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Scalar differential concomitants of first order of a symmetric connexion Γ^i_{jk} in the two-dimensional space and their applications

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Abstract. Let A^n be the n-dimensional space with a symmetric connexion Γ^i_{jk}. As is known (cf. [5], p. 105, also [3], p. 279), if a purely differential geometric object of first class is a differential concomitant of first order of the symmetric connexion Γ^i_{jk}, then this object is an algebraic concomitant of the curvature tensor $R_{ijk}^{\ \ \ l}$ in A^n. In particular, determination of first order scalar differential concomitants of Γ^i_{jk} is important from the geometric point of view, because we can characterize the spaces A^n by means of such scalars.

L. Bieszk (cf. [1]) has determined (in the functions class C_1) the general form of scalar and density concomitants of the curvature tensor in the two-dimensional space A^2.

In the present note we determine in A^2 all scalar (density) differential concomitants of first order of Γ^i_{jk}, but we assume no regularity conditions concerning those functions (Section 1).

In Section 2 a certain classification of spaces A^2 is given.

In Section 3 we investigate one of the classes found in Section 2.

1. In the space A^n with a symmetric connexion Γ^i_{jk} the curvature tensor $R_{ijk}^{\ \ \ l}$ has the form

$$R_{jkl}^{\ \ \ i} = 2(\partial_{[j}\Gamma^i_{k]l} + \Gamma^i_{[j|s|}\Gamma^s_{k]l}). \tag{1.1}$$

The problem of finding the first order scalar differential concomitants of Γ^i_{jk} rests on solving the equation

$$f(R_{i'j'k'}^{\ \ \ \ \ l'}) = f(R_{ijk}^{\ \ \ l}), \qquad i, j, k, l, i', j', k', l' = 1, 2, \ldots, n. \tag{1.2}$$

Tensor (1.1) has the following concomitants:

$$R_{ik} = R_{lik}^{\ \ \ l}, \tag{1.3}$$

$$V_{ik} = R_{ikl}^{\ \ \ l}, \tag{1.4}$$

and for $n = 2$ we have

$$V_{ik} = 2R_{[ik]} = R_{ik} - R_{ki} \tag{1.5}$$

(cf. [2], p. 213).

In the n-dimensional space the number N of essential components of the tensor $R_{ijk}{}^{l}$ is equal to $n^2\binom{n}{2}$. For $n = 2$ we consider the following essential components:

$$R_{121}{}^{1},\ R_{122}{}^{1},\ R_{121}{}^{2},\ R_{122}{}^{2}.$$

From (1.3) it follows that

(1.6) $R_{121}{}^{1} = R_{21}, \quad R_{122}{}^{1} = R_{22}, \quad R_{121}{}^{2} = -R_{11}, \quad R_{122}{}^{2} = -R_{12}.$

We introduce the following notation:

$$K_{ik} = R_{(ik)} = \tfrac{1}{2}(R_{ik}+R_{ki}), \tag{1.7}$$

$$\omega = \mathrm{Det}\,\|K_{ik}\|, \tag{1.8}$$

$$\sigma = \mathrm{Det}\,\|V_{ik}\|. \tag{1.9}$$

First we prove

LEMMA. *In A^2 every scalar differential concomitant of first order of Γ is a scalar concomitant of the tensor K_{ij} and of the density σ.*

Proof. As follows from (1.5), (1.6), (1.7), our problem leads to determination of scalar concomitants of the pair of tensors (K_{ij}, V_{ij}). In A^2 the tensor V_{ij} has the single essential component $V_{12} = a$. If $\sigma = \mathrm{Det}\,\|V_{ij}\|$, then $a = \sqrt{\sigma}$ or $a = -\sqrt{\sigma}$. We notice that the following pairs of matrices

$$\left(\|K_{ij}\|,\ \left\|\begin{matrix}0 & \sqrt{\sigma}\\ -\sqrt{\sigma} & 0\end{matrix}\right\|\right),\quad \left(\|K_{ij}\|,\ \left\|\begin{matrix}0 & -\sqrt{\sigma}\\ \sqrt{\sigma} & 0\end{matrix}\right\|\right)$$

are congruent. Hence every scalar concomitant of the pair (K_{ij}, V_{ij}) is a scalar concomitant of the tensor K_{ij} and of the density σ.

This completes the proof.

Now we shall determine the general solution of the equation

$$h(K_{i'j'}, \sigma') = h(K_{ij}, \sigma), \tag{1.10}$$

where

$$K_{i'j'} = A^{i}_{i'}A^{j}_{j'}K_{ij}, \qquad \sigma' = J^{-2}\sigma$$

and $\|A^{i'}_{i}\| \in \mathrm{GL}(2)$, $J = \mathrm{Det}\,\|A^{i'}_{i}\|$.

We assume that $\omega \neq 0$. It is known that we can always find a nonsingular matrix $\|A^{i}_{i'}\|$ such that

$$\|K_{i'j'}\| = \left\|\begin{matrix}\varepsilon_1 & 0\\ 0 & \varepsilon_2\end{matrix}\right\|, \tag{1.11}$$

where $\varepsilon_i = \pm 1$.

It follows from (1.11) that

$$J^{-2} = \varepsilon_1\varepsilon_2/\omega. \tag{1.12}$$

Inserting (1.11) and (1.12) into (1.10), we obtain

$$h(K_{ij},\sigma) = h\left(\begin{Vmatrix}\varepsilon_1 & 0\\ 0 & \varepsilon_2\end{Vmatrix}, \varepsilon_1\varepsilon_2\frac{\sigma}{\omega}\right), \tag{1.13}$$

where σ/ω is a scalar.

Let us put $\omega = 0$. Then the matrix $\|K_{ij}\|$ has the following eigenvalues: $\lambda_1 = 0$, $\lambda_2 = K_{11}+K_{22}$. Therefore there exists an orthogonal matrix $\|A^i_{i'}\|$ such that

$$\|K_{i'j'}\| = \begin{Vmatrix}K_{11}+K_{22} & 0\\ 0 & 0\end{Vmatrix}.$$

If $K_{11}+K_{22} \neq 0$, we put

$$\|A^{i'}_{j''}\| = G(\sigma)\begin{Vmatrix}\dfrac{1}{\sqrt{|K_{11}+K_{22}|}} & 0\\ 0 & \sqrt{|K_{11}+K_{22}|}\end{Vmatrix},$$

where

$$G(\sigma) = \begin{cases}\begin{Vmatrix}0 & 1\\ 1 & 0\end{Vmatrix}, & \sigma = 0,\\[2ex] \begin{Vmatrix}0 & \dfrac{1}{\sqrt{\sigma}}\\ 1 & 0\end{Vmatrix}, & \sigma \neq 0.\end{cases}$$

Inserting the matrix $\|A^i_{i''}\| = \|A^i_{i'}A^{i'}_{i''}\|$ into (1.10), we get

$$h(K_{ij},\sigma) = h\left(\begin{Vmatrix}0 & 0\\ 0 & \operatorname{sgn}(K_{11}+K_{22})\end{Vmatrix}, \eta(\sigma)\right), \tag{1.14}$$

where

$$\eta(\sigma) = \begin{cases}0, & \sigma = 0,\\ 1, & \sigma \neq 0.\end{cases}$$

If $K_{11}+K_{22} = 0$, then

$$h(K_{ij},\sigma) = h\left(\begin{Vmatrix}0 & 0\\ 0 & 0\end{Vmatrix}, \eta(\sigma)\right). \tag{1.15}$$

Thus we have obtained the following

THEOREM 1. *Every scalar differential concomitant f of first order of the symmetric connexion Γ^i_{jk}, for $n = 2$, has the form*

$$f = \begin{cases}\varphi(r, s, \sigma/\omega), & \omega \neq 0,\\ \psi(r, s, \eta(\sigma)), & \omega = 0,\end{cases}$$

where s is the signature of the tensor K_{ij}, r is the rank of the tensor K_{ij}, φ and ψ are arbitrary functions.

Remark. With aid of similar methods we can obtain the following corollaries:

COROLLARY 1. *If $\omega^2+\sigma^2 = 0$, then there do not exist differential concomitants of first order of the symmetric connexion Γ^i_{jk}, for $n = 2$, which are G- or W-densities of a weight p, $p \neq 0$.*

COROLLARY 2. *If $\sigma = 0$, $\omega \neq 0$, then there do not exist differential concomitants of the first order of the symmetric connexion Γ^i_{jk}, for $n = 2$, which are G-densities of a weight p, $p \neq 0$; however, every differential concomitant of first order of Γ^i_{jk} which is a W-density of a weight p has the form*

$$|\omega|^{-p/2}\gamma,$$

where γ is an arbitrary function of the signature s of the tensor K_{ij} (see also [4], p. 74–75).

COROLLARY 3. *If $\sigma \neq 0$, then every differential concomitant of first order of the symmetric connexion Γ^i_{jk}, for $n = 2$, which is a density of a weight p, $p \neq 0$, has the form*

$$\varepsilon|a|^{-p}g,$$

where

$$a = V_{12},$$

$$\varepsilon = \begin{cases} 1 & \text{for a } W\text{-density}, \\ \operatorname{sgn} a & \text{for a } G\text{-density}, \end{cases}$$

$$g = \begin{cases} \varphi(r, s, \sigma/\omega), & \omega \neq 0, \\ \psi(r, s), & \omega = 0. \end{cases}$$

2. We use the results of Section 1 to a certain classification of spaces A^2 From our considerations it follows that the tensor R_{ij} in A^2 (consequently $R_{ijk}{}^l$) can have the following canonical forms:

For $\sigma = 0$:

I. $\begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix}$,

II. $\begin{Vmatrix} 1 & 0 \\ 0 & -1 \end{Vmatrix}$,

III. $\begin{Vmatrix} -1 & 0 \\ 0 & -1 \end{Vmatrix}$,

IV. $\begin{Vmatrix} 1 & 0 \\ 0 & 0 \end{Vmatrix}$,

$$\text{V.}\ \begin{Vmatrix} -1 & 0 \\ 0 & 0 \end{Vmatrix},$$

$$\text{VI.}\ \begin{Vmatrix} 0 & 0 \\ 0 & 0 \end{Vmatrix}.$$

For $\sigma \neq 0$:

$$\left.\begin{array}{l} \text{I.}\ \begin{Vmatrix} 1 & \dfrac{\sigma}{\omega} \\ -\dfrac{\sigma}{\omega} & 1 \end{Vmatrix} \\[2ex] \text{II.}\ \begin{Vmatrix} 1 & -\dfrac{\sigma}{\omega} \\ \dfrac{\sigma}{\omega} & -1 \end{Vmatrix} \\[2ex] \text{III.}\ \begin{Vmatrix} -1 & \dfrac{\sigma}{\omega} \\ -\dfrac{\sigma}{\omega} & -1 \end{Vmatrix} \end{array}\right\} \omega \neq 0,$$

$$\left.\begin{array}{l} \text{IV.}\ \begin{Vmatrix} 1 & 1 \\ -1 & 0 \end{Vmatrix} \\ \text{V.}\ \begin{Vmatrix} -1 & 1 \\ -1 & 0 \end{Vmatrix} \\ \text{VI.}\ \begin{Vmatrix} 0 & 1 \\ -1 & 0 \end{Vmatrix} \end{array}\right\} \omega = 0$$

(see also [5], p. 224).

The type of a space A^2 is given by the following quantities:

$$\begin{array}{ll} r,\ s,\ \sigma/\omega, & \text{if } \omega \neq 0, \\ r,\ s & \text{if } \sigma \neq 0,\ \omega = 0. \end{array}$$

On the other hand, a suitable space A^2 exists for every type of the tensor R_{ij} ($R_{ijk}{}^{l}$). In fact, we have

For $\sigma = 0$:

$$\begin{array}{llll} \text{I.} & \Gamma^1_{11} \overset{*}{=} 1, & \Gamma^1_{12} \overset{*}{=} 1, & \Gamma^1_{22} \overset{*}{=} 0, \\ & \Gamma^2_{11} \overset{*}{=} 1, & \Gamma^2_{12} \overset{*}{=} 0, & \Gamma^2_{22} \overset{*}{=} 2, \\ \text{II.} & \Gamma^1_{11} \overset{*}{=} 1, & \Gamma^1_{12} \overset{*}{=} -1, & \Gamma^1_{22} \overset{*}{=} 0, \\ & \Gamma^2_{11} \overset{*}{=} -1, & \Gamma^2_{12} \overset{*}{=} 0, & \Gamma^2_{22} \overset{*}{=} 0, \end{array}$$

$$\text{III.}\quad \begin{array}{lll} \Gamma^1_{11} \overset{*}{=} 1, & \Gamma^1_{12} \overset{*}{=} -1, & \Gamma^1_{22} \overset{*}{=} 0, \\ \Gamma^2_{11} \overset{*}{=} 1, & \Gamma^2_{12} \overset{*}{=} 0, & \Gamma^2_{22} \overset{*}{=} 0, \end{array}$$

$$\text{IV.}\quad \begin{array}{lll} \Gamma^1_{11} \overset{*}{=} 1, & \Gamma^1_{12} \overset{*}{=} 0, & \Gamma^1_{22} \overset{*}{=} 0, \\ \Gamma^2_{11} \overset{*}{=} 1, & \Gamma^2_{12} \overset{*}{=} 0, & \Gamma^2_{22} \overset{*}{=} 1, \end{array}$$

$$\text{V.}\quad \begin{array}{lll} \Gamma^1_{11} \overset{*}{=} 1, & \Gamma^1_{12} \overset{*}{=} 0, & \Gamma^1_{22} \overset{*}{=} 0, \\ \Gamma^2_{11} \overset{*}{=} 1, & \Gamma^2_{12} \overset{*}{=} 0, & \Gamma^2_{22} \overset{*}{=} -1, \end{array}$$

$$\text{VI.}\quad \Gamma^1_{11} \overset{*}{=} \Gamma^1_{12} \overset{*}{=} \Gamma^1_{22} \overset{*}{=} \Gamma^2_{11} \overset{*}{=} \Gamma^2_{12} \overset{*}{=} \Gamma^2_{22} = 0.$$

For $\sigma \neq 0$:

$$\text{I.}\quad \begin{array}{lll} \Gamma^1_{11} \overset{*}{=} au^2, & \Gamma^1_{12} \overset{*}{=} 1, & \Gamma^1_{22} \overset{*}{=} 0, \\ \Gamma^2_{11} \overset{*}{=} 1, & \Gamma^2_{12} \overset{*}{=} 0, & \Gamma^2_{22} \overset{*}{=} 2, \end{array}$$

$$\text{II.}\quad \begin{array}{lll} \Gamma^1_{11} \overset{*}{=} au^2, & \Gamma^1_{12} \overset{*}{=} 1, & \Gamma^1_{22} \overset{*}{=} 0, \\ \Gamma^2_{11} \overset{*}{=} 1, & \Gamma^2_{12} \overset{*}{=} 0, & \Gamma^2_{22} \overset{*}{=} 0, \end{array}$$

$$\text{III.}\quad \begin{array}{lll} \Gamma^1_{11} \overset{*}{=} au^2, & \Gamma^1_{12} \overset{*}{=} 1, & \Gamma^1_{22} \overset{*}{=} 0, \\ \Gamma^2_{11} \overset{*}{=} -1, & \Gamma^2_{12} \overset{*}{=} 0, & \Gamma^2_{22} \overset{*}{=} 0, \end{array}$$

$$\text{IV.}\quad \begin{array}{lll} \Gamma^1_{11} \overset{*}{=} au^2, & \Gamma^1_{12} \overset{*}{=} 0, & \Gamma^1_{22} \overset{*}{=} 0, \\ \Gamma^2_{11} \overset{*}{=} 1, & \Gamma^2_{12} \overset{*}{=} 0, & \Gamma^2_{22} \overset{*}{=} 1, \end{array}$$

$$\text{V.}\quad \begin{array}{lll} \Gamma^1_{11} \overset{*}{=} au^2, & \Gamma^1_{12} \overset{*}{=} 0, & \Gamma^1_{22} \overset{*}{=} 0, \\ \Gamma^2_{11} \overset{*}{=} 1, & \Gamma^2_{12} \overset{*}{=} 0, & \Gamma^2_{22} \overset{*}{=} -1, \end{array}$$

$$\text{VI.}\quad \begin{array}{lll} \Gamma^1_{11} \overset{*}{=} au^2, & \Gamma^1_{12} \overset{*}{=} 0, & \Gamma^1_{22} \overset{*}{=} 0, \\ \Gamma^2_{11} \overset{*}{=} 1, & \Gamma^2_{12} \overset{*}{=} 0, & \Gamma^2_{22} \overset{*}{=} 0, \end{array}$$

where u^1, u^2 are local coordinates in the space A^2, $a = \text{const} \neq 0$.

3. Let A^2 be a space with a symmetric connexion Γ^i_{jk}. We assume that there exists a symmetric regular tensor g_{ij} and that

$$\nabla_k g_{ij} = 0. \tag{3.1}$$

(∇ denotes the covariant differentiation with respect to Γ^i_{jk}). It is known (cf. [2], p. 224) that in this case the parameters Γ^i_{jk} are Christoffel's symbols of the tensor g_{ij} and that such a space A^2 is a Riemann space.

Now we shall investigate spaces A^2 with $\sigma = 0$. In this case we shall seek a symmetric and regular tensor g_{ij} which fulfils (3.1). If $r = 0$, i.e., $R_{ij} = R_{ijk}{}^{l} = 0$, then (3.1) has the following solution:

$$g_{ij} = \underset{ij}{C},$$

where $\underset{ij}{C} = \text{const}$ (cf. [2], p. 228) and such a space A^2 is a Riemann space. For $r = 1$ or $r = 2$ we consider the system of equations

$$\begin{array}{lll} R_{12}g_{11} - R_{11}g_{12} & & = 0, \\ & R_{22}g_{12} - R_{12}g_{22} & = 0, \\ R_{22}g_{11} & \quad - R_{11}g_{22} & = 0. \end{array} \tag{3.2}$$

We know (cf. [2], p. 227) that (3.2) is a necessary condition for (3.1) to hold.

We now prove the following

THEOREM 2. *A space A^2 with $\sigma = 0$ and $r = 1$ is not a Riemann space.*

Proof. If $r = 1$, then at a point p of the space A^2 the symmetric tensor R_{ij} has the canonical form

$$\|R_{ij}\|_p \overset{*}{=} \begin{Vmatrix} \varepsilon & 0 \\ 0 & 0 \end{Vmatrix}, \qquad \varepsilon = \pm 1.$$

The system of equations (3.2) has at a point p the form

$$\varepsilon g_{12} = 0, \qquad \varepsilon g_{22} = 0.$$

Hence $g_{12} = g_{22} = 0$ and every solution of (3.2) is singular.

This completes the proof.

Now we consider the case $r = 2$. In this case we prove the following

THEOREM 3. *The space A^2 with $\sigma = 0$ and $r = 2$ is a Riemann space if and only if*

$$\nabla_k R_{ij} = -a_k R_{ij}, \tag{3.3}$$

where a_k is a gradient field.

Proof. If a space A^2 is a Riemann space with $\sigma = 0$ and $r = 2$, then

$$R_{ij} = -K g_{ij},$$

where K is Gauss' curvature and $K \neq 0$. Thus we have

$$\nabla_k R_{ij} = (\partial_k \ln |K|) R_{ij}.$$

We assume that $\sigma = 0$, $r = 2$ and (3.3). If $r = 2$, then the general solution of (3.2) has the form

$$g_{ij} = \tau R_{ij}. \tag{3.4}$$

Putting (3.4), where $\tau \neq 0$, into (3.1), we obtain

$$\nabla_k (\tau R_{ij}) = 0. \tag{3.5}$$

By covariant differentiation of (3.5) we obtain

$$(\partial_k \tau) R_{ij} + \tau \nabla_k R_{ij} = 0$$

and thus

$$\nabla_k R_{ij} = -(\partial_k \ln |\tau|) R_{ij}. \tag{3.6}$$

It follows from (3.3) and (3.6) that

$$\partial_k \ln |\tau| = a_k.$$

Let us put $a_k = \partial_k \beta$, where β is a scalar field. Then we have

$$\partial_k(\ln|\tau| - \beta) = 0$$

and thus

$$\tau = e^{\beta + c},$$

where c = const.

This completes the proof.

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