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ON MONOTONE UNIONS OF CLOSED n-CELLS

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1. Introduction. In [1] Fort showed that every open connected set in E^n is a monotone union of closed n-cells. We extend this property to a larger class of spaces and study some properties of such monotone unions. All spaces are Hausdorff.

2. Definitions. A simplicial complex K^n is *monotonic* if

$$K^n = \bigcup_{i=1}^{p} K_i,$$

where each K_i is a subcomplex of K^n, K_i is an n-simplex and K_{i+1} $(1 \leqslant i \leqslant p-1)$ is obtained from K_i by adding just one n-simplex to K_i that has an $(n-1)$-simplex in common with K_i. If F is a set, $|F|$ is its cardinality.

3. Monotonic complexes.

THEOREM 1. *If K^n is a monotonic complex of dimension n $(n \geqslant 2)$, then*

$$K^n = \bigcup_{i=1}^{\infty} C_i,$$

where C_i is a closed n-cell and $C_i \subset C_{i+1}$ for $i = 1, 2, 3, \ldots$

Proof. Let K^n be monotonic so that

$$K^n = \bigcup_{1}^{p} K_i.$$

The proof is by induction on p. The inductive hypothesis we carry along is the following: For $p < a$ we have proved our result with the hypothesis that if $\{\sigma_j^{n-1}\}_{j=1}^{q}$ is any collection of $(n-1)$-simplices in K^n, then the union

$$\bigcup_{1}^{\infty} C_i = K^n$$

can be chosen so that Bd C_i meets each σ_j^{n-1} in an $(n-1)$-dimensional proper set for all i. For $p = 1$ the result follows. So assume it for $p < a$. Let

$$K^n = \bigcup_1^a K_i.$$

$\bigcup_1^{a-1} K_i$ is monotonic and has an $(n-1)$-simplex in common with σ^n, the n-simplex added to K_{a-1} to get K_a. Call it σ^{n-1}. By the inductive hypothesis we have $K_{a-1} = \bigcup_{i=1}^{\infty} C_i$ and Bd $C_i \cap \sigma^{n-1}$ has dimension $n-1$. There is on each Bd C_i a point x and a neighborhood of x in Int σ^{n-1}. To each C_i we attach a monotone union of closed n-cells in $(\sigma^n - K_{a-1}) \cup \Delta^{n-1}$ where Δ^{n-1} is an $(n-1)$-simplex neighborhood of x in Int σ^{n-1}. Call the resulting n-cells C_i^1 and then $K^n = \bigcup_{i=1}^{\infty} C_i^1$. The inductive hypothesis follows immediately.

By way of example consider a 3-book B^3; that is $B^3 = T \times [0, 1]$, where T is a triod. By looking at Fig. 1 we can see the right monotone union since B^3 is a 1-1 continuous image of B_1^3.

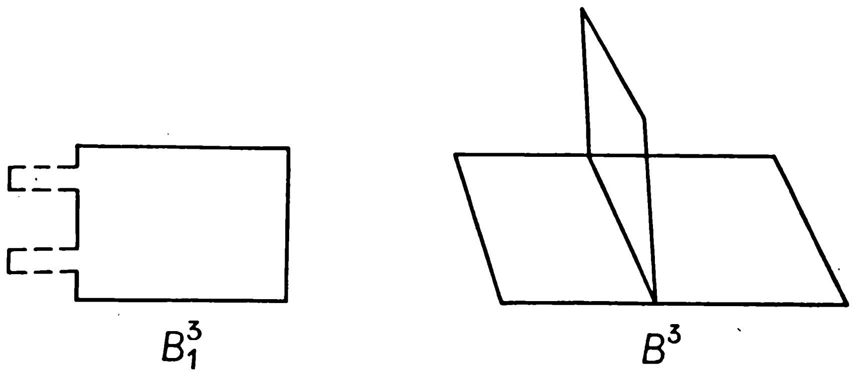

Fig. 1

4. Complexes of dimension $n \leqslant 1$. A monotonic 0-complex is a point. So let K be a monotonic 1-complex. We ask when

$$K = \bigcup_1^{\infty} I_i,$$

where $I_i = [a_i, b_i]$ is an arc with endpoints a_i, b_i while $I_i \subset I_{i+1}$. The triod shows this is not always the case, while every connected 1-complex is monotonic. Without loss of generality let $K = \bigcup_1^{\infty} I_i$, where $a_i \to a$ and $b_i \to b$. Then K is a very special continuous image of the closed interval $[0, 1]$ under $f\colon [0, 1] \to K$ such that $f|(0, 1)$ is a homeomorphism while $f(0) = a, f(1) = b$, and so $F = f^{-1}(a \cup b)$ contains at most four points.

If F contains four points then f identifies 0 or 1 with interior points c and d respectively. Then K is determined entirely by the order 0, c, d, 1. The two figures appear in Fig. 2.

If F contains 3 points, then f identifies a single interior point. The other figures appear in Fig. 2.

These figures represent the termination of a selfavoiding walk discussed in [2], if growth occurs at each end. Similar figures are found in the study of 1-1 maps as seen in [3].

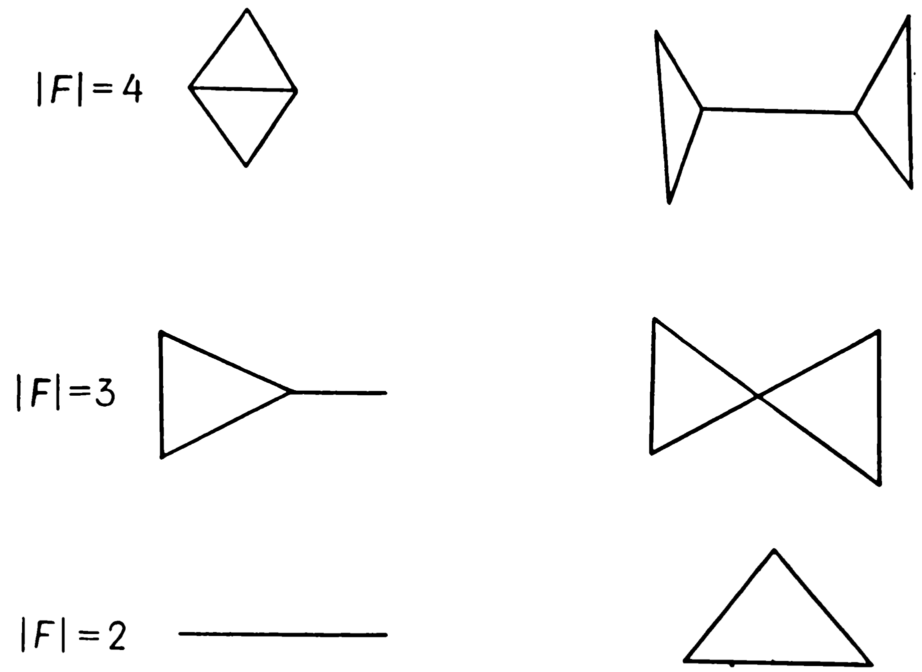

Fig. 2

5. Products and 1-1 maps. Suppose X is a topological space and X is a monotone union of closed n-cells. Then if Y is a 1-1 continuous image of X, Y has the property as well. If X and Y both have the property, then $X \times Y$ is a monotone union of closed cells. The converse of the last statement if false as shown by the product of a triod and an interval. From this remark and Theorem 1 we have.

THEOREM 2. *Let K_1 and K_2 be complexes of dimension at least 2. Then $K_1 \times K_2$ is a monotone union of closed cells if and only if each K_i is.*

6. The monotone topology. If $X = \bigcup C_i$, where $C_i \subset C_{i+1}$ is a closed n-cell, the *monotone* topology is obtained by defining a new topology on the set X. If X is an interior point of some C_i, then a basis at x is any basis at x in Int C_i. If X is never in the interior of a cell we take as basis at x all sets of form $\bigcup_{a_x}^{\infty} U_j^x$ where $x \in C_{a_x}$ and U_j^x is a neighborhood of x in C_j. Let $\tilde{X}$ be the resulting space. Then the natural map $N: \tilde{X} \to X$ is continuous. $\tilde{X}$ has a weak topology.

The above results can be generalized to infinite complexes.

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[2] H. Kesten, *On the number of self-avioiding walks,* Journal of Mathematical Physic 4 (1963), p. 960 - 969.

[3] A. Lelek and L. F. McAuley, *On hereditarily locally connectet spaces and one-to-one continuous images of a line,* Colloquium Mathematicum 27 (1967), p. 319 - 324

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