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On invariant measures for piecewise convex transformations

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Abstract. It is shown that a class of piecewise convex transformations on $[0, 1]^n$ has an absolutely continuous invariant measure.

1. Introduction. The purpose of this note is to show the existence of an absolutely continuous invariant measure for a transformation τ: $[0, 1]^n \to [0, 1]^n$. Our theorem is a generalization of some results of A. Rényi [9], A. O. Gel'fond [3], W. Parry [4] and A. Lasota [6] to an n-dimensional space. In the proof, as in [7], we explore the fact that the Frobenius–Perron operator corresponding to τ has the property of shrinking the variation of the function.

In Section 2 we recall some basic definitions and state the main theorem. In Section 3 we prove some necessary lemmas and the theorem. In Section 4 we show a certain property of invariant measures under τ.

2. Let $I^n = [0, 1]^n$. Denote by $(L^1, \| \ \|)$ the space of all integrable functions defined on I^n. The n-dimensional Lebesgue measure on I^n will be denoted by m, and we shall write $m(dx) = dx = dx_1 \ldots dx_n$.

We say that a measurable transformation τ: $I^n \to I^n$ is *non-singular* if $m(\tau^{-1}(A)) = 0$ whenever $m(A) = 0$ for any measurable set A.

For non-singular τ: $I^n \to I^n$ we define the Frobenius–Perron operator P_τ: $L^1 \to L^1$ by the formula

$$\int_A P_\tau f dx = \int_{\tau^{-1}(A)} f dx,$$

which is valid for each measurable set $A \subset I^n$. It is well known that the operator P_τ is linear and satisfies the following conditions:

(a) P_τ is positive: $f \geqslant 0 \Rightarrow P_\tau \geqslant 0$;

(b) P_τ preserves integrals:

$$\int_{I^n} P_\tau f dx = \int_{I^n} f dx, \quad f \in L^1;$$

(c) $P_{\tau^k} = P_\tau^k$ (τ^k denotes the n-th interate of τ);

(d) $P_\tau f = f$ if and only if the measure $d\mu = f dx$ is invariant under τ, i.e., $\mu(\tau^{-1}(A)) = \mu(A)$ for each measurable A.

We shall not make a distinction between functions $f\colon I^n \to R$ defined on I^n and functions $f\colon I^n \to R$ taken as elements of the space L^p for $p \geq 1$. This difference will become clear in the context.

A function $f\colon I^n \to R$ is said to be *decreasing* if

$$f(x_1, \ldots, x_n) \leqslant f(y_1, \ldots, y_n)$$

for

$$(x_1, \ldots, x_n) \geqslant (y_1, \ldots, y_n) \quad (x_i \geqslant y_i,\ i = 1, \ldots, n).$$

For a decreasing function $f\colon I^n \to R$ we define the variation by the formula

$$\bigvee f = \sum_{i=1}^{n} \bigvee_i f,$$

where

$$\bigvee_i f = \int_{I^{n-1}} \big(f(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n) - \\ - f(x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n)\big) dx_1 \ldots dx_{i-1} dx_{i+1} \ldots dx_n.$$

Denote by $\prod_{i=1}^{n} [a_i, b_i)$ the Cartesian product of the intervals $[a_i, b_i)$, $i = 1, \ldots, n$.

THEOREM 1. *Let* $A_j = \prod_{i=1}^{n} A_{ij}$, $j = 1, \ldots, K$, *where* $A_{ij} = [a_{ij}, b_{ij})$ *if* $b_{ij} < 1$ *and* $A_{ij} = [a_{ij}, b_{ij}]$ *if* $b_{ij} = 1$, *be a partition of the* I^n *such that for* $j \neq k$ *the set* $A_j \cap A_k$ *is empty and*

$$\bigcup_{j=1}^{K} A_j = I^n.$$

Let the transformation $\tau\colon I^n \to I^n$ *be given by the formula*

$$\tau(x_1, \ldots, x_n) = \big(\varphi_{1j}(x_1), \ldots, \varphi_{nj}(x_n)\big), \quad (x_1, \ldots, x_n) \in A_j,$$

where the functions $\varphi_{ij}\colon A_j \to [0, 1]$ *satisfy the following conditions*:

(1) $$\varphi_{ij}(a_{ij}) = 0,$$

(2) $$\varphi'_{ij}(a_{ij}) > 0,$$

(3) $$\varphi'_{ij}(a_{ij}) > 1 \quad \text{if } a_{ij} = 0,$$

(4) φ'_{ij} *are increasing.*

Then there exists a decreasing function $f \in L^1$ ($\|f\| = 1, f \geqslant 0$) *such that the measure* $d\mu = f dx$ *is invariant under* τ.

EXAMPLE. Let $A_1 = [0, 1/2) \times [0, 1/2)$, $A_2 = [1/2, 1] \times [0, 1/2)$, $A_3 = [0, 1] \times [1/2, 1]$ be the partition of the I^2. For the transformation given by the formula

$$\tau(x, y) = \begin{cases} (2x, 2y) & \text{for } (x, y) \in A_1, \\ (2x-1, 2y) & \text{for } (x, y) \in A_2, \\ (x, 2y) & \text{for } (x, y) \in A_3, \end{cases}$$

there exists an absolutely continuous non-trivial invariant measure.

3. In the proof of Theorem 1 we will use the following lemmas.

LEMMA 1. *If functions $F_i: I^n \to R$, $i = 1, \ldots, n$, do not depend on x_i and $F_i^{n-1} \in L^1(I^n)$, then*

$$\int_{I^n} |F_1 \ldots F_n| \, dx \leqslant \int_{I^n} |F_1^{n-1}| \, dx \ldots \int_{I^n} |F_n^{n-1}| \, dx.$$

The proof of this lemma is given in [3].

LEMMA 2. *The set S of functions $f: I^n \to R$ such that*

(e) *$f: I^n \to R$ is decreasing,*

(f) $\bigvee f \leqslant M$,

(g) $\int_{I^n} f dx \leqslant 1$,

is weakly relatively compact in L^1.

Proof. Let $f: I^n \to R$ satisfy (e), (f), (g). Since f is decreasing, Lemma 1 implies that

$$(5) \qquad \int_{I^n} f^{n/(n-1)}(x) \, dx \leqslant \int_{I^n} f^{1/(n-1)}(0, x_2, \ldots, x_n) \ldots f^{1/(n-1)}(x_1, \ldots, x_{n-1}, 0) \, dx$$
$$\leqslant \int_{I^{n-1}} f(0, x_2, \ldots, x_n) \, dx_2 \ldots dx_n \ldots \int_{I^{n-1}} f(x_1, \ldots, x_{n-1}, 0) \, dx_1 \ldots dx_{n-1}.$$

Since the function

$$g_i(x_i) = \int_{I^{n-1}} f(x_1, \ldots, x_n) \, dx_1 \ldots dx_{i-1} dx_{i+1} \ldots dx_n$$

is decreasing, we have

$$(6) \qquad \int_{I^{n-1}} f(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n) \, dx_1 \ldots dx_{i-1} dx_{i+1} \ldots dx_n$$
$$= g_i(0) \leqslant \|g\| + \bigvee g_i(x_i) \leqslant \|f\| + \bigvee f.$$

From (5) and (6) it follows that

$$\Big(\int_{I^n} f^{n/(n-1)}(x) \, dx \Big)^{(n-1)/n} \leqslant (\|f\| + \bigvee f)^{n-1} \leqslant (1+M)^{n-1}.$$

Since $n/(n-1) > 1$, from the last inequality it follows that the set S is weakly relatively compact in $L^{n/(n-1)}$ and consequently in L^1. This completes the proof.

LEMMA 3. *If a function $h\colon [0,1] \to R^+$ is decreasing, then*

$$h(s) \leqslant \frac{\|h\|_{L^1}}{s}, \quad 0 \leqslant s \leqslant 1.$$

Proof. The proof is a consequence of the following inequality:

$$\|h\|_{L^1} \geqslant \int_0^s h(u)\,du \geqslant \int_0^s h(s)\,du = sh(s).$$

LEMMA 4. *The linear subspace E generated by all decreasing functions $f\colon I^n \to R$ is dense in L^1 (I^n).*

Proof. Let χ_A denote the characteristic function of a measurable set A. Consider the function

$$g = \sum_{r=1}^{s} a_r \chi_{A_r}, \tag{7}$$

where

$$A_r = \prod_{i=1}^{n} [a_{ri}, b_{ri}].$$

For each A_r there exist sets $B_{rj} = \prod_{i=1}^{n} (-\infty, b_{rji}]$, $j = 1, \ldots, N_r$, such that

$$\chi_{A_r} = \sum_{j=1}^{N_r} \beta_{rj} \chi_{B_r}, \quad \beta_{rj} \in R.$$

It is clear that $\chi_{B_{rj}}$ is decreasing; therefore $g \in E$. It is known that any L^1 function may be approximated by functions of form (7). Thus the lemma is completely proved.

Proof of Theorem 1. Let

$$\psi_{ij}(x_i) = \begin{cases} \varphi_{ij}^{-1}(x_i) & \text{for } x_i \in \varphi_{ij}([a_{ij}, b_{ij})), \\ b_{ij} & \text{for } x_i \in [0,1] \setminus \varphi_{ij}([a_{ij}, b_{ij})). \end{cases}$$

A simple computation shows that the Frobenius–Perron operator corresponding to τ may be written in the form:

$$(P_\tau f)(x_1, \ldots, x_n) = \sum_{j=1}^{K} f(\psi_{1j}(x_1), \ldots, \psi_{nj}(x_n)) \psi'_{ij}(x_1) \ldots \psi'_{nj}(x_n).$$

By its very definition the operator P_τ is a mapping from L^1 into L^1, but the last formula enables us to consider P_τ as a map from the space of functions defined on I^n into itself. It is easy to verify that $P_\tau f$ is decreasing for any decreasing f. For any decreasing $f \geqslant 0$ we have, moreover,

$$\bigvee_i P_\tau f = \sum_{j=1}^{K} \int_{I^{n-1}} \big[f\big(\psi_{1j}(x_1), \ldots, \psi_{ij}(0), \ldots, \psi_{nj}(x_n)\big) \times \times \psi'_{1j}(x_1)\ldots\psi'_{ij}(0)\ldots\psi'_{nj}(x_n) - -f\big(\psi_{1j}(x_1), \ldots, \psi_{ij}(1), \ldots, \psi_{nj}(x_n)\big) \times \times \psi'_{1j}(x_1)\ldots\psi'_{ij}(1)\ldots\psi'_{nj}(x_n)\big]\, dx_1\ldots dx_{i-1}dx_{i+1}\ldots dx_n$$

$$= \sum_{j=1}^{K} \int_{a_{1j}}^{b_{1j}} \ldots \int_{a_{i-1,j}}^{b_{i-1,j}} \int_{a_{i+1,j}}^{b_{i+1,j}} \ldots \int_{a_{nj}}^{b_{nj}} [f(x_1, \ldots, a_{ij}, \ldots, x_n)\psi'_{ij}(0) - -f(x_1, \ldots, b_i{}^j, \ldots, x_n)\psi'_{ij}(1)]\, dx_1\ldots dx_{i-1}dx_{i+1}\ldots dx_n$$

and consequently

$$(8) \quad \bigvee_i P_\tau f \leqslant a_i \int_{I^{n-1}} [f(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n) - -f(x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n)]\, dx_1\ldots dx_{i-1}dx_{i+1}\ldots dx_n + + a_i \int_{I^{n-1}} f(x_1, \ldots, x_{i-1}, 1, x_{i-1}, \ldots, x_n)\, dx_1\ldots dx_{i-1}dx_{i+1}\ldots dx_n + + \sum_{j:a_{ij}>0} \int_{I^{n-1}} f(x_1, \ldots, x_{i-1}, a_{ij}, x_{i+1}, \ldots, x_n) \times \times \psi'_{ij}(0)\, dx_1\ldots dx_{i-1}dx_{i+1}\ldots dx_n + + \sum_{j=1}^{K} \int_{I^{n-1}} f(x_1, \ldots, x_{i-1}, b_{ij}, x_{i+1}, \ldots, x_n) \times \times \psi'_{ij}(1)\, dx_1\ldots dx_{i-1}dx_{i+1}\ldots dx_n,$$

where

$$a_i = \max_{j:a_{ij}=0} \psi'_{ij}(0) < 1.$$

Lemma (3) implies

$$(9) \quad \int_{I^{n-1}} f(x_1, \ldots, x_n)\, dx_1\ldots dx_{i-1}dx_{i+1}\ldots dx_n \leqslant \frac{\|f\|}{x_i}.$$

Applying (9) to (8), we obtain

$$\bigvee_i P_\tau f \leqslant a_i \bigvee_i f + a_i\|f\| + \sum_{j:a_{ij}>0} \frac{\psi'_{ij}(0)}{a_{ij}}\|f\| + + \sum_{j=1}^{K} \frac{\psi'_{ij}(1)}{b_{ij}}\|f\| = a_i \bigvee_i f + M_i\|f\|,$$

where

$$M_i = \sum_{j:a_{ij}>0} \frac{\psi'_{ij}(0)}{a_{ij}} + \sum_{j=1}^{K} \frac{\psi'_{ij}(1)}{b_{ij}} + a_i.$$

Therefore

$$\limsup_{k\to\infty} \bigvee_i P_\tau^k f \leqslant \frac{M_i}{1-a_i}\|f\|$$

and finally

$$\limsup_{r\to\infty} \bigvee_i \left(\frac{1}{r}\sum_{k=0}^{r-1} P_\tau^k f\right) \leqslant \frac{M_i}{1-a_i}\|f\|. \tag{10}$$

The last inequality and Lemma 2 imply that the set

$$\left\{\frac{1}{r}\sum_{k=0}^{r-1} P_\tau^k f\right\}_{r=1}^{\infty} \tag{11}$$

is weakly compact in L^1. This conclusion and Lemma 4 enable us to use the Kakutani–Yosida ergodic theorem. For any $f \in L^1$ sequence (11) converges strongly to the function f^*, which is invariant under P_τ. From (a) and (b) it follows that $f^* \geqslant 0$ and $\|f^*\| = \|f\| > 0$.

4. Final remarks. Let $A \subset I^n$ and $m(A) = 0$. Given $f\colon I^n \to R$ and $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$, write

$$(\sup_{i,A} f)(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) = \sup\{f(x_1, \ldots, x_n)\colon x_i \in (0,1), (x_1, \ldots, x_i, \ldots, x_n) \notin A\},$$

$$(\inf_{i,A} f)(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) = \inf\{f(x_1, \ldots, x_n)\colon x_i \in (0,1), (x_1, \ldots, x_i, \ldots, x_n) \notin A\}.$$

The functions $\sup_{i,A} f$ and $\inf_{i,A} f$ depend upon the $n-1$ variables $(x_1, \ldots \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$.

If the function $f\colon I^n \to R$ is decreasing, we define the variation $\bigvee_A f$ by the formula

$$\bigvee_A f = \sum_{i=1}^{n} \bigvee_{i,A} f,$$

where

$$\bigvee_{i,A} f = \int_{I^{n-1}} (\sup_{i,A} f - \inf_{i,A} f)\, dx_1, \ldots, x_{i-1}, x_{i+1}, \ldots, dx_n$$

(functions $\sup_{i,A} f$ and $\inf_{i,A} f$ are measurable because f is decreasing and $m(A) = 0$). It is easy to see that

$$\bigvee_A f \leqslant \bigvee f.$$

LEMMA 5. *If a sequence of functions $f_k: I^n \to R^+$ satisfies the conditions*

(h) *f_k is convergent in L^1 norm,*

(i) *f_k is decreasing,*

(j) *$\bigvee f_k \leqslant M_1$,*

then there exists a set $A \subset I^n$, $m(A) = 0$, such that

$$\bigvee_A f \leqslant \limsup_{k\to\infty} \bigvee f_k .$$

Proof. There exists a subsequence f_{k_j} convergent to f almost everywhere in I^n. Let A be the set of points from I^n for which f_{k_j} is not convergent. It is easy to see that

$$\sup_{i,A} f - \inf_{i,A} f \leqslant \liminf_{j\to\infty} (\sup_{i,A} f_{k_j} - \inf_{i,A} f_{k_j}).$$

From the last inequality and the Fatou lemma it follows that

$$\bigvee_{i,A} f \leqslant \liminf_{j\to\infty} \bigvee_{i,A} f_{k_j} \leqslant \liminf_{j\to\infty} \bigvee_{i} f_{k_j}.$$

Therefore

$$\bigvee_A f \leqslant \liminf_{j\to\infty} \bigvee f_{k_j}$$

and consequently

$$\bigvee_A f \leqslant \limsup_{k\to\infty} \bigvee f_k .$$

This completes the proof.

Using this lemma, we are enable to prove the following

THEOREM 2. *Assume that τ satisfies the condition of Theorem 1. Let $f: I^n \to R$ be a given integrable decreasing function (not an element of L^1). Then there exists a set $A \subset I^n$, $m(A) = 0$, and there exists a constant c independent of the choice of the initial decreasing f such that*

$$\bigvee_A f^* \leqslant c\|f\|, \tag{12}$$

where

$$f^* = \lim_{k\to\infty} \frac{1}{k} \sum_{r=0}^{k-1} P_\tau^r f .$$

Proof. Writing

$$Q = \lim_{k\to\infty} \frac{1}{k} \sum_{r=0}^{k-1} P_\tau^r ,$$

from (10) and Lemma 5 we have

$$\bigvee_A Qf \leqslant c\|f\|$$

for a decreasing f of bounded variation and a certain set A such that $m(A)$ $= 0$. Applying Lemma 5 once more, we have inequality (12) for any integrable decreasing function.

This finishes the proof.

References

[1] N. Dunford and J. T. Schwartz, *Linear operators*. I. *General Theory*, Pure Appl. Math. 7, Intercience, New York 1958.

[2] E. Gagliardo, *Propieta di alcune classi di funzioni in piu variabili*, Ricerche Mat. 7 (1958), p. 102–137.

[3] A. O. Gel'fond, *A common property of number systems*, Izv. Akad. Nauk SSSR Ser. Mat. 23 (1959), p. 809–814.

[4] K. Krzyżewski and W. Szlenk, *On invariant measures for expanding differentiable mappings*, Studia Math. 33 (1969), p. 83–92.

[5] A. Lasota, *Invariant measures and functional equations*, Aequationes Math. 9 (1973), p. 193–200.

[6] — *On the existence of invariant measures for Markov processes*, Ann. Polon. Math. 28 (1973), p. 207–211.

[7] — and I. A. Yorke, *On the existence of invariant measures for piecevise monotonic transformations*, Trans. Amer. Math. Soc. 186 (1973), p. 481–488.

[8] W. Parry, *On the β-expansion of real numbers*, Acta Math. Acad. Sci. Hungar. 11 (1960), p. 401–416.

[9] A. Rényi, *Representation for real numbers and their ergodic properties*, ibidem 8 (1957), p. 477–493.

[10] V. A. Rohlin, *Exact endomorphism of Lebesgue spaces*, Izv. Akad. Nauk SSSR Ser. Mat. 25 (1961), p. 499–530.

[11] S. M. Ulam, *A collection of mathematical problems*, Interscience Tracts in Pure Appl. Math. 8, Interscience, New York 1960.

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