ANNALES
POLONICI MATHEMATICI
XLIV (1984)

A note on the volume of balls on Riemannian manifolds of non-negative curvature

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Abstract. In this note we study the volume of balls on Riemannian manifolds of non-negative sectional curvature. We prove that the function V: $[0, +\infty) \to R$ defined by

$$V(r) = \text{the supremum of volumes of balls with the radius } r$$

increases "uniformly" in the sense that the ratio $V(ar)/V(r)$ is bounded by a number $c(a)$ depending only on a, not on r.

Let (M, g) be a complete Riemannian manifold. Denote by d_M the Riemannian distance on (M, g) and let

$$B(x, r) = \{y \in M;\ d_M(x, y) \leqslant r\}$$

be the ball with centre x and radius r. Bishop [1] (see also [2] and [3]) proved that the volume $V(x, r)$ of $B(x, r)$ satisfies the inequality

$$V(x, r) \leqslant \omega_m \cdot r^m,$$

where $m = \dim M$ and ω_m is the volume of the unit ball in the m-dimensional Euclidean space, providing that the Ricci curvature of (M, g) is non-negative.

Put

$$V(r) = \sup_{x \in M} V(x, r).$$

In this note we prove the following

THEOREM. *If the sectional curvature of (M, g) is non-negative, then for any $a > 0$ there exists a number $c(a)$ such that*

$$V(ar) \leqslant c(a) \cdot V(r) \quad \textit{for each } r \geqslant 0.$$

In the proof we shall use the following version of the Topogonov comparison theorem (see [4], Section 6.4).

(*) *Let c_i: $[0, 1] \to M$, $i = 1, 2, 3$, be geodesics such that $p_1 = c_1(0) = c_3(1)$, $p_2 = c_2(0) = c_1(1)$, $p_3 = c_3(0) = c_2(1)$. If $L(c_3) = d_M(p_3, p_1)$, $L(c_1) = d_M(p_1, p_2)$, $L(c_2) \leqslant L(c_1) + L(c_3)$, and the sectional curvature of (M, g) is non-negative, then*

$$L(c_1)^2 \leqslant L(c_2)^2 + L(c_3)^2 - 2L(c_2)\,L(c_3) \cos \sphericalangle \big(c_3(0), c_2(1)\big).$$

Proof of the Theorem. Let us take a number $\varepsilon \in (0, \frac{1}{2}\pi)$ such that $\cos\varepsilon \geqslant \frac{9}{10}$. We can find a collection $e_1, \ldots, e_k$ of vectors of the tangent space $T_x M$, $x \in M$, such that

(i) $\|e_i\| = 1$ for $i = 1, \ldots, k$,

(ii) if $i \neq j$, then $\sphericalangle(e_i, e_j) \geqslant \varepsilon$,

(iii) $\min\{\sphericalangle(v, e_1), \ldots, \sphericalangle(v, e_k)\} \leqslant \varepsilon$ for any v of $T_x M$, $v \neq 0$, where the angle $\sphericalangle(v_1, v_2)$ is taken always in the interval $[0, \pi]$. Then

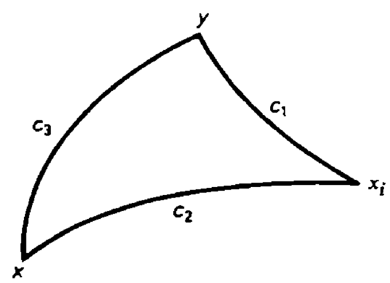
$$k \leqslant \sigma/\sigma_\varepsilon, \tag{1}$$

where σ is the measure of the sphere S^{m-1} and σ_ε the measure of the set $\{u \in S^{m-1};\ \sphericalangle(u, u_0) < \frac{1}{2}\varepsilon\}$ $(u_0 \in S^{m-1})$.

Let us take $r \geqslant 0$, $x \in M$. Then

$$B(x, \sqrt{3}r) \subset B(x, r) \cup \bigcup_{i=1}^{k} B(x_i, r), \tag{2}$$

where $x_i = \exp re_i$. In fact, if $y \in B(x, \sqrt{3}r)$, then $y = \exp tv$, where $v \in T_x M$, $\|v\| = 1$, $0 \leqslant t = d_M(x, y) \leqslant \sqrt{3}r$. If $t \leqslant r$, then $y \in B(x, r)$. If $t \geqslant r$, then we can apply (*) to the triangle $\triangle xyx_i$ sketched below;



here c_1 is a geodesic satisfying $L(c_1) = d_M(y, x_i)$, $c_2(s) = \exp se_i$ for $s \in [0, r]$, and $c_3(s) = \exp sv$ for $s \in [0, t]$. Clearly, $L(c_2) = r \leqslant L(c_3) \leqslant L(c_1) + L(c_3)$. Taking i such that $\sphericalangle(v, e_i) \leqslant \varepsilon$ we get

$$\begin{aligned} d_M(y, x_i)^2 &= L(c_1)^2 \leqslant t^2 + r^2 - 2tr\cos\sphericalangle(v, e_i) \\ &\leqslant (t-r)^2 + 2tr(1-\cos\varepsilon) \leqslant r^2\left[(\sqrt{3}-1)^2 + 2\sqrt{3}\tfrac{1}{10}\right] < r^2. \end{aligned}$$

This proves (2).

Inequality (1) and relation (2) yield

$$V(x, \sqrt{3}r) \leqslant \left(\frac{\sigma}{\sigma_\varepsilon} + 1\right) V(r) \quad \text{and} \quad V(\sqrt{3}r) \leqslant \left(\frac{\sigma}{\sigma_\varepsilon} + 1\right) V(r)$$

for any $r \geqslant 0$, x of M.

Finally, if $a > 0$ and n is a natural number such that $a \leqslant \sqrt{3}^n$, then

$$V(ar) \leqslant V(\sqrt{3}^n r) \leqslant \left(\frac{\sigma}{\sigma_\varepsilon} + 1\right)^n V(r),$$

i.e. we obtained the statement of the Theorem with

$$c(a) = \left(\frac{\sigma}{\sigma_\varepsilon} + 1\right)^{1 + [\log_{\sqrt{3}} a]}$$

The following problems arise.

1. Find an example of a Riemannian manifold (M, g) of non-negative Ricci curvature and such that the function

$$[0, +\infty) \ni r \mapsto V(r)$$

satisfies the following condition.

(**) There exists a number $a > 0$ such that for any $c > 0$ we can find $t \in [0, +\infty)$ such that

$$V(at) > c \cdot V(t).$$

2. Estimate $V(r)$ in the case of a Riemannian manifold with the sectional curvature non-negative outside a compact set.

We end this note with an example of a non-negative increasing differentiable function f: $[0, +\infty) \to R$ which is bounded by a given increasing function φ: $[0, +\infty) \to R$ such that $\varphi(0) = 0$ and $\lim_{t \to +\infty} \varphi(t) = +\infty$ and satisfies condition (**).

If φ is an increasing function on the interval $[0, +\infty)$ and $\lim_{t \to \infty} \varphi(t) = +\infty$, then we can construct a sequence $(t_1, t_2, \ldots)$ of positive numbers such that

$$2t_n < t_{n+1} \quad \text{and} \quad (n+1)\varphi(t_n) < \varphi(t_{n+1})$$

for $n = 1, 2, \ldots$ Put

$$a_n = \frac{1}{n+1}\varphi(t_n), \qquad b_n = \frac{n+1}{n+2}\varphi(t_n), \qquad n = 1, 2, \ldots,$$

and define a function h: $[0, +\infty) \to R$ by the conditions

(i) $h(t_n) = a_n$, $h(2t_n) = b_n$ for $n = 1, 2, \ldots$,

(ii) $h(t) = \frac{1}{2}\varphi(t)$ for $0 \leqslant t \leqslant t_1$,

(iii) h is linear on each of the intervals $[t_n, 2t_n]$, $[2t_n, t_{n+1}]$, where $n = 1, 2, \ldots$

Clearly, h is a non-negative increasing function, $h(t) \leqslant \varphi(t)$ for any $t \geqslant 0$, and

$$h(2t_n) > n \cdot h(t_n).$$

A slight modification of h yields a differentiable function f having all the required properties.

References

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[4] D. Gromoll, W. Klingenberg, W. Meyer, *Riemannsche Geometrie im Grossen*, Springer Verlag, 1968.

Reçu par la Rédaction le 6. 11. 1979