi dx = 1$, $\operatorname{supp}\psi \subset \{x: |x| \leqslant 1\}$. Set $\lambda^{\{\varepsilon\}}(\eta) = \varepsilon^{-n}\psi(\eta/\varepsilon)$ for $\eta \in E^n$, $\varepsilon > 0$, and observe that

$$(10) \qquad \lim_{\varepsilon\to 0} u*\lambda^{\{\varepsilon\}} = u \quad \text{in } D'(E^n).$$

Moreover, the functions $\lambda^{\{\varepsilon\}}$ are rotation invariant and consequently, by Lemma 2, so are the functions $u*\lambda^{\{\varepsilon\}}$. Thus there exists a function h such that $(u*\lambda^{\{\varepsilon\}})(x) = h(|x|)$ for $x \in E^n$. Therefore

$$(11) \qquad (u*\lambda^{\{\varepsilon\}})_s(x) = \frac{1}{\omega_n}\int_{|\eta|=1} h(|x\eta|)\,d\sigma_\eta = h(|x|) = (u*\lambda^{\{\varepsilon\}})(x).$$

Let $\varphi \in C_0^\infty(E^n \setminus \{0\})$. Applying (10), (11), Definition 5 and Lemma 1 we obtain

$$u[\varphi] = \lim_{\varepsilon\to 0}(u*\lambda^{\{\varepsilon\}})[\varphi] = \lim_{\varepsilon\to 0}(u*\lambda^{\{\varepsilon\}})_s[\varphi] = \lim_{\varepsilon\to 0}(u*\lambda^{\{\varepsilon\}})[\varphi_s] = u[\varphi_s],$$

i.e., the desired assertion: $u[\varphi] = u[\varphi_s]$ for any $\varphi \in C_0^\infty(E^n \setminus \{0\})$.

Remark 3. It is possible to replace in Lemma 3 the assumption $u \in D'(E^n)$ by the more natural $u \in D'(E^n \setminus \{0\})$. In fact, we have the following

PROPOSITION 4. *If $u \in D'(E^n \setminus \{0\})$ is a rotation invariant distribution, then $u = u_s$ in $E^n \setminus \{0\}$.*

Proof([8]). Fix arbitrarily $\varphi \in C_0^\infty(E^n \setminus \{0\})$. Let $0 < \varepsilon < \mu$ be such that $\operatorname{supp}\varphi \subset \{x: \varepsilon < |x| < \mu\}$. Choose a rotation invariant C^∞-function $\tilde{\psi}$ such that $\tilde{\psi}(x) = 1$ for $|x| \leqslant \varepsilon/4$, $\tilde{\psi}(x) = 0$ for $|x| \geqslant \varepsilon/2$. Let $\psi(x) = 1 - \tilde{\psi}(x)$ for $x \in E^n$. Observe that $\psi \in C^\infty(E^n)$ is rotation invariant and $\psi\varphi = \varphi$, $\psi\varphi_s = \varphi_s$. It is easy to see that the formal definition

$$\tilde{u}[\chi] = u[\psi\chi] \quad \text{for } \chi \in C_0^\infty(E^n)$$

defines a distribution $\tilde{u} \in D'(E^n)$ which is rotation invariant. Moreover, $\tilde{u}[\varphi] = u[\varphi]$, $\tilde{u}[\varphi_s] = u[\varphi_s]$. From Lemma 3 we immediately obtain $\tilde{u}[\varphi] = \tilde{u}_s[\varphi]$; hence $u[\varphi] = \tilde{u}[\varphi] = \tilde{u}_s[\varphi] = \tilde{u}[\varphi_s] = u[\varphi_s] = u_s[\varphi]$.

([8]) I have invited my student B. Ziemian to prove this proposition. Here I reproduce his proof.

Now we can give a full characterization of rotation invariant distributions.

THEOREM 4. *Let* $u \in D'(E^n \setminus \{0\})$. *The following three properties are equivalent*:

(i) u *is rotation invariant.*

(ii) $u_s = u$ *on* $E^n \setminus \{0\}$.

(iii) *There exists a distribution* $h \in D'(E^1_+)$ *unique in* $D'(E^1_+)$ *such that*

$$u[\varphi] = h[T\varphi] \quad \textit{for } \varphi \in C_0^\infty(E^n \setminus \{0\}), \tag{12}$$

where

$$(T\varphi)(r) = \int_{|x|=r} \varphi(x)\, d\sigma_x \quad \textit{for } \varphi \in C_0^\infty(E^n \setminus \{0\}). \tag{13}$$

Proof. (i)⇒(ii) by Proposition 4.

(ii)⇒(i): Let f be any rotation. Then for every function $\varphi \in C_0^\infty(E^n \setminus \{0\})$ we have

$$(u \circ f)[\varphi] = u[\varphi \circ f^{-1}] = u_s[\varphi \circ f^{-1}] = u[(\varphi \circ f^{-1})_s] = u[\varphi_s] = u_s[\varphi] = u[\varphi].$$

(iii)⇒(i): Let f be a rotation. Then by (12) we have

$$(u \circ f)[\varphi] = u[\varphi \circ f^{-1}] = h[T(\varphi \circ f^{-1})] = h[T\varphi] = u[\varphi].$$

(ii)⇒(iii): Let

$$(P\psi)(x) = \frac{\psi(|x|)}{\omega_n |x|^{n-1}} \quad \text{for } \psi \in C_0^\infty(E^1_+). \tag{14}$$

We see that $P\psi \in C_0^\infty(E^n \setminus \{0\})$ and that the formal equality

$$h[\psi] = u[P\psi] \quad \text{for } \psi \in C_0^\infty(E^1_+)$$

in fact defines a distribution $h \in D'(E^1_+)$.

Let $\varphi \in C_0^\infty(E_n \setminus \{0\})$. By (ii) and (8) we have

$$u[\varphi] = u_s[\varphi] = u[\varphi_s] = u\left[\frac{1}{\omega_n} \frac{(T\varphi)(|x|)}{|x|^{n-1}}\right] = u[P(T\varphi)] = h[T\varphi].$$

Now suppose that there is $h_1 \in D'(E^1_+)$ such that $u[\varphi] = h_1[T\varphi]$ for $\varphi \in C_0^\infty(E_n \setminus \{0\})$. By (12) we obtain: $h[T\varphi] = h_1[T\varphi]$ for $\varphi \in C_0^\infty(E^n \setminus \{0\})$. Let $a \in C_0^\infty(E^1_+)$. Then the formula $\tilde{\varphi}(x) = a(|x|)/\omega_n |x|^{n-1}$ for $|x| > 0$ defines $\tilde{\varphi} \in C_0^\infty(E^n \setminus \{0\})$ such that $T\tilde{\varphi} = a$. Therefore $h[a] = h[T\tilde{\varphi}] = h_1[T\tilde{\varphi}] = h_1[a]$.

4. Homogeneous rotation invariant distributions in $E^n \setminus \{0\}$, $n \geqslant 2$.

THEOREM 5 ([9]). *If $u \in D'(E^n \setminus \{0\})$ is rotation invariant and homogeneous of order k, then u is of the form $C|x|^k$ outside the origin.*

Proof. By Theorem 4, $u[\varphi] = h[T\varphi]$ for $\varphi \in C_0^\infty(E^n \setminus \{0\})$, where $h[\psi] = u[P\psi]$ for $\psi \in C_0^\infty(E_+^1)$ and $P\psi$ is defined by (14). Let

$$g_\varepsilon(t) = \varepsilon t \quad \text{for } t \in E^1, \quad G_\varepsilon(x) = \varepsilon x \quad \text{for } x \in E^n \quad (\varepsilon > 0).$$

Observe that for every $\psi \in C_0^\infty(E_+^1)$ we have

$$\big(P(\psi \circ g_\varepsilon)\big)(x) = \frac{\psi(|\varepsilon x|)}{\omega_n |x|^{n-1}} = \varepsilon^{n-1}(P\psi \circ G_\varepsilon)(x).$$

Hence by the homogeneity of u we derive that

$$h \circ g_\varepsilon = \varepsilon^k h \quad (\varepsilon > 0). \tag{15}$$

Indeed, for every $\psi \in C_0^\infty(E_+^1)$ we have

$$(h \circ g_\varepsilon)[\psi] = \frac{1}{\varepsilon} h[\psi \circ g_{1/\varepsilon}] = \frac{1}{\varepsilon} u[P(\psi \circ g_{1/\varepsilon})] = \varepsilon^{-n} u[P\psi \circ G_{1/\varepsilon}]$$
$$= (u \circ G_\varepsilon)[P\psi] = \varepsilon^k u[P\psi] = \varepsilon^k h[\psi].$$

We know that $h \in D'(E_+^1)$. Hence, by (15) and Theorem 2, h is given by a function cr^k for $r > 0$. Thus

$$u[\varphi] = h[T\varphi] = c\int_0^\infty r^k (T\varphi)(r)\,dr = c\int_{E^n} |x|^k \varphi(x)\,dx$$
$$\text{for } \varphi \in C_0^\infty(E^n \setminus \{0\}).$$

5. Fundamental solution of the Laplace operator Δ_n for $n \geqslant 2$. After performing the Fourier transformation $\hat{E} = FE$ the equation $\Delta_n E_n = \delta$ takes the form([10])

$$-|x|^2 \hat{E}_n = (2\pi)^{-\frac{1}{2}n}. \tag{16}$$

Let H_n denote the function

$$H_n(x) = -(2\pi)^{-\frac{1}{2}n} \frac{1}{|x|^2} \quad \text{for } x \in E^n \setminus \{0\}. \tag{17}$$

From now on we distinguish two cases: (i) $n \geqslant 3$, (ii) $n = 2$.

([9]) This Theorem is a slight generalization of a theorem proved in [1], p. 112, in which u was assumed to be a distribution on E^n. The proof given by W. F. Donoghue makes use of the Haar measure on the orthogonal group.

([10]) We define $\hat{\varphi}(x) = (F\varphi)(x) = (2\pi)^{-\frac{1}{2}n} \int_{E^n} e^{-ix\xi}\varphi(\xi)\,d\xi$ for $\varphi \in S(E^n)$.

(i) $n \geqslant 3$. If $n \geqslant 3$, the function H_n is in $L_1^{\mathrm{loc}}(E^n)$ and defines a tempered solution E_n of equation (16). It is a rotation invariant distribution, homogeneous of order -2. Thus, by Corollaries 1 and 2, $E_n = F^{-1}H_n$ is rotation invariant and homogeneous of order $2-n$. Applying Theorem 5 and Proposition 2 we conclude that E_n is a function, namely $E_n(x) = C_n \frac{1}{|x|^{n-2}}$. It is well known that $C_n = -\frac{1}{(n-2)\omega_n}$. To prove this we put $\varphi(x) = \exp(-\frac{1}{2}|x|^2)$ in the equality

$$C_n \int\limits_{E^n} \frac{1}{|x|^{n-2}} \varphi(x)\,dx = F^{-1}H_n[\varphi].$$

(ii) $n = 2$. If $n = 2$, the function (17) is not locally integrable on E^2, but it is so on $E^2 \setminus \{0\}$. The distribution $H_2 \in L_1^{\mathrm{loc}}(E^2 \setminus \{0\})$ has an extension $\tilde{H} \in S'(E^2)$ defined by

$$\tilde{H}[\psi] = -\frac{1}{2\pi}\left\{\int\limits_{|x|\leqslant 1} \frac{\psi(x)-\psi(0)}{|x|^2}\,dx + \int\limits_{|x|>1} \frac{\psi(x)}{|x|^2}\,dx\right\} \quad \text{for } \psi \in S(E^2). \tag{18}$$

The tempered distribution $\tilde{H}$ satisfies equation (16) with $n = 2$, thus $F^{-1}\tilde{H}$ is a fundamental solution of the operator Δ_2. It is well known that, by employing Bessel functions[11], a direct computation of $F^{-1}\tilde{H}$ leads to the result

$$(F^{-1}\tilde{H})(x) = C + \frac{1}{2\pi}\log|x| \tag{19}$$

with some constant C. One can avoid this computation taking into consideration some properties of $\tilde{H}$. First of all, it is evident that $\tilde{H}$ is rotation invariant. Furthermore, for any homothety f_r we have

$$\tilde{H}\circ f_r = r^{-2}(\tilde{H} - \log r\cdot\delta) \quad (r > 0).$$

These properties imply that the distribution $E = F^{-1}\tilde{H}$ is rotation invariant and satisfies the equation

$$E\circ f_r = E + \frac{1}{2\pi}\log r \quad (r > 0). \tag{20}$$

By Theorem 4 there exists a distribution $h \in D'(E^1_+)$ such that $E[\varphi] = h[T\varphi]$ for $\varphi \in C_0^\infty(E^2 \setminus \{0\})$. Let $g_\varepsilon(t) \coloneqq \varepsilon t$ for $t \in E^1$. Proceeding as in the proof of Theorem 5, but applying this time equation (20) instead

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(11) See [6], § 9 or [3], § 21.

of the homogeneity condition, we obtain $(h\circ g_\varepsilon)[\psi] = h[\psi]+\frac{\log\varepsilon}{2\pi}1[\psi]$ for $\psi\in C_0^\infty(E_+^1)$ $(\varepsilon>0)$, i.e.:

$$h[\psi\circ g_{1/\varepsilon}] = \varepsilon h[\psi]+\frac{\varepsilon\log\varepsilon}{2\pi}1[\psi] \quad \text{for } \psi\in C_0^\infty(E_+^1)\ (\varepsilon>0).$$

Let us differentiate both sides of the last equation with respect to ε and then set $\varepsilon = 1$. We obtain $-h[t\psi'(t)] = h[\psi]+\frac{1}{2\pi}1[\psi]$ and consequently $th'[\psi] = \frac{1}{2\pi}1[\psi]$ for $\psi\in C_0^\infty(E_+^1)$. Hence $h(t) = \frac{1}{2\pi}\log t+C$ for $t>0$, and therefore $E = \frac{1}{2\pi}\log|x|+C$ outside the origin. Thus there are constants a_α such that

$$E = \frac{1}{2\pi}\log|x|+C+\sum_{|\alpha|\leqslant k} a_\alpha D^\alpha\delta.$$

By (20) we conclude that all a_α are equal to zero and so we obtain formula (19). Set

$$E_2(x) = \frac{1}{2\pi}\log|x| \quad \text{for } x\in E^2\setminus\{0\}.$$

Clearly $E_2\in L_1^{\rm loc}(E^2)$ is a fundamental solution of Δ_2.

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