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An implicit function theorem in Banach spaces

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Abstract. Let E, F, G be real Banach spaces, let $U \subseteq E$ and $V \subseteq F$ be open neighbourhoods of 0, and let $T\colon U \to V$, $S\colon V \to G$ be continuously differentiable maps such that $T(0) = 0$, $S(0) = 0$, $T(U) \subseteq S^{-1}(0)$, $\mathrm{image}(T_0') = \mathrm{kernel}(S_0')$ and $\mathrm{image}(S_0')$ is topologically closed in G. Then, for all sufficiently small $x \in S^{-1}(0)$, the equation $Ty = x$ has a solution.

If T is a differentiable map between open sets in Banach spaces, then we denote by T_x' the derivative at x. In this paper we prove

1. THEOREM. *Let E, F, G be real Banach spaces, let $U \subseteq E$, $V \subseteq F$ be open neighbourhoods of* 0, *and let $T\colon U \to V$, $S\colon V \to G$ be continuously differentiable maps such that*:

(i) $T(0) = 0$, $S(0) = 0$, $T(U) \subseteq S^{-1}(0)$;

(ii) $\mathrm{image}(T_0') = \mathrm{kernel}(S_0')$;

(iii) $\mathrm{image}(S_0')$ *is topologically closed in* G.

Then there exist $\varepsilon > 0$ and a continuous map

$$R\colon \{y \in S^{-1}(0)\colon \|y\| < \varepsilon\} \to U$$

such that $TRy = y$ for all $y \in S^{-1}(0)$ with $\|y\| < \varepsilon$.

In the case that $S = 0$ and $\mathrm{kernel}(T_0')$ is complemented in E, this theorem is well-known. For the case that $S = 0$ but $\mathrm{kernel}(T_0')$ is non-complemented, see, for instance, [1]. Theorem 1 is not valid without condition (iii). However, I. Reaburn [3] proved the following result: Let E, F, G be *complex* Banach spaces, let $U \subseteq E$, $V \subseteq F$ be open neighbourhoods of 0, and let $T\colon U \to V$, $S\colon V \to G$ be *holomorphic* maps such that conditions (i) and (ii) are fulfilled (whereas (iii) can be violated). Then there exists $\varepsilon > 0$ such that, for each open set $D \subseteq C^n$ and each *holomorphic* map $h\colon D \to S^{-1}(0)$ with $\|h(x)\| < \varepsilon$, $x \in D$, there is a *holomorphic* map $H\colon D \to U$ with $h(x) = T(H(x))$ for all $x \in D$.

Observe that Theorem 1 can be applied in the theory of holomorphic vector bundles (see [2], Section 2). This is the motivation for the present paper.

In the proof of Theorem 1 we need some lemmas. First we introduce some notations.

Let E, F be real Banach spaces. Denote by $L(E, F)$ the Banach space of bounded linear operators from E to F. Let $A \in L(E, F)$. We use the abbreviations $\operatorname{Ker} A = \operatorname{kernel}(A)$, $\operatorname{Im} A = \operatorname{image}(A)$, and we set

$$k(A) = \sup_{x \in E \setminus \operatorname{Ker} A} \frac{\operatorname{dist}(x, \operatorname{Ker} A)}{\|Ax\|},$$

where $\operatorname{dist}(x, \operatorname{Ker} A) := \inf\{\|x-y\|: y \in \operatorname{Ker} A\}$. Observe that $k(A) = \|\hat{A}^{-1}\|$, where $\hat{A}$: $E/\operatorname{Ker} A \to \operatorname{Im} A$ is the operator induced by A. By Banach's open mapping theorem, $k(A) < \infty$ precisely when $\operatorname{Im} A$ is topologically closed in F.

2. Lemma. *Let E, F be real Banach spaces, and let $A, B \in L(E, F)$. Then*:

(i) $\operatorname{dist}(x, \operatorname{Ker} B) \leqslant k(B)\|A-B\|\,\|x\|$ *for all* $x \in \operatorname{Ker} A$;

(ii) $\operatorname{dist}(x, \operatorname{Im} B) \leqslant k(A)\|A-B\|\,\|x\|$ *for all* $x \in \operatorname{Im} A$.

Proof. (i) Let $x \in \operatorname{Ker} A$. Then $\|Bx\| \leqslant \|A-B\|\,\|x\|$ and thus $\operatorname{dist}(x, \operatorname{Ker} B) \leqslant k(B)\|A-B\|\,\|x\|$.

(ii) Let $x \in \operatorname{Im} A$. Choose a sequence $y_n \in E$ such that $Ay_n = x$ and $\|y_n\| \leqslant (k(A)+1/n)\|x\|$. Then $\operatorname{dist}(x, \operatorname{Im} B) \leqslant \|x - By_n\| \leqslant \|A-B\|(k(A)+ + 1/n)\|x\|$.

3. Lemma. *Let E, F be real Banach spaces, and let $A_n, A \in L(E, F)$ such that $k(A) < \infty$ and $\lim\|A_n - A\| = 0$. Then $\underline{\lim}\, k(A_n) \geqslant k(A)$.*

Proof. Choose $x_n \in E$ such that $\|x_n\| \leqslant 2$, $\operatorname{dist}(x_n, \operatorname{Ker} A) = 1$ and $k(A) \leqslant 1/\|Ax_n\| + 1/n$. Then it follows from Lemma 2 (i) that $\underline{\lim}\, \operatorname{dist}(x_n, \operatorname{Ker} A_n) \geqslant 1$. Hence $\underline{\lim}\, k(A_n) \geqslant \underline{\lim}\, 1/\|Ax_n\| \geqslant k(A)$.

4. Lemma. *Let E, F be real Banach spaces, and let $A \in L(E, F)$. Suppose there are a map α: $F \to E$ and numbers $0 < q < 1$, $C < \infty$ such that $\|\alpha x\| \leqslant C\|x\|$ and $\|x - A\alpha x\| \leqslant q\|x\|$ for all $x \in F$. Then $\operatorname{Im} A = F$ and $k(A) = C/(1-q)$.*

Proof. Let $x \in F$. Set $y_0 = 0$ and $y_{k+1} = y_k + \alpha(x - Ay_k)$ for $k = 1, 2, \ldots$ Then $\|y_{k+1} - y_k\| \leqslant q^k C\|x\|$ and $\|x - Ay_k\| \leqslant q^k\|x\|$. Therefore $y := \lim y_k$ exists and $Ay = x$, $\|y\| \leqslant \sum \|y_{k+1} - y_k\| \leqslant C\|x\|/(1-q)$.

5. Lemma. *Let E, F, G be real Banach spaces, and let $A \in L(E, F)$, $B \in L(F, G)$ such that*

(1) $$\operatorname{Im} A = \operatorname{Ker} B$$

and

(2) $$k(B) < \infty.$$

If $A_n \in L(E, F)$ *and* $B_n \in L(F, G)$ *are sequences with*

$$\lim \|A_n - A\| = \lim \|B_n - B\| = 0 \tag{3}$$

and

$$\operatorname{Im} A_n \subseteq \operatorname{Ker} B_n \quad \textit{for all } n, \tag{4}$$

then

$$\lim k(A_n) = k(A), \tag{5}$$

$$\lim k(B_n) = k(B), \tag{6}$$

and, for sufficiently large n,

$$\operatorname{Im} A_n = \operatorname{Ker} B_n. \tag{7}$$

Proof. By (1) and (2), $k(A) < \infty$ and $k(B) < \infty$. By Lemma 2 (i), we can find maps γ_n: $\operatorname{Ker} B_n \to \operatorname{Ker} B$ such that $\|\gamma_n x - x\| \leqslant 2k(B)\|B - B_n\| \|x\|$ for all $x \in \operatorname{Ker} B_n$. In view of (1) there are maps α_n: $\operatorname{Ker} B \to E$ such that $A\alpha_n x = x$ and $\|\alpha_n x\| \leqslant (1 + 1/n) k(A)\|x\|$ for all $x \in \operatorname{Ker} B$. Set $\vartheta_n = \alpha_n \circ \gamma_n$. Then for all $x \in \operatorname{Ker} B_n$

$$\|\vartheta_n x\| \leqslant \frac{n+1}{n} k(A)\big(1 + 2k(B)\|B - B_n\|\big)\|x\|$$

and

$$\|x - A_n \vartheta_n x\| \leqslant 2k(B)\|B - B_n\| \|x\|.$$

By Lemmas 3 and 4 this implies (5) and (7). It remains to prove (6). To do this we choose $x_n \in F$ with

$$\operatorname{dist}(x_n, \operatorname{Ker} B_n) = 1, \qquad \|x_n\| \leqslant 2, \tag{8}$$

and

$$k(B_n) \leqslant \frac{1}{\|B_n x_n\|} + \frac{1}{n}. \tag{9}$$

In view of (1) and (7) it follows from Lemma 2 (ii) that $\operatorname{dist}(x, \operatorname{Ker} B_n) \leqslant k(A)\|A_n - A\| \|x\|$ for all $x \in \operatorname{Ker} B$. By (8) this implies that

$$\underline{\lim} \operatorname{dist}(x_n, \operatorname{Ker} B) \geqslant 1.$$

Taking into account (9) and (8), therefore it follows

$$\overline{\lim}\, k(B_n) = \overline{\lim} \frac{\operatorname{dist}(x_n, \operatorname{Ker} B_n)}{\|Bx_n\| - 2\|B - B_n\|} \leqslant k(B).$$

Together with Lemma 3 this gives (6).

Proof of Theorem 1. Since $STx = 0$ for $x \in U$, it follows that $\operatorname{Im} T'_x \subseteq \operatorname{Ker} S'_{Tx}$ for all $x \in U$. Hence, by Lemma 5, after shrinking U we can assume that

$$\operatorname{Im} T'_x = \operatorname{Ker} S'_{Tx} \quad \text{for all } x \in U, \tag{10}$$

and, for some constant $1 < C < \infty$,

$$k(T'_x),\ k(S'_{Tx}) \leqslant \tfrac{1}{32}C \quad \text{for all } x \in U. \tag{11}$$

By Taylor's formula, after shrinking U and V, we can further assume that

$$\|Ty - Tx - T'_x(y-x)\| \leqslant \frac{1}{C}\|y-x\| \quad \text{for all } x, y \in U, \tag{12}$$

and

$$\|Sy - Sx - S'_x(y-x)\| \leqslant \frac{1}{C}\|y-x\| \quad \text{for all } x, y \in V. \tag{13}$$

Choose $\varepsilon > 0$ so small that all $y \in E$ with $\|y\| \leqslant \varepsilon C$ belong to U. Set

$$M = \{x \in S^{-1}(0)\colon \|x\| \leqslant \varepsilon\}.$$

Claim. Let $\delta \leqslant \varepsilon$, and let $f\colon M \to E$ be a continuous map such that

$$\|f(x)\| \leqslant \tfrac{1}{2}\varepsilon C \quad \text{for } x \in M, \tag{14}$$

and

$$\|Tf(x) - x\| \leqslant \delta \quad \text{for } x \in M. \tag{15}$$

Then there exists a continuous map $\tilde{f}\colon M \to E$ with

$$\|\tilde{f}(x) - f(x)\| \leqslant \tfrac{1}{4}\delta C \quad \text{for } x \in M, \tag{16}$$

and

$$\|T\tilde{f}(x) - x\| \leqslant \tfrac{1}{2}\delta \quad \text{for } x \in M. \tag{17}$$

Proof of the claim. Since $STf = 0$, it follows from (13) and (15) that

$$\|S'_{Tf(x)}(Tf(x) - x)\| \leqslant \delta/C \quad \text{for } x \in M.$$

By (11) and a partition of unity argument, we can therefore find a continuous map $w\colon M \to E$ such that

$$\|w(x)\| \leqslant \tfrac{1}{16}\delta \quad \text{for } x \in M, \tag{18}$$

and

$$\|S'_{Tf(x)}(Tf(x) - x - w(x))\| \leqslant \delta/C \quad \text{for } x \in M.$$

Hence by (11)

$$\operatorname{dist}(Tf(x) - x - w(x),\ \operatorname{Ker} S'_{Tf(x)}) \leqslant \tfrac{1}{32}\delta \quad \text{for } x \in M.$$

By (10), (11) and a partition of unity argument, we thus obtain a continuous map $v\colon M \to E$ such that

$$(19)\qquad \|v(x)\| \leqslant \tfrac{1}{16}C\big(\|Tf(x)-x-w(x)\|+\tfrac{1}{16}\delta\big) \quad \text{for } x \in M,$$

and

$$(20)\qquad \big\|T'_{f(x)}v(x)-\big(Tf(x)-x-w(x)\big)\big\| \leqslant \tfrac{1}{16}\delta \quad \text{for } x \in M.$$

(19), (18) and (15) imply

$$(21)\qquad \|v(x)\| \leqslant \tfrac{1}{4}\delta C \quad \text{for } x \in M.$$

Set $\tilde{f} = f - v$. Then (16) is (21), and (17) follows from (12), (20), (18) and (21):

$$\|T\tilde{f}(x)-x\| \leqslant \|T\tilde{f}(x)-Tf(x)+T'_{f(x)}v(x)\|+\|Tf(x)-T'_{f(x)}v(x)-x\|$$
$$\leqslant \frac{1}{C}\|v(x)\|+\|w(x)\|+\tfrac{1}{16}\delta \leqslant \tfrac{1}{2}\delta \quad \text{for } x \in M.$$

The claim is proved.

Set $f_0(x) = 0$ for $x \in M$. Then by the claim we can find a sequence of continuous maps $f_k\colon M \to E$ such that

$$\|f_k(x)-f_{k-1}(x)\| \leqslant 2^{-k-1}\varepsilon C \quad \text{for } x \in M,$$

and

$$\|Tf_k(x)-x\| \leqslant 2^{-k}\varepsilon \quad \text{for } x \in M.$$

Then $R := \lim f_k$ exists uniformly on M and $TRx = x$ for $x \in M$.

References

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