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*APPROXIMATING PAVINGS AND CONSTRUCTION OF MEASURES**

BY

FLEMMING TOPSØE (COPENHAGEN)

It is well known that for the construction of non-trivial measures one usually, if not always, reduces the problem to a purely set-theoretical property. In Marczewski's paper [4], the set-theoretical aspects were isolated by the introduction of the notion of a compact paving, and the relation to the set functions under consideration was achieved by assuming that an approximation property was satisfied.

Recall that a paving $\mathscr{K}$ on the set X is *compact* if, for any countable subpaving $\mathscr{K}_0$ of $\mathscr{K}$ with empty intersection, there exists a finite subpaving $\mathscr{K}_{00}$ of $\mathscr{K}_0$ with empty intersection.

A slight but useful variation of Marczewski's definition was recently suggested by Mallory [3]. We formulate his definition as follows. The paving $\mathscr{K}$ is *monocompact* if any decreasing sequence of $\mathscr{K}$-sets with empty intersection contains the empty set.

Let us mention some facts which are all easy to establish and which illustrate the differences between compact and monocompact pavings.

A compact paving is monocompact but the converse is not true. If $\mathscr{K}$ is monocompact, then neither the closure of $\mathscr{K}$ under finite unions nor the closure of $\mathscr{K}$ under finite disjoint unions nor the closure of $\mathscr{K}$ under countable decreasing intersections need be monocompact. So, in some respects, the property of monocompactness is not so well behaved as that of compactness. On the other hand, in other respects, we have the reverse situation; e.g., if $\mathscr{K}_1$ and $\mathscr{K}_2$ are monocompact, so is $\mathscr{K}_1 \cup \mathscr{K}_2$. Also, we may mention that if $(\mathscr{K}_i)_{i\in I}$ is a family of algebraically σ-independent pavings (i.e., for a countable subset I_0 of I and for a family $(K_i)_{i\in I_0}$ of non-empty sets from the corresponding pavings $\mathscr{K}_i$ we have $\bigcap_{I_0} K_i \neq \emptyset$) and if all the pavings $\mathscr{K}_i$ are monocompact, then so is $\bigcup_I \mathscr{K}_i$ (this property also holds for compact pavings provided we add the condition that all

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the pavings $\mathscr{K}_i$ be closed under finite intersections). Finally, we mention a property shared by both concepts: If $\pi: X \to Y$ is surjective and if $\mathscr{K}$ is a monocompact [compact] paving on Y, then $\pi^{-1}(\mathscr{K})$ is a monocompact [compact] paving on X.

Let $(X, \mathscr{A}, \mu)$ be a finite, finitely additive measure, i.e., $\mathscr{A}$ is an algebra and $\mu: \mathscr{A} \to [0, \infty[$ is finitely additive. A paving $\mathscr{K}$ on X is an *approximating paving* for μ if, for all $A \in \mathscr{A}$ and for all $\varepsilon > 0$, there exist $K \in \mathscr{K}$ and $B \in \mathscr{A}$ with $B \subseteq K \subseteq A$ and $\mu(B) > \mu(A) - \varepsilon$.

LEMMA 1. *Let $(X, \mathscr{A}, \mu)$ be a finite, finitely additive measure, let $\mathscr{A}_1 \subseteq \mathscr{A}_2 \subseteq \ldots$ be subalgebras of $\mathscr{A}$ and assume that, for each n, $\mathscr{K}_n$ is an approximating paving for $\mu \mid \mathscr{A}_n$. Then, for any decreasing sequence $A_1 \supseteq A_2 \supseteq \ldots$ with $A_n \in \mathscr{A}_n$, $n \geqslant 1$, and for any $\varepsilon > 0$, there exist sequences (B_n) and (K_n) with $B_n \in \mathscr{A}_n$, $K_n \in \mathscr{K}_n$, $n \geqslant 1$, such that the inclusions*

$$\begin{array}{ccccccccc} A_1 & & & & A_2 & & & & A_3 \\ \cup\!\mid & & & & \cup\!\mid & & & & \cup\!\mid \\ K_1 & \supseteq & B_1 & \supseteq & K_2 & \supseteq & B_2 & \supseteq & K_3 \supseteq \ldots \end{array}$$

hold and such that

$$\lim_{n\to\infty} \mu(B_n) \geqslant \lim_{n\to\infty} \mu(A_n) - \varepsilon.$$

The proof follows a well-known pattern and is left to the reader.

It follows from the lemma that a finite, finitely additive measure with a monocompact approximating paving is countably additive (and hence extends to a measure on the generated σ-algebra). Also, we see that a measure with an approximating compact paving has an approximating compact paving consisting of measurable sets. These results are, essentially, due to Aleksandroff ([1], Theorem 3.5) and to Marczewski ([4], (i) and (iii) of Section 4). For refinements see [3], Theorem 1.2, and [8], Theorem 6.1.

With Marczewski we call a measure *compact* if it has a compact approximating paving. And we call a measure *monocompact* if it has a monocompact approximating paving.

In order to formulate a further consequence of Lemma 1, extending one of the above-mentioned results, we first introduce a definition. Let $I = (I, \leqslant)$ be an upward directed set and let, for each $i \in I$, $\mathscr{K}_i$ be a paving in X. We say that the family of pavings $(\mathscr{K}_i)_{i\in I}$ is *asymptotically monocompact* if for every sequence $(i_n)_{n\geqslant 1}$ from I there exists a sequence $(j_n)_{n\geqslant 1}$ from I with $j_1 \leqslant j_2 \leqslant \ldots$ such that $j_n \geqslant i_n$ for all n and such that, for every decreasing sequence of non-empty sets $K_1 \supseteq K_2 \supseteq \ldots$ with $K_n \in \mathscr{K}_{j_n}$ for all n,

$$\bigcap_1^\infty K_n \neq \emptyset.$$

Of course, this is a sequential notion so that the terminology "sequentially asymptotically monocompact" might have been more suggestive.

We notice that $(\mathscr{K}_i)_{i\in I}$ is asymptotically monocompact if the following simpler condition is fulfilled:

For $i_1 \leqslant i_2 \leqslant \ldots$ and $K_1 \supseteq K_2 \supseteq \ldots$ with $K_n \in \mathscr{K}_{i_n}$, $K_n \neq \emptyset$ for all n, we have $\bigcap K_n \neq \emptyset$.

The drawback — if you consider it a drawback — with this condition is that it requires each individual paving $\mathscr{K}_i$ to be monocompact. Note, however, that if $\mathscr{K}_i \subseteq \mathscr{K}_j$ for all $i \leqslant j$, then the two conditions are equivalent and, in fact, amount to monocompactness of the paving $\bigcup_I \mathscr{K}_i$.

The corollary to Lemma 1, we wish to point out, is

THEOREM 1. *Let $(X, \mathscr{A}, \mu)$ be a finite, finitely additive measure. Assume that there exist an upward directed set I and pavings $\mathscr{A}_i$, $\mathscr{K}_i$ for $i \in I$ such that*

$\mathscr{A}_i$ is a subalgebra of $\mathscr{A}$, $i \in I$;

$\mathscr{A}_i \uparrow \mathscr{A}$, i.e., $\mathscr{A}_i \subseteq \mathscr{A}_j$ for $i \leqslant j$ and $\bigcup \mathscr{A}_i = \mathscr{A}$;

$\mathscr{K}_i$ is an approximating paving for $\mu|\mathscr{A}_i$, $i \in I$;

$(\mathscr{K}_i)_{i\in I}$ is asymptotically monocompact.

Then μ is countably additive.

We shall see in the sequel that μ of Theorem 1 even extends to a perfect measure.

Let $(X, \mathscr{A}, \mu)$ be a measure. If there exist I, $(\mathscr{A}_i)$ and $(\mathscr{K}_i)$ satisfying the conditions of Theorem 1, the measure μ is called *asymptotically monocompact.*

THEOREM 2. *A finite measure is perfect if and only if it is asymptotically monocompact.*

Proof. Let $(X, \mathscr{B}, \mu)$ be a perfect finite measure. By Theorem III of Ryll-Nardzewski [7], the restriction of μ to every countably generated sub-σ-algebra of $\mathscr{B}$ is compact. This easily implies that μ is asymptotically monocompact — the defining property even holds in a strengthened form, since each $\mathscr{K}_i$ may be taken compact and since the sequence (j_n) from the definition may always be chosen constant.

To prove the converse, we first state a result due to Musiał:

A finitely additive probability measure $(X, \mathscr{A}, \mu)$ extends to a perfect measure if and only if, for every sequence $\mathscr{A}_1 \subseteq \mathscr{A}_2 \subseteq \ldots$ of finite subalgebras of $\mathscr{A}$ and for every $\varepsilon > 0$, there exists a sequence $A_1 \subseteq A_2 \subseteq \ldots$ with $A_k \in \mathscr{A}_k$, $\mu(A_k) \geqslant 1-\varepsilon$ for all $k \geqslant 1$ and such that $\bigcap_1^\infty \Delta_k \neq \emptyset$ for every sequence $\Delta_1 \supseteq \Delta_2 \supseteq \ldots$, where Δ_k is an atom in $\mathscr{A}_k$ with $\Delta_k \subseteq A_k$ for all $k \geqslant 1$.

This follows from Theorem 7.2 of [5].

What we shall prove is the following strengthening of Theorem 1:

If $(X, \mathscr{A}, \mu)$ is a finitely additive probability measure and if I, $(\mathscr{A}_i)_{i\in I}$ and $(\mathscr{K}_i)_{i\in I}$ have the same properties as stated in Theorem 1, then μ extends to a perfect measure.

To prove this, we shall appeal to the above-mentioned result of Musiał. So consider a sequence $\mathscr{A}_1 \subseteq \mathscr{A}_2 \subseteq \dots$ of finite subalgebras of $\mathscr{A}$ and an $\varepsilon > 0$. First determine a Souslin scheme $\Delta(n_1, \dots, n_k)$ such that

1° For each k and each atom Δ of $\mathscr{A}_k$ there exists precisely one multi-index $(n_1, \dots, n_k)$ such that

$$\Delta(n_1, \dots, n_k) = \Delta.$$

2° If $\Delta(n_1, \dots, n_k)$ is not an atom of $\mathscr{A}_k$, then

$$\Delta(n_1, \dots, n_k) = \emptyset.$$

3° For each $n_1, \dots, n_k$,

$$\Delta(n_1, \dots, n_k) = \bigcup_1^\infty (n_1, \dots, n_k, n).$$

Now choose — using, among other things, the finiteness of the algebras $\mathscr{A}_k$ — a sequence $j_1 \leqslant j_2 \leqslant \dots$ from I such that $\mathscr{A}_k \subseteq \mathscr{A}_{j_k}$ for a l $k \geqslant 1$ and such that, for every sequence of non-empty sets $K_1 \supseteq K_2 \supseteq \dots$ with $K_k \in \mathscr{K}_{j_k}$, $k \geqslant 1$, we have

$$\bigcap_1^\infty K_k \neq \emptyset.$$

Then we construct two Souslin schemes $K(n_1, \dots, n_k)$ and $A(n_1, \dots, n_k)$ such that

4° $K(n_1, \dots, n_k) \in \mathscr{K}_{j_k}$ and $A(n_1, \dots, n_k) \in \mathscr{A}_{j_k}$.

5° $A(n_1, \dots, n_k) \subseteq K(n_1, \dots, n_k) \subseteq \Delta(n_1, \dots, n_k) \cap A(n_1, \dots, n_{k-1})$.

6° $\mu\left(\bigcup \{A(n_1, \dots, n_k) \mid (n_1, \dots, n_k) \in N^k\}\right) > 1 - \varepsilon$.

If $k = 1$ in 5°, we take $A(n_1, \dots, n_{k-1}) = X$.

Now put

$$A_k = \bigcup \{\Delta(n_1, \dots, n_k) \mid K(n_1, \dots, n_k) \neq \emptyset\}.$$

It is a matter of simple checking to show that the sets A_k have all the properties which were demanded in Musiał's result.

It follows from the theorem that every monocompact measure is perfect. This result was first obtained by Pachl, using a different method, in an oral discussion with the author. It is not unlikely that a monocompact measure is even compact, but we do not know this.

Note that, for finite measures defined on countably generated σ-algebras, the concepts "compact", "monocompact" and "perfect" coincide.

We turn to a study of projective limits of probability spaces. We refer to Musiał [5], and to the references mentioned therein. Mallory's paper [3] is particularly important for what we have in mind.

For the purposes of this paper, we call $(X_i, \mathscr{B}_i, \mu_i, \pi_{ij})_I$ a *projective system* if $I = (I, \leqslant)$ is an upward directed set, if $(X_i, \mathscr{B}_i, \mu_i)$ is a probability space for each $i \in I$, if $\pi_{ij}: X_j \to X_i$ is a measurable measure-preserving map for all $i \leqslant j$, if $\pi_{ik} = \pi_{ij}\pi_{jk}$ for all $i \leqslant j \leqslant k$, and if π_{ii} is the identity map $X_i \to X_i$ for each i.

We say that (X, π_i) is a *target space* (associated with the projective system $(X_i, \mathscr{B}_i, \mu_i, \pi_{ij})_I$) if X is a set and if, for each $i \in I$, π_i is a surjective map $X \to X_i$ such that $\pi_i = \pi_{ij}\pi_j$ for all $j \geqslant i$. Note the requirement of surjectivity. It implies — if there at all exists a target space — that all the π_{ij}'s are surjective. Had we wanted to, the requirement could have been replaced by the weaker condition that $\pi_i(X)$ be a thick subset of X_i for each i.

With regard to a given projective system and an associated target space, we consider the algebra $\mathscr{C}$ of cylinder sets, which consists of all sets of the form $\pi_i^{-1}(A)$ with $i \in I$ and $A \in \mathscr{B}_i$. On $\mathscr{C}$ we consider the finitely additive probability measure μ defined by

$$\mu(\pi_i^{-1}A) = \mu_i(A), \qquad i \in I, A \in \mathscr{B}_i.$$

We write

$$\mu = \varprojlim \mu_i.$$

This notation is not meant to imply that μ is countably additive. In fact, it is our major concern to find conditions ensuring that this is the case. The extra conditions we shall consider will involve a family of pavings $\mathscr{K}_i$ approximating the measures μ_i.

From Lemma 1 we deduce

LEMMA 2. *Let $(X_i, \mathscr{B}_i, \mu_i, \pi_{ij})_I$ be a projective system and let (X, π_i) be an associated target space. Assume that, for each i, $\mathscr{K}_i$ is an approximating paving for μ_i. Then a necessary and sufficient condition that $\varprojlim \mu_i$ be countably additive is that, for any sequence $i_1 \leqslant i_2 \leqslant \ldots$ on I, for any sequence (K_n) with $K_n \in \mathscr{K}_{i_n}$ and for any sequence (B_n) with $B_n \in \mathscr{B}_{i_n}$ such that*

$$\pi_{i_1}^{-1}(K_1) \supseteq \pi_{i_1}^{-1}(B_1) \supseteq \pi_{i_2}^{-1}(K_2) \supseteq \pi_{i_2}^{-1}(B_2) \supseteq \ldots$$

and

$$\bigcap_1^\infty \pi_{i_n}^{-1}(B_n) = \emptyset,$$

the equality

$$\lim_{n\to\infty} \mu\left(\pi_{i_n}^{-1}(B_n)\right) = 0$$

holds.

From Theorem 1 or from a slight variation of Lemma 2 we get

LEMMA 3. *If under the assumptions of Lemma 2 the family of pavings* $(\pi_i^{-1}(\mathscr{K}_i))_{i\in I}$ *is asymptotically monocompact, then* $\varprojlim \mu_i$ *is countably additive.*

The condition of this lemma involves both the family $(\mathscr{K}_i)$ and the target space. It is convenient to split the condition into two. Before doing so, we introduce some terminology.

We consider a projective system $(X_i, \mathscr{B}_i, \mu_i, \pi_{ij})_I$ and a target space (X, π_i). Let $i_1 \leqslant i_2 \leqslant \ldots$ and let $A_1, A_2, \ldots$ be sets with $A_n \subseteq X_{i_n}$, $n \geqslant 1$. Then (A_n) is called *subconsistent* if $\pi_{i_n i_m}(A_m) \subseteq A_n$ for all $n \leqslant m$ or, equivalently, if $\pi_{i_n i_{n+1}}(A_{n+1}) \subseteq A_n$ for all $n \geqslant 1$. By surjectivity of the π_i's, this is also equivalent to the sequence $(\pi_{i_n}^{-1}(A_n))$ being decreasing. We call (B_n) *subordinated* to (A_n) if $B_n \subseteq A_n$, $n \geqslant 1$. The sequence (A_n) is *consistent* if $\pi_{i_n i_m}(A_m) = A_n$ for all $n \leqslant m$.

The following notion is due to Bochner [2] (a special form was considered by Marczewski in [4], Section 6):

A projective system and an associated target space satisfy the condition of *sequential maximality* if, for all sequences $i_1 \leqslant i_2 \leqslant \ldots$ from I and for all consistent sequences $(x_n)_{n\geqslant 1}$ with $x_n \in X_{i_n}$, $n \geqslant 1$, there exists $x \in X$ such that $\pi_{i_m}(x) = x_n$ for all n.

The other notion we shall consider depends only on the projective system and is as follows.

Let, for each $i \in I$, $\mathscr{K}_i$ be a paving on X_i; then the family $(\mathscr{K}_i)_{i\in I}$ is called *projectively monocompact* if, for every sequence $i_1 \leqslant i_2 \leqslant \ldots$ from I and for every subconsistent sequence $(K_n)_{n\geqslant 1}$ of non-empty sets with $K_n \in \mathscr{K}_{i_n}$, $n \geqslant 1$, there exists a consistent sequence of points $(x_n)_{n\geqslant 1}$ subordinated to $(K_n)_{n\geqslant 1}$.

Quite clearly, combining the two concepts, we get

LEMMA 4. *Let* $(X_i, \mathscr{B}_i, \mu_i, \pi_{ij})_I$ *be a projective system and assume that* $(\mathscr{K}_i)_{i\in I}$ *is a projectively monocompact family of pavings. Then, for any target space* $(X, \pi_i)_I$ *for which the condition of sequential maximality is fulfilled, the family* $(\pi_i^{-1}(\mathscr{K}_i))_{i\in I}$ *is asymptotically monocompact.*

If, furthermore, $\pi_{ij}(\mathscr{K}_j) \subseteq \mathscr{K}_i$ for $i \leqslant j$, the projective monocompactness is also necessary for the above conclusion to hold.

In the formulation of the next result, a *chain* in a paving $\mathscr{K}$ is a subpaving $\mathscr{K}_0$ of $\mathscr{K}$ such that, for all $K_1, K_2 \in \mathscr{K}_0$, either $K_1 \subseteq K_2$ or $K_2 \subseteq K_1$ holds.

THEOREM 3. *Let* $(X_i, \mathscr{B}_i, \mu_i, \pi_{ij})_I$ *be a projective system of probability spaces and let* $(\mathscr{K}_i)_{i\in I}$ *be a family of pavings such that each* $\mathscr{K}_i$ *is an approximating paving for* μ_i *and such that* $\pi_{ij}(\mathscr{K}_j) \subseteq \mathscr{K}_i$ *for* $i \leqslant j$. *Assume that one of the following two conditions is satisfied:*

1° *For every* i *and for every chain of non-empty sets in* $\mathscr{K}_i$, *the intersection of the sets in the chain is a non-empty member of* $\mathscr{K}_i$.

2° *For each* i, $\mathcal{K}_i$ *is monocompact, and for each* $i \leqslant j$, *each* $y \in X_i$ *and each decreasing sequence* $(K_n)_{n\geqslant 1}$ *of sets in* $\mathcal{K}_j$ *with* $K_n \cap \pi_{ij}^{-1}(y) \downarrow \emptyset$, *there exists an* n *such that* $K_n \cap \pi_{ij}^{-1}(y) = \emptyset$.

Then, for every target space $(X, \pi_i)_I$ *such that the condition of sequential maximality is fulfilled,* $\overleftarrow{\lim}\,\mu_i$ *is countably additive.*

Proof. In both cases we verify projective monocompactness of $(\mathcal{K}_i)_{i\in I}$. So, let $i_1 \leqslant i_2 \leqslant \ldots$ together with a subconsistent sequence $(K_n)_{n\geqslant 1}$ of non-empty sets from the pavings $\mathcal{K}_{i_n}$ be given. To save notation, we write π_{nm} and $\mathcal{K}_n$ in place of $\pi_{i_n i_m}$ and $\mathcal{K}_{i_n}$, respectively.

Case 1°. Choose by Zorn's lemma a minimal subconsistent sequence (K_n') of non-empty sets from the pavings $\mathcal{K}_n$ subordinated to (K_n). As (K_n'') defined by

$$(*) \qquad K_n'' = \bigcap_{m\geqslant n} \pi_{nm}(K_m')$$

is of the same type, $K_n' = K_n''$ for all n. It follows that (K_n') is consistent. A consistent sequence (x_n) subordinated to (K_n'), hence also to (K_n), is now easily constructed.

Case 2°. Define K_n'' as in $(*)$ with $K_n' = K_n$. Then $K_n'' \neq \emptyset$ for all n. By a well-known argument, (K_n'') is consistent. The proof is then completed in the same way as in case 1°.

It seems difficult to formulate a simple condition sufficient for projective monocompactness (or just for countable additivity of $\overleftarrow{\lim}\,\mu_i$) and covering case 1° as well as case 2° of the theorem. Case 1° is very close to Mallory's Theorem 2.4 of [3] (but our proof is simpler).

We remark that if, in case 1°, we assume only that each of the pavings $\mathcal{K}_i$ is monocompact, we cannot conclude that $(\mathcal{K}_i)_{i\in I}$ is projectively monocompact. To see this, take $I = N$, $X_i = N^i$ for $i \geqslant 1$ and let π_{ij} be the usual projection maps. Then

$$N^{(N)} = \bigcup_1^\infty N^i$$

is a tree (we may add the empty multi-index to get a tree with a root). Let (K_n) be a subconsistent sequence. Then $T = \bigcup K_n$ is a tree. The condition that the K_n's be non-empty means that T contains arbitrarily long branches. In order that the sets K_n be members of monocompact pavings $\mathcal{K}_n$ such that $\pi_{nm}(\mathcal{K}_m) \subseteq \mathcal{K}_n$, $n \leqslant m$, it is necessary and sufficient that $K_n^* \neq \emptyset$ for all n, where

$$K_n^* = \bigcap_{m\geqslant n} \pi_{nm}(K_m).$$

In terms of T this means that at each level there is a point in T which supports arbitrarily long branches. Projective monocompactness means that T contains an infinite branch. Having translated the relevant properties into the language of trees, it is easy to construct an example as announced — the tree in the figure will do. We point out that since the X_i's in this example are countable, we cannot contradict the conclusion of Theorem 3 that $\varprojlim \mu_i$ is countably additive.

It would be interesting if one could obtain sensible results assuming that the following property, a little stronger than monocompactness, is satisfied for each paving $\mathscr{K}_i$:

The intersection of the sets in any descending sequence of non-empty members of $\mathscr{K}_i$ is a non-empty member of $\mathscr{K}_i$. With this condition, examples as the one given above are ruled out in case where the spaces X_i are countable (cf. Added in proof).

Finally, we remark that the conclusion of Theorem 3 may be strengthened. Firstly, the measures $\varprojlim \mu_i$ that appear in the conclusion are not only countably additive, they even extend to perfect measures. Secondly, the sequential maximality condition may be weakened to almost sequential maximality (cf. [5], Definition 4.5), which means that for $i_1 \leqslant i_2 \leqslant \ldots$ and $\varepsilon > 0$ there exists a subconsistent sequence (A_n) of measurable sets with $\mu_{i_n}(A_n) \geqslant 1-\varepsilon$ for all n and such that, for any consistent sequence (x_n) subordinated to (A_n), there exists $x \in X$ such that $\pi_{i_n}(x) = x_n$ for all n.

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Added in proof. David Fremlin has pointed out that examples of the type under consideration do exist with all the sets X_i of cardinality ω_1. Fremlin's construction is as follows: Take $X_n = \omega_1^n$, $n \geqslant 1$, and let π_{nm} be the usual projections. Then the sets K_n, $n \geqslant 1$, defined by

$$K_n = \{(\xi_1, \xi_2, \ldots, \xi_n) \mid \xi_1 > \xi_2 > \ldots > \xi_n\}$$

turn out to have the claimed properties.

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