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THE HOMEOMORPHISM GROUP OF A THREE-DIMENSIONAL POLYHEDRON IS LOCALLY CONTRACTIBLE

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1. Introduction. Siebenmann has proved in [4] that the space of homeomorphisms of a compact polyhedron is locally contractible. The author reproves this result* for 3-dimensional polyhedra using the same techniques that Brown [1] uses to prove the Hauptvermutung for 3-complexes, the most important one being the Munkres notion [3] of a composition space associated with a complex. With this tool we can analyze the structure of a 3-complex so that we reduce the theorem to the result for 3-manifolds which has been proved in [2] in much greater generality.

A. *Composition space.* If we call those points of the 3-complex K that have no open cell neighborhood the *singular points* of K, then the composition space $(\tilde{K}, p)$ is the result of tearing apart the original complex along the singular set. In 3-dimensions, $\tilde{K}$ is a complex such that the star of each vertex is the cone over a 2-manifold. To be more precise we define the following functions on $|K|$: $d(x)$ is the *local dimension* of $|K|$ at x, i.e., $d(x)$ is the largest integer such that x belongs to the closure of a $d(x)$-simplex. $i(x)$ is the *index* at x. It is the largest integer such that x belongs to the interior of an $i(x)$-simplex in some triangulation of some neighborhood of x. Let $b|K| = \{x \mid i(x) < d(x)\}$, where $b|K|$ is called the *singular set* of $|K|$. Notice that components of $|K|-b|K|$ are manifolds, for $b|K|$ consists exactly of those points which have no neighborhood homeomorphic to the interior of a cell. In addition, one can show $b|K|$ is the polyhedron of a subcomplex bK of K. We define $s(x)$, the *singularity* of x, to be the smallest integer k so that, for arbitrarily small neighborhoods U of x in $|K|$, $U-b|K|$ has k components. In fact, it is not hard to show that if σ is a simplex of K, then the constant value of $s(x)$ on σ is equal to the number of components in $\mathrm{st}(\sigma; K)-b|K|$. Following [1] closely we can now define the composition space $(\tilde{K}, p)$ of K as follows.

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Let σ be a simplex of K' (the derived complex of K). For each component $C(\sigma, i)$ of $\mathrm{st}(\sigma; K') - b|K'|$ we associate a simplex σ_i of $\tilde{K}$ having the same dimension as σ. We write $p(\sigma_i) = \sigma$. Now, I_j will be a face of σ_i provided $p(I_j)$ is a face of $p(\sigma_i)$ and $C(I, j) \supset C(\sigma, i)$. One can check now that, in fact, $\tilde{K}$ is a simplicial complex, and thus p is a simplicial map.

Let us denote by $|K(r)|$ the points of index less than or equal to r. It is the polyhedron of a subcomplex $K(r)$ of K. We denote by $|k(r, m)|$ the points of index r and singularity m. Now, $|K(r, m)|$ is the union of a set $k(r, m)$ of simplexes of K, but is not, in general, closed. We now quote two of Brown's results in [1] which we will be using extensively. We will give it with the numbering that he uses.

(3.9) *If K is a complex with composition space $(\tilde{K}, p)$, then the restriction of p to $p^{-1}|K(r, m)|$ is an m-sheeted covering map onto $|K(r, m)|$.*

(2.8) (Isotopy Lifting Theorem). *Let K and L be complexes, and let $f_t\colon |K| \to |L|$, $0 \leqslant t \leqslant 1$, be an isotopy. For each t there exists a unique continuous map $\tilde{f}_t\colon |K| \to |\tilde{L}|$ so that the diagram*

$$\begin{array}{ccc} |\tilde{K}| & \xrightarrow{\tilde{f}_t} & |\tilde{L}| \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ |K| & \xrightarrow{f_t} & |L| \end{array}$$

commutes. The map $\tilde{f}_t$ is a homeomorphism, and if f_t is piecewise linear on the subcomplex M of K, then $\tilde{f}_t$ is piecewise linear on $p^{-1}(M')$. Finally, the family $\tilde{f}_t$ is an isotopy.

B. *Deformations.* We follow Edwards and Kirby [2] here. Suppose X is a space and A and B are subspaces. A *deformation* of A into B is a map $\varphi\colon A \times I \to X$ such that $\varphi|A \times 0 = 1_A$ and $\varphi(A \times \{1\}) \subset B$. All the deformations in this paper will be deformations of a neighborhood P of 1_X in $H(X)$, the *space of homeomorphisms* of X. For a subspace M of X, a deformation $\varphi\colon P \times I \to H(X)$ is *modulo M* if $\varphi(h, t)|M = h|M$ for all $h \in P$ and $t \in I$. Suppose

$$\varphi\colon P \times I \to H(X) \quad \text{and} \quad \psi\colon Q \times I \to H(X)$$

are deformations of subsets of $H(X)$ such that $\varphi(P \times \{1\}) \subset Q$. Then the composition of ψ with φ, denoted by $\psi * \varphi$, is defined by

$$(\psi*\varphi)(h, t) = \begin{cases} \varphi(h, 2t) & \text{if } t \in [0, \tfrac{1}{2}], \\ \psi(\varphi(h, 1),\ 2t-1) & \text{if } t \in [\tfrac{1}{2}, 1]. \end{cases}$$

C. *Miscellaneous notation.* If $|L| \subseteq |K|$, $P \subseteq H(|K|)$, and each homeomorphism of P takes $|L|$ into $|L|$, then it makes sense to regard P

in $H(|L|)$. Denote it by $P \mid |L|$. Using (2.8), (3.10) and (3.11) of [1] we have a map from $H(|K|)$ into $H(|\tilde{K}|)$. Denote the image of $P \subseteq H(|K|)$ by $\tilde{P}$. Moreover, we will see that the above-mentioned results of Brown allows us to use a deformation in $H(|K|)$ to define one in $H(|\tilde{K}|)$.

2. Proof of the Theorem. In this section we first give the thread of the argument used to prove the result. Next we state and prove the necessary lemmas, and end with the proof of the main result.

If $H_1(|K|, |M|)$ denotes the space of homeomorphisms of K which are the identity on a subcomplex M, then we are looking for a neighborhood P of 1_K in $H_1(|K|, |M|)$ and a deformation of P to 1_K. Denote by K_1 the set of points of K of index less than or equal to 1. Lemma 1 below provides us with a neighborhood P in $H_1(|K|, |M|)$ and a deformation φ of P into $H_1(|K|, |M \cup K_1|)$.

Next restrict the homeomorphisms in $\varphi(P \times I)$ to the total singular set $b|K|$, where $b|K|$ is at most 2-dimensional. Thus we can apply the 2-dimensional result given in Lemma 2 below to the homeomorphisms restricted to $b|K|$. If one has been sufficiently careful at this point to choose the deformation "small" enough, then one can lift each of the homeomorphisms in P to $|\tilde{K}|$ and also the deformation to $p^{-1}(b|K|)$, where p is the projection of $\tilde{K}$ onto K. The deformation is extended to all of $|\tilde{K}|$. Further deformations isotope the homeomorphisms in $\tilde{P}$ to the identity on a neighborhood of $p^{-1}(b|K| \cup |M|)$. Removal of this neighborhood from $|\tilde{K}|$ leaves a manifold with boundary. Here the Kirby and Edward result [2] for manifolds provides the final deformation. Then (3.11) of [1] allows us to push everything down to K, concluding the proof.

LEMMA 1. *Suppose L is a 1-dimensional subcomplex of a complex K such that $h(|L|) = |L|$ for all h in $H(|K|)$. Then there exist a neighborhood P of 1_K in $H_1(|K|, |M|)$ and a deformation φ of P into $H_1(|K|, |M \cup L|)$. Furthermore, φ is modulo $|K\text{-st}^{(2)}(L; K)|$, where $\text{st}^{(2)}$ means a star in the 2-nd barycentric subdivision of K.*

Proof. If $\sigma \in M \cap \text{st}(L; K)$, then the deformation must remain fixed on σ. Thus, if $\sigma \in L$, we must write $(\varphi_t h)(x) = h(x) = x$ for x in $|\sigma|$. If σ has a 1-simplex of L as a face, we must again write $\varphi_t h(x) = x$ for $x \in |\sigma|$. Finally, if σ has only a vertex v which is in L, then v will, in fact, be an end of an interval I_α on which the deformation will be fixed. Initially choose a neighborhood P of 1_K in $H_1(|K|, |M|)$ so that for all x in $|K|$, and $h \in P$, $\varrho(h(x), x)$ is less than the distance between any two vertices of K. Let

$$S = \{x \mid x \in L, h(x) = x \text{ for } h \in P\}.$$

Each component of $L-S$ has a constant index and singularity with respect to K.

$L-S$ being 1-dimensional, we infer that each component is either an open interval I_a or a simple closed curve C_a on which P is invariant. Let p_a be a point on C_a. We will first show there is a deformation of P in $H_1(|K|, |M|)$ into $H_1(|K|, |M \cup \bigcup_a p_a|)$. This will be accomplished by a composition of deformations φ_a, one for each C_a. Instead of keeping track of these deformations and their end result on P (i.e., $\varphi_a(P \times \{1\})$, we will start fresh each time calling P the resulting deformed neighborhood of 1.

Case 1. Suppose some C_a is the total set of singular points in st C_a. Since $|\tilde{K}-\tilde{K}_0|$ is a manifold ((2.9) of [1]), regular neighborhoods of components of $p^{-1}(C_a)$ are either homeomorphic to $D \times C_a$, where D is a disc, or to $I \times C_a$, where $I = [0,1]$. In the latter case C_a sits on the boundary of $|\tilde{K}-\tilde{K}_0|$. By (2.8) of [1], each homeomorphism h in $H(|K|)$ has a unique lift $\tilde{h}$ to $H(|\tilde{K}|)$. Moreover, by (3.10) of [1], we can, in fact, lift h in $H_1(|K|, |M|)$ to $\tilde{h}$ in $H_1(|\tilde{K}|, p^{-1}(|M|))$. Call by $\tilde{P}$ the image of P in $H_1(|\tilde{K}|, p^{-1}(|M|))$. Let $p_a^j \epsilon p^{-1}(p_a)$ for $p_a \epsilon C_a$. It is clear from the structure of the neighborhoods of components of $p^{-1}(C_a)$ that there is a deformation $\tilde{\varphi}_a$ of $\tilde{P}$ in $H_1(\tilde{K}, \tilde{M})$ into $H_1(\tilde{K}, \tilde{M} \cup p^{-1}(p_a))$ which is modulo the boundary of the regular neighborhoods. Furthermore, the deformation can be done in such a way as to give rise to a desirable deformation φ_a on $H_1(|K|, |M|)$. We do this for each C_a which locally is the total singular set and use the composition of these deformations.

Case 2. If some C_a lies on a singular set of dimension two, then each component C_a^j of $p^{-1}(C_a)$ is in the boundary of $|\tilde{K}-\tilde{K}_0|$. Either C_a^j is again a bounding curve of $|\tilde{K}-\tilde{K}_0|$ or it can be considered as the center curve of a 2-dimensional regular neighborhood of C_a^j on the boundary. In either case, simple deformations can be performed, and in the latter case one can use collar neighborhoods of the boundary to extend the deformation to all of $\tilde{K}$, so that now we may assume that our neighborhood P has been deformed into

$$H_1\left(|K|, \left|M \cup \bigcup_a p_a\right|\right).$$

Using our convention we continue the argument calling this resulting neighborhood P. We continue assuming now that each component of $L-S$ is an open interval.

For each component I_a, there is a deformation ψ_a of $P|I_a$ in $H_1(I_a, I_a)$ to 1_{I_a}. Now $P|L$ makes sense and each ψ_a can be regarded as a deformation in $H(L)$. The composition of the ψ_a's gives a deformation ψ_L on

$P|L$ in $H_1(L, S)$ to 1_L. Note that ψ_L preserves index and singularity with respect to K.

We will now extend ψ_L to $H_1(|K|, |M|)$ by extending it to $H(\mathrm{cl}(\mathrm{st}^{(2)} I_a))$ modulo $\mathrm{cl}(\mathrm{st}^{(2)} I_a) - \mathrm{st}^{(2)} I_a$. We do this by separating out three cases.

(1) There are no singular points on $\mathrm{st}^{(2)} I_a$.

(2) There are only points of index one on $\mathrm{st}^{(2)} I_a$.

(3) $\mathrm{st}^{(2)} I_a$ has points of index two.

These cases are exhaustive, since I_a has no points of index zero.

Case 1. Clearly, by regular neighborhood theory, (1) presents no difficulties in extending ψ_L to $H(\mathrm{cl}(\mathrm{st}^{(2)} I_a))$.

Case 2. We infer that $\mathrm{st}^{(2)} I_a$ has no singular 2-simplexes. Moreover, $\mathrm{st}^{(2)} I_a - I_a$ is non-singular, since all singular 1-simplexes are on I_a. Hence each component I_a^j of $p^{-1}(I_a)$ has a regular neighborhood $\mathrm{st}^{(2)} I_a^j$. Now, $\mathrm{st}^{(2)} I_a^j - I_a^j$ is homeomorphic *via* $p: \tilde{K} \to K$ to a component C_j of $\mathrm{st}^{(2)} I_a - I_a$. Using the fact that $\mathrm{cl}\, C_j$ is homeomorphic to $\mathrm{cl}(\mathrm{st}^{(2)} I_a^j - I_a^j)$ we infer, in the case where $\dim \mathrm{st}^{(2)} I_a = 2$, that $\mathrm{cl}\, C_j$ is homeomorphic to $(\mathrm{cl}\, I_a) \times I$ if $\mathrm{cl}\, I_a$ is an interval or to $(\mathrm{cl}\, I_a) \times (I/\partial I_a) \times I$, where I_a is identified with $I_a \times \{0\}$, if $\mathrm{cl}\, I_a$ is a simple closed curve. In the case where $\dim \mathrm{st}^{(2)} I_a = 3$, $\mathrm{cl}\, C_j$ is a solid cylinder with I_a as core or a solid cylinder with ∂I_a identified if $\mathrm{cl}\, I_a$ is a simple closed curve. By (2.8) of [1], we can lift the deformation ψ_L to a deformation $\tilde{\psi}_L$ of $\tilde{P}| \bigcup_{j,a} \mathrm{cl}\, I_a^j$ to the identity on $\bigcup_{j,a} \mathrm{cl}\, I_a^j$. Using regular neighborhoods of $(\mathrm{cl}\, I_a^j)$'s in $\tilde{K}$, which are homeomorphic to the C_j's described above, we can extend $\tilde{\psi}_L$ to all of $\tilde{K}$ so that it is modulo $\mathrm{cl}(\tilde{K} - \bigcup C_j)$. Since we can be careful to preserve index and singularity, (3.11) of [1] allows us to "lower" this deformation to K and we now have the desired deformation in the case where $\mathrm{st}^{(2)} I_a$ has no singular 2-simplexes.

We start Case 3 assuming that P is the deformed neighborhood gotten from Cases 1 and 2.

Case 3. Here we treat the situation in which there are singular 2-simplexes. $I_a \subseteq \mathrm{cl}\, K(2, s)$ for some s, since the singularity function $s(x)$ is constant on open simplexes of K.

Hence all components of $p^{-1}(I_a)$ are homeomorphic, by the composition function p, to I_a. Note that $p^{-1}(K(2, s))$ is contained in the boundary of $|\tilde{K} - \tilde{K}_0|$. So, for each component I_a^j of $p^{-1}(I_a)$, we can define a deformation first on a neighborhood of I_a^j in the boundary of $|\tilde{K} - \tilde{K}_0|$ and then extend it to all of $|\tilde{K}|$ by using a collar neighborhood of the set. Taking care to use the "same" deformation on each neighborhood of I_a^j will ensure that we can "lower" this deformation to get our final desired peformation.

The next result is the relative version of the local contractibility of the homeomorphism group of a complex of dimension 2. The following proposition is the statement of Corollary 7.3 in [2] for dimension 2.

PROPOSITION. *Let M be a compact 2-manifold with boundary and let A be a set of isolated points interior to M. Then the homeomorphism group $H(M)$ is locally contractible in such a way that the contraction takes $H_1(M, \partial M \cup A)$ into itself.*

LEMMA 2. *Let K be a 2-complex and M a subcomplex of K. Then $H_1(|K|, |M|)$ is locally contractible.*

Proof. We begin as Brown [1] does in his proof of the Hauptvermutung for 3-complexes. First eliminate $\operatorname{int}|M|$, since all maps must be the identity there. Start anew then assuming $\dim M \leqslant 1$. We further change K so that all 1-simplexes of M are in bK by adding a vertex v_σ for each 1-simplex of M not in bK, and adding $\operatorname{cl}\sigma * v_\sigma$ to K. Extend all homeomorphisms of $H_1(|K|, |M|)$ to be the identity on these new simplexes. Notice during the proof that, since the deformations are small, our original complex will be mapped into itself at each stage and that the deformation will be modulo our original subcomplex M.

Let $K_1 = \{x \mid i(x) < 2\}$. K_1 is invariant under $H(|K|)$, since homeomorphisms preserve index. By Lemma 1, there is a neighborhood P of $H_1(|K|, |M|)$ and a deformation φ of P into a subset of $H_1(K, |M \cup K_1|)$. We continue, calling this deformed set P.

Taking $(\tilde{K}, p)$ to be the composition space of K, we let $N = \{x \mid x \in \tilde{K}, d(x) = 2\}$. By the Munkres result (see (2.9) in [1]), N is a 2-manifold with boundary. From (3.10) of [1], the image $\tilde{P}$ of P in $H(|\tilde{K}|)$ is, in fact, contained in

$$H_1\big(|K|, p^{-1}(b|K| \cup |M|) \cap N\big).$$

The set $p^{-1}(b|K| \cup |M|) \cap N$ consists of ∂N, and a set A of isolated points interior to N. We restrict $\tilde{P}$ to N and apply the Proposition to get a deformation ψ_N of a neighborhood Q of 1_N in $\tilde{P}$ to 1_N in $H_1(N, \partial N \cup A)$. Since

$$N \cap \operatorname{cl}(|\tilde{K}| - N) \subseteq p^{-1}(b|K|)$$

and each h in $\tilde{P}$ is already the identity on $p^{-1}(b|K|)$, the deformation ψ_N trivially extends to a deformation $\tilde{\psi}$ of a neighborhood R of $1_{\tilde{K}}$ ($R \subseteq \tilde{P}$) in $H_1\big(K, p^{-1}(bK \cup M)\big)$. But since $\dim K = 2$, we have $\tilde{K}_1 = p^{-1}(b|K|)$. Hence $\tilde{\psi}$ contracts R to $1_{\tilde{K}}$. By (3.11) of [1], $\tilde{\psi}$ can be lowered to K which completes the proof.

THEOREM. *Let K be a 3-complex and M a subcomplex. Then $H_1(|K|, |M|)$ is locally contractible.*

Proof. Step 1. Reductions. As in the proof of Lemma 2, we may assume that $\dim M \leqslant 2$ and that all 2-simplexes of M are singular. If K_1 is the set of points of index less than two, then there are, by Lemma 1, a neighborhood P and a deformation φ of P into $H_1(|K|, |M \cup K_1|)$. Now this will allow us to start afresh under the assumption that all points of $|K|$ have dimension three. For if A is the set of points of dimension less than three, $\mathrm{cl}\, A - A \subseteq K_1$ on which $\varphi(P \times \{1\})$ is already the identity. Hence, by Lemma 2, there is a deformation ψ_A of some smaller neighborhood of $1_{\mathrm{cl}\, A}$ in $H_1(\mathrm{cl} A, \mathrm{cl} A - A)$ to $1_{\mathrm{cl}\, A}$. This deformation ψ_A, obviously, extends to a deformation on $H(|K|)$.

Step 2. Let us start here assuming P has been chosen small enough so that the deformations used so far made sense on it. Further, as before, let us call P the result of the deformations performed so far. Let $K(2, s)$ be those points of index 2 and singularity s. Note that if $h \in P$, then $h(\mathrm{cl} K(2, s)) \subseteq \mathrm{cl} K(2, s)$, and that $\mathrm{cl} K(2, s) - K(2, s) \subseteq K_1$. So it makes sense to restrict P to the pair

$$\big(\mathrm{cl} K(2, s), (K_1 \cup M) \cap \mathrm{cl} K(2, s)\big).$$

Lemma 2 essentially gives us a deformation $\psi_{K(2,s)}$ of $P | \mathrm{cl} K(2, s)$ to $1_{\mathrm{cl} K(2,s)}$. Each $\psi_{K(2,s)}$ can be extended trivially to $P \mid b|K|$. The composition of all these deformations gives a deformation ψ_{bK} of $P| bK$ to 1_{bK}.

Step 3. Since ψ_{bK} preserves index and singularity with respect to K, we use (3.10) of [1] to lift this deformation to $\tilde{\psi}_{bK}$ on a neighborhood of the identity in $H_1\big(p^{-1}(bK), p^{-1}(M \cup K_1)\big)$. We now extend $\tilde{\psi}_{bK}$ to a deformation of a neighborhood of $1_{\tilde{K}}$ in $H(\tilde{K})$ as follows.

Let K_0 be the points of $|K|$ of index zero. By (2.9) of [1], $|\tilde{K} - \tilde{K}_0|$ is a manifold with boundary. $\tilde{\psi}_{bK}$ is already defined on $\tilde{P} | \partial(\tilde{K} - \tilde{K}_0)$, since $\partial(\tilde{K} - \tilde{K}_0)$ are points of index 2, and hence singular.

Let

$$N = \mathrm{cl}\big(p^{-1}(|bK \cup M|) \cap \mathrm{int}\, |K - K_0|\big),$$

where $\mathrm{int}\, |K - K_0|$ means the manifold interior of $|K - K_0|$. This is a 1-complex lying mostly interior to the manifold $|\tilde{K} - \tilde{K}_0|$. The deformations must be modulo N. Let R be the $\tilde{K}$-closure of the manifold boundary of $|\tilde{K} - \tilde{K}_0|$. The closed star of $R - N$ in the 2-nd barycentric subdivision of $\tilde{K}$ is a regular neighborhood of the manifold boundary together with the points $N \cap R$. The structure of this set around a point $N \cap R$ is a cone over a 2-manifold. Since it is a regular neighborhood around the manifold boundary points, give these points a product structure, namely, a homeomorphism f from $[\mathrm{cl}\, \partial\, |\tilde{K} - \tilde{K}_0| - N] \times I$ onto this regular neighborhood which respects the cone structure. Let $f(x, 0) = x$. We now extend $\tilde{\psi}_{bK}$ to all of $\tilde{P}$ as follows (it is denoted by $\tilde{\psi}$):

$$\tilde{\psi}(h, t)f(x, s) = hf\big(h^{-1}[\tilde{\psi}_{bK}(h, t-s)](x), s\big) \quad \text{for } 0 \leqslant s \leqslant t,$$

$$\tilde{\psi}(h, t)(p) = h(p) \quad \text{for } p \notin [\mathrm{cl}\,\partial(\tilde{K} - \tilde{K}_0) - N] \times [0, t].$$

Since $f(x, 0) = x$, $\tilde{\psi}$ is an extension of $\tilde{\psi}_{bK}$, and since $\tilde{\psi}_{bK}(h, 0) = h$, $\tilde{\psi}$ is a well-defined deformation. Note also that $\tilde{\psi}$ is modulo $p^{-1}(|M|)$. Thus $\tilde{\psi}$ takes $\tilde{P}$ into $H_1\big(|\tilde{K}|, p^{-1}(bK \cup |M|)\big)$.

Step 4. We now find a further deformation of $\tilde{P}$ to a set in which each homeomorphism is the identity in a neighborhood of $p^{-1}(|bK \cup M|)$. Small neighborhoods in $|\tilde{K}|$ of vertices of $p^{-1}(|bK \cup M|)$ are homeomorphic to cones over 2-manifolds. Using Theorem 5.1 of [2] we get a neighborhood $\tilde{Q}$ of 1 in $H_1\big(|\tilde{K}|, p^{-1}(|bK \cup M|)\big)$ and a deformation $\tilde{u}_1$ of $\tilde{Q}$ such that, for each vertex v in $p^{-1}(|bK \cup M|)$, there is a "cone" neighborhood N_v, where $h(N_v) = N_v$ for $h \in u_1(\tilde{Q} \times \{1\})$. Moreover, each h is the identity on the "top" of the cone. Using the cone structure of N_v it is easy now to get a deformation $\tilde{u}_2$ so that $\tilde{u}_2 * \tilde{u}_1$ contracts $\tilde{Q}$ to

$$H_1\big(|\tilde{K}|, p^{-1}(|bK \cup M|) \cup \bigcup_v N_v\big).$$

Let

$$\tilde{K}^1 = |\tilde{K}| - \bigcup_v \mathrm{int}\, N_v \quad \text{for all } v \text{ in } p^{-1}(bK \cup M).$$

Notice that the homeomorphisms of $\tilde{u}_2 * \tilde{u}_1(Q \times \{1\})$ are the identity on the boundary of $\tilde{K}^1$. The intersection $p^{-1}(M \cup K_1) \cap \tilde{K}^1$ is a family of properly imbedded arcs on which these homeomorphisms are already the identity. The 2-nd regular neighborhood of such an arc is homeomorphic to the product of the arc with a disc, the end discs belonging to the boundary. Again apply Theorem 5.1 of [2] to find a neighborhood $\tilde{R}$ of 1 in $H\big(\tilde{K}, p^{-1}(bK \cup M)\big)$ and a deformation $\tilde{u}_3$ so that homeomorphisms in $\tilde{u}_3(\tilde{R} \times \{1\})$ are the identity on these neighborhoods of the arcs in $p^{-1}(M \cup K_1) \cap \tilde{K}^1$.

Let $\tilde{K}^2$ be the complement of the interior of these neighborhoods in $\tilde{K}^1$. Homeomorphisms in $\tilde{u}_3(R \times \{1\})$ are the identity on $\partial \tilde{K}^2$. A final application of Theorem 5.1 of [1] yields a neighborhood $\tilde{S}$ of 1 in $H_1(\tilde{K}^2, \partial \tilde{K}^2)$ and a deformation $\tilde{u}_4$ of $\tilde{S}$ to $1_{\tilde{K}^2}$. Continuing to use the same names, $\tilde{S}$ can be regarded as homeomorphisms on $\tilde{K}$ which are contracted by $\tilde{u}_4$ to homeomorphisms which are the identity on a neighborhood of $p^{-1}(|bK \cup M|)$. Thus $\tilde{u}_4 * \tilde{u}_3 * \tilde{u}_2 * u_1$ takes a neighborhood of 1 in $H_1(\tilde{K}, p^{-1}(M))$ to $1_{\tilde{K}}$. According to (3.11) of [1], this composition deformation covers a deformation of a neighborhood of 1_K in $H_1(K, M)$ to 1_K.

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