A note on the volume of balls on Riemannian manifolds of non-negative curvature

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Abstract. In this note we study the volume of balls on Riemannian manifolds of non-negative sectional curvature. We prove that the function $V : [0, +\infty) \to \mathbb{R}$ defined by

$$V(r) = \text{the supremum of volumes of balls with the radius } r$$

increases "uniformly" in the sense that the ratio $V(ar)/V(r)$ is bounded by a number $c(a)$ depending only on $a$, not on $r$.

Let $(M, g)$ be a complete Riemannian manifold. Denote by $d_M$ the Riemannian distance on $(M, g)$ and let

$$B(x, r) = \{ y \in M : d_M(x, y) \leq r \}$$

be the ball with centre $x$ and radius $r$. Bishop [1] (see also [2] and [3]) proved that the volume $V(x, r)$ of $B(x, r)$ satisfies the inequality

$$V(x, r) \leq \omega_m r^m,$$

where $m = \dim M$ and $\omega_m$ is the volume of the unit ball in the $m$-dimensional Euclidean space, providing that the Ricci curvature of $(M, g)$ is non-negative.

Put

$$V(r) = \sup_{x \in M} V(x, r).$$

In this note we prove the following

Theorem. If the sectional curvature of $(M, g)$ is non-negative, then for any $a > 0$ there exists a number $c(a)$ such that

$$V(ar) \leq c(a) \cdot V(r) \quad \text{for each } r \geq 0.$$

In the proof we shall use the following version of the Topogonov comparison theorem (see [4], Section 6.4).

(*) Let $c_i : [0, 1] \to M$, $i = 1, 2, 3$, be geodesics such that $p_1 = c_1(0) = c_3(1)$, $p_2 = c_2(0) = c_1(1)$, $p_3 = c_3(0) = c_2(1)$. If $L(c_3) = d_M(p_3, p_1)$, $L(c_1) = d_M(p_1, p_2)$, $L(c_2) \leq L(c_1) + L(c_3)$, and the sectional curvature of $(M, g)$ is non-negative, then

$$L(c_1)^2 \leq L(c_2)^2 + L(c_3)^2 - 2L(c_2)L(c_3) \cos \angle (c_3(0), c_2(1)).$$
Proof of the Theorem. Let us take a number $\varepsilon \in (0, \frac{1}{2} \pi)$ such that $\cos \varepsilon \geq \frac{1}{10}$. We can find a collection $e_1, \ldots, e_k$ of vectors of the tangent space $T_x M$, $x \in M$, such that

(i) $|e_i| = 1$ for $i = 1, \ldots, k$,

(ii) if $i \neq j$, then $\langle e_i, e_j \rangle \geq \varepsilon$,

(iii) $\min \{ \langle v, e_1 \rangle, \ldots, \langle v, e_k \rangle \} \leq \varepsilon$ for any $v$ of $T_x M$, $v \neq 0$, where the angle $\langle v_1, v_2 \rangle$ is taken always in the interval $[0, \pi]$. Then

$$k \leq \sigma / \sigma_{e},$$

where $\sigma$ is the measure of the sphere $S^{m-1}$ and $\sigma_{e}$ the measure of the set $\{ u \in S^{m-1}; \langle u, u_0 \rangle < \frac{1}{2} \varepsilon \}$ ($u_0 \in S^{m-1}$).

Let us take $r \geq 0$, $x \in M$. Then

$$B(x, \sqrt{3} r) \subset B(x, r) \cup \bigcup_{i=1}^{k} B(x_i, r),$$

where $x_i = \exp_{x} e_i$. In fact, if $y \in B(x, \sqrt{3} r)$, then $y = \exp v$, where $v \in T_x M$, $|v| = 1$, $0 \leq t = d_M(x, y) \leq \sqrt{3} r$. If $t \leq r$, then $y \in B(x, r)$. If $t \geq r$, then we can apply ($\ast$) to the triangle $\Delta x x_i y$ sketched below;

![Diagram](image-url)

here $c_1$ is a geodesic satisfying $L(c_1) = d_M(x_i, x)$, $c_2(s) = \exp s e_i$ for $s \in [0, r]$, and $c_3(s) = \exp s v$ for $s \in [0, t]$. Clearly, $L(c_2) = r \leq L(c_3) \leq L(c_1) + L(c_3)$. Taking $i$ such that $\langle v, e_i \rangle \leq \varepsilon$ we get

$$d_M(y, x_i)^2 = L(c_1)^2 \leq t^2 + r^2 - 2tr \cos \langle v, e_i \rangle \leq (t-r)^2 + 2tr(1-\cos \varepsilon) \leq r^2 [((\sqrt{3}-1)^2 + 2\sqrt{3} \frac{1}{10})] < r^2.$$

This proves (2).

Inequality (1) and relation (2) yield

$$V(x, \sqrt{3} r) \leq \left( \frac{\sigma}{\sigma_{e}} + 1 \right) V(r) \quad \text{and} \quad V(\sqrt{3} r) \leq \left( \frac{\sigma}{\sigma_{e}} + 1 \right) V(r)$$

for any $r \geq 0$, $x$ of $M$. 


Finally, if \( a > 0 \) and \( n \) is a natural number such that \( a \leq \sqrt[3]{n} \), then
\[
V(ar) \leq V(\sqrt[3]{n} r) \leq \left( \frac{\sigma}{\sigma_\epsilon} + 1 \right)^n V(r),
\]
i.e. we obtained the statement of the Theorem with
\[
c(a) = \left( \frac{\sigma}{\sigma_\epsilon} + 1 \right)^{1+\left[ \log_3 \sigma \right]}
\]

The following problems arise.

1. Find an example of a Riemannian manifold \((M, g)\) of non-negative Ricci curvature and such that the function
\[
[0, + \infty) \ni r \mapsto V(r)
\]
satisfies the following condition.

\((***)\) There exists a number \( a > 0 \) such that for any \( c > 0 \) we can find \( t \in [0, + \infty) \) such that
\[
V(at) > c \cdot V(t).
\]

2. Estimate \( V(r) \) in the case of a Riemannian manifold with the sectional curvature non-negative outside a compact set.

We end this note with an example of a non-negative increasing differentiable function \( f: [0, + \infty) \rightarrow \mathbb{R} \) which is bounded by a given increasing function \( \varphi: [0, + \infty) \rightarrow \mathbb{R} \) such that \( \varphi(0) = 0 \) and \( \lim_{t \rightarrow + \infty} \varphi(t) = + \infty \) and satisfies condition \((***)\).

If \( \varphi \) is an increasing function on the interval \([0, + \infty)\) and \( \lim_{t \rightarrow + \infty} \varphi(t) = + \infty \), then we can construct a sequence \((t_1, t_2, \ldots)\) of positive numbers such that
\[
2t_n < t_{n+1} \quad \text{and} \quad (n+1) \varphi(t_n) < \varphi(t_{n+1})
\]
for \( n = 1, 2, \ldots \). Put
\[
a_n = \frac{1}{n+1} \varphi(t_n), \quad b_n = \frac{n+1}{n+2} \varphi(t_n), \quad n = 1, 2, \ldots,
\]
and define a function \( h: [0, + \infty) \rightarrow \mathbb{R} \) by the conditions
(i) \( h(t_n) = a_n, \) \( h(2t_n) = b_n \) for \( n = 1, 2, \ldots \),
(ii) \( h(t) = \frac{1}{2} \varphi(t) \) for \( 0 \leq t \leq t_1 \),
(iii) \( h \) is linear on each of the intervals \([t_n, 2t_n], [2t_n, t_{n+1}]\), where \( n = 1, 2, \ldots \).
Clearly, $h$ is a non-negative increasing function, $h(t) \leq \varphi(t)$ for any $t \geq 0$, and

$$h(2r_n) > n \cdot h(r_n).$$

A slight modification of $h$ yields a differentiable function $f$ having all the required properties.

References


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