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On continuous solutions of systems
of functional equations

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The purpose of the present paper is to prove some theorems concerning the existence and uniqueness of continuous solutions of the system of N functional equations of order p

$$(1) \qquad \hat{\varPhi}(x) = \hat{H}\big(x, \hat{\varPhi}[f(x)], \hat{\varPhi}[f^2(x)], \ldots, \hat{\varPhi}[f^p(x)]\big),$$

in which the function $\hat{\varPhi}(x) = \big(\varphi_1(x), \ldots, \varphi_N(x)\big)$ is unknown, the function $f(x)$ and the function $\hat{H}(x, \hat{y}) = \big(h_1(x, \hat{y}), \ldots, h_N(x, \hat{y})\big)$, where $\hat{y} = (y_1, \ldots, y_{pN})$, are given, and $f^\nu(x)$ for $\nu = 2, \ldots, p$ denotes the νth iterate of the function $f(x)$, i.e., $f^\nu(x) = f\big(f^{\nu-1}(x)\big)$ for $\nu = 2, \ldots, p$ with $f^1(x) = f(x)$.

In the sequel we shall keep to the following notational convention: letters with hats denote vectors, small letters denote n-dimensional vectors (where $n = pN$) and capital letters denote less than n-dimensional vectors; in other cases we use letters without hats, in particular, small letters denote real numbers (λ, which may also be complex, is an exception), capital German letters (Fraktur) denote matrices, and other capital letters denote sets (with the exception of N, which is a natural parameter).

Section 1 contains a theorem on the equivalence of systems of equations which allows us to replace the investigation of system (1) by the investigation of a certain system of $n = pN$ functional equations of the first order

$$(2) \qquad \hat{\varphi}(x) = \hat{h}\big(x, \hat{\varphi}[f(x)]\big),$$

in which $\hat{\varphi}(x) = \big(\varphi_1(x), \ldots, \varphi_n(x)\big)$ is the unknown function and the functions $f(x)$ and $\hat{h}(x, \hat{y}) = \big(h_1(x, \hat{y}), \ldots, h_n(x, \hat{y})\big)$, where $\hat{y} = (y_1, \ldots, y_n)$, are given.

In section 2 we formulate some assumptions concerning the function $f(x)$ and give some lemmas determining the properties of this function. In section 3 we formulate some assumptions concerning the function

$\hat{h}(x, \hat{y})$ and prove some lemmas and theorems concerning the existence
of continuous solutions of system (2), whereas section 4 contains theorems,
preceded by some lemmas, concerning the existence and uniqueness of
continuous solutions of this system. In section 5 we formulate assumptions
regarding the function $\hat{H}(x, \hat{y})$ and theorems concerning the properties
of system (1), resulting from those concerning the properties of system
(2). Finally, section 6 contains a short comparison of the results obtained
with those known previously.

1. The investigation of the system of equations (1) may be reduced
to the investigation of a certain system of equations of the first order.
This is the consequence of the following

THEOREM 1. *The system of equations* (1) *is equivalent to the system
of equations* (2), *where* $n = pN$, *the function* $f(x)$ *and the functions* $h_\mu(x, \hat{y}$
for $\mu = 1, \ldots, N$ *are the same functions in both systems,* (1) *and* (2), *and*

(3) $h_{\nu N+\mu}(x, \hat{y}) = y_{(\nu-1)N+\mu}$ *for* $\nu = 1, \ldots, p-1$ *and* $\mu = 1, \ldots, N$.

*Here, the equivalence is understood in the following sense: if a system
of functions* $\varphi_1(x), \ldots, \varphi_N(x)$ *satisfies the system of equations* (1), *then the
system of functions* $\varphi_1(x), \ldots, \varphi_n(x)$, *where the functions* $\varphi_{N+1}(x), \ldots, \varphi_n(x)$
are defined by

(4) $\varphi_{\nu N+\mu}(x) = \varphi_\mu[f^\nu(x)]$ *for* $\nu = 1, \ldots, p-1$ *and* $\mu = 1, \ldots, N$,

satisfies the system of equations (2); *and if a system of functions* $\varphi_1(x),$
$\ldots, \varphi_n(x)$ *satisfies the system of equations* (2), *then the system of functions*
$\varphi_1(x), \ldots, \varphi_N(x)$ *satisfies the system of equations* (1) *and relation* (4) *holds.*

Proof. 1° Let a system of functions $\varphi_1(x), \ldots, \varphi_N(x)$ satisfy the
system of equations (1) and let the functions $\varphi_\varkappa(x)$ be defined for
$\varkappa = N+1, \ldots, n$ by formula (4). Since by definition

$$\hat{\Phi}(x) = (\varphi_1(x), \ldots, \varphi_N(x)),$$

we have

(5) $\hat{\Phi}[f(x)] = (\varphi_1[f(x)], \ldots, \varphi_N[f(x)])$

and, by (4), we have for $\nu = 2, \ldots, p,$

(6) $\hat{\Phi}[f^\nu(x)] = \hat{\Phi}[f^{\nu-1}(f(x))]$

                 $= (\varphi_1[f^{\nu-1}(f(x))], \ldots, \varphi_N[f^{\nu-1}(f(x))])$

                 $= (\varphi_{(\nu-1)N+1}[f(x)], \ldots, \varphi_{(\nu-1)N+N}[f(x)]).$

Similarly, we have by definition

$$\hat{\varphi}(x) = (\varphi_1(x), \ldots, \varphi_n(x)),$$

whence

(7)
$$\hat{\varphi}[f(x)] = \big(\varphi_1[f(x)], \ldots, \varphi_n[f(x)]\big).$$

We get by (5), (6) and (7)

(8)
$$\hat{\varphi}[f(x)] = \big(\hat{\Phi}[f(x)], \hat{\Phi}[f^2(x)], \ldots, \hat{\Phi}[f^p(x)]\big).$$

(We shall make use of relation (8) also in the second part of the proof; therefore, let us note that it has been derived under the sole assumption that relation (4) holds.) From relation (8) and from the assumed fulfilment of system (1) we obtain for $\mu = 1, \ldots, N$

(9)
$$h_\mu\big(x, \hat{\varphi}[f(x)]\big) = h_\mu\big(x, \hat{\Phi}[f(x)], \hat{\Phi}[f^2(x)], \ldots, \hat{\Phi}[f^p(x)]\big) = \varphi_\mu(x),$$

and by (3) and (4) we have for $\mu = 1, \ldots, N$

(10)
$$h_{N+\mu}\big(x, \hat{\varphi}[f(x)]\big) = \varphi_\mu[f(x)] = \varphi_{N+\mu}(x)$$

and for $\nu = 2, \ldots, p-1$ and $\mu = 1, \ldots, N$

(11)
$$h_{\nu N+\mu}\big(x, \hat{\varphi}[f(x)]\big) = \varphi_{(\nu-1)N+\mu}[f(x)] = \varphi_\mu\big[f^{\nu-1}\big(f(x)\big)\big]$$
$$= \varphi_\mu[f^\nu(x)] = \varphi_{\nu N+\mu}(x).$$

Thus in virtue of (9), (10) and (11) the system of functions $\varphi_1(x), \ldots$ $\ldots, \varphi_n(x)$ satisfies the system of equations (2).

2^0 Let a system of functions $\varphi_1(x), \ldots, \varphi_n(x)$ satisfy the system of equations (2). Then, according to (3), we have for $\nu = 1, \ldots, p-1$ and $\mu = 1, \ldots, N$

(12)
$$\varphi_{\nu N+\mu}(x) = h_{\nu N+\mu}\big(x, \hat{\varphi}[f(x)]\big) = \varphi_{(\nu-1)N+\mu}[f(x)].$$

From relation (12) we shall derive relation (4); the proof goes by induction with respect to ν.

For $\nu = 1$ and $\mu = 1, \ldots, N$ both relations, (4) and (12), are identical. Let us assume that relation (4) holds for $\nu = m$ and $\mu = 1, \ldots, N$, where $1 \leqslant m \leqslant p-2$, i.e.,

$$\varphi_{mN+\mu}(x) = \varphi_\mu[f^m(x)].$$

Hence, in view of (12), we get for $\mu = 1, \ldots, N$

$$\varphi_{(m+1)N+\mu}(x) = \varphi_{mN+\mu}[f(x)] = \varphi_\mu\big[f^m\big(f(x)\big)\big] = \varphi_\mu[f^{m+1}(x)],$$

which means that relation (4) is fulfilled for $\nu = m+1$ and $\mu = 1, \ldots, N$. By the induction principle we infer hence that relation (4) is valid for all $\nu = 1, \ldots, p-1$ and $\mu = 1, \ldots, N$.

Relation (4) implies — as has been shown in the first part of the proof — relation (8). Hence and from the fulfilment of system (2) we get for $\mu = 1, \ldots, N$

$$h_\mu\big(x, \hat{\Phi}[f(x)], \hat{\Phi}[f^2(x)], \ldots, \hat{\Phi}[f^p(x)]\big) = h_\mu\big(x, \hat{\varphi}[f(x)]\big) = \varphi_\mu(x).$$

Consequently, the system of functions $\varphi_1(x), \ldots, \varphi_N(x)$ satisfies the system of equations (1).

This completes the proof of Theorem 1.

2. In the sequel we shall assume that the function $f(x)$ fulfils the following

HYPOTHESIS 1. *The function $f(x)$ is defined and continuous in an interval I and we have $x < f(x) < \xi$ for $x \in I, x < \xi$, and $\xi < f(x) < x$ for $x \in I, x > \xi$; moreover, if the point ξ does not belong to the interval I, then it is one of the end-points of this interval, and if ξ belongs to I and $\xi = \infty$, then the value of the function $f(x)$ at the point ξ is to be understood as its limit at this point.*

The properties of the function $f(x)$ which will be used in the sequel are listed in the four lemmas below. Lemma 1 is an immediate consequence of Hypothesis 1 and thus it will be given without proof and it will not be quoted when we make use, in what follows, of the properties contained therein.

LEMMA 1. *Let the function $f(x)$ fulfil Hypothesis 1. Then: 1° $f(x) \in I$ for $x \in I$; 2° If $\xi \in I$, then $f(\xi) = \xi$; 3° If the function $f(x)$ is strictly monotonic in the interval I, then it is strictly increasing in I. Moreover, let $f(x)$ be strictly increasing in the interval I. Then: 4° There exists an inverse function $f^{-1}(x)$ to $f(x)$ in I; 5° The function $f^{-1}(x)$ is defined, continuous and strictly increasing in the interval $f(I)$; 6° $f^{-1}(x) < x$ for $x \in f(I), x < \xi$ and $f^{-1}(x) > x$ for $x \in f(I), x > \xi$.*

LEMMA 2. *Let the function $f(x)$ fulfil Hypothesis 1, let $\xi \notin I$ and let the function $f(x)$ be strictly increasing in the interval I; moreover, let x_0 be an arbitrarily fixed point of the interval I. If the point ξ is the right endpoint of the interval I, then the relation*

$$(13) \qquad x_{k+1} = f^{-1}(x_k)$$

defines for $k \geq 0$ an infinite or finite, strictly decreasing sequence x_k with elements belonging to the interval I,

$$(14) \qquad f(\langle x_{k+2}, x_{k+1}\rangle) = \langle x_{k+1}, x_k\rangle \quad \text{for } k \geq 0,$$

$$f(\langle x_1, x_0\rangle) = \langle x_0, f(x_0)\rangle;$$

if the sequence x_k is infinite, then

$$(15) \qquad I = \bigcup_0^\infty \langle x_{k+1}, x_k\rangle \cup \langle x_0, \xi\rangle,$$

and if the sequence x_k is finite and x_{k_0}, where $k_0 \geq 0$, is its last existing element, then there exists an interval Δ such that

$$(16) \qquad I = \Delta \cup \bigcup_0^{k_0-1} \langle x_{k+1}, x_k\rangle \cup \langle x_0, \xi\rangle \quad \text{for } k_0 > 0,$$

$$I = \Delta \cup \langle x_0, \xi) \quad for \quad k_0 = 0,$$

with $f(\Delta) \subset \langle x_{k_0}, x_{k_0-1})$ *for* $k_0 > 0$, $f(\Delta) \subset \langle x_0, f(x_0))$ *for* $k_0 = 0$ *(in particular, Δ may be an empty set); if the point ξ is the left end-point of the interval I, then relation* (13) *defines for* $k \geqslant 0$ *an infinite or finite, strictly increasing sequence* x_k *with elements belonging to the interval* I,

$$f\big((x_{k+1}, x_{k+2})\big) = (x_k, x_{k+1}) \quad for \quad k \geqslant 0,$$

$$f\big((x_0, x_1)\big) = \big(f(x_0), x_0\big);$$

if the sequence x_k *is infinite, then*

$$I = (\xi, x_0) \cup \bigcup_0^\infty (x_k, x_{k+1}),$$

and if the sequence x_k *is finite and* x_{k_0}, *where* $k_0 \geqslant 0$, *is its last existing element, then there exists an interval* Δ *such that*

$$I = (\xi, x_0) \cup \bigcup_0^{k_0-1} (x_k, x_{k+1}) \cup \Delta \quad for \quad k_0 > 0,$$

$$I = (\xi, x_0) \cup \Delta \quad for \quad k_0 = 0,$$

with $f(\Delta) \subset (x_{k_0-1}, x_{k_0})$ *for* $k_0 > 0$, $f(\Delta) \subset (f(x_0), x_0)$ *for* $k_0 = 0$ *(in particular, Δ may be an empty set).*

Proof. Let ξ be the right end-point of the interval I and let the sequence x_k be defined for $k > 0$ by relation (13). Since $f^{-1}(x) < x$ for $x \epsilon f(I)$, the sequence x_k for $k \geqslant 0$ is strictly decreasing. It follows from relation (13) that $x_k = f(x_{k+1})$ and $x_{k+1} = f(x_{k+2})$; thus relation (14) results from the monotonicity of the function $f(x)$. If $f(I) = I$, then the sequence x_k is infinite and $x_k \epsilon I$ for $k \geqslant 0$. Since the sequence x_k is decreasing, it converges to a certain limit c (c may equal $-\infty$). If we had $c \epsilon I$, then passing to the limit in the relation $x_k = f(x_{k+1})$ we would obtain $c = f(c)$, and in virtue of the monotonicity of the sequence x_k we would have $c < \xi$. But it follows from Hypothesis 1 that for $x \epsilon I$ and $x < \xi$ we have $f(x) > x$, which proves that $c \notin I$. Consequently, since $x_k \epsilon I$ for $k \geqslant 0$, relation (15) holds. If $f(I) \neq I$, then $f(I) \subset I$, the sequence x_k is finite and $x_k \epsilon I$ for $0 \leqslant k \leqslant k_0$, where $k_0 \geqslant 0$ is such that $f^{-1}(x_{k_0})$ is not defined or does not belong to the interval I. Then relation (16) holds.

If the point ξ is the left end-point of the interval I, the proof of the lemma is analogous.

LEMMA 3. *Let the function* $f(x)$ *fulfil Hypothesis* 1, *let* $\xi \notin I$ *and let the function* $f(x)$ *be strictly increasing in the interval* I; *moreover, let* x_0 *be an arbitrarily fixed point of the interval* I. *If the point* ξ *is the right end-point of the interval* I, *then the relation*

$$(17) \qquad\qquad x_{k+1} = f(x_k)$$

defines for $k \geqslant 0$ an infinite, strictly increasing sequence x_k with elements belonging to the interval I,

(18) $f^{-1}\big(\langle x_{k+1}, x_{k+2}\rangle\big) = \langle x_k, x_{k+1}\rangle$ *for* $k \geqslant 0$,

and

(19) $$\langle x_0, \xi) = \bigcup_0^\infty \langle x_k, x_{k+1});$$

if the point ξ is the left end-point of the interval I, then relation (17) *defines for $k \geqslant 0$ an infinite, strictly decreasing sequence x_k with elements belonging to the interval I,*

                $f^{-1}\big((x_{k+2}, x_{k+1}\rangle\big) = (x_{k+1}, x_k\rangle$ *for* $k \geqslant 0$,

and

$$(\xi, x_0\rangle = \bigcup_0^\infty (x_{k+1}, x_k\rangle.$$

Proof. Let the point ξ be the right end-point of the interval I and let the sequence x_k be defined for $k > 0$ by relation (17). Since $f(x) > x$ for $x \in I$, the sequence x_k for $k \geqslant 0$ is strictly increasing. Relation (17) implies that $f^{-1}(x_{k+1}) = x_k$ and $f^{-1}(x_{k+2}) = x_{k+1}$. Thus relation (18) results from the monotonicity of the function $f^{-1}(x)$. Since $f(I)$ is contained in I, the sequence x_k is infinite and $x_k \in I$ for $k \geqslant 0$. Since the sequence x_k is increasing, it converges to a certain limit c (c may also equal ∞). If we had $c \in I$, then passing to the limit in relation (17) we would obtain $c = f(c)$, which is imposible, since, by Hypothesis 1, $f(x) > x$ for $x \in I$ and $x < \xi$. Since all x_k belong to the interval I, c must be the end-point of the interval I, i.e. $c = \xi$. Thus we obtain relation (19), since $x_k \in I$ for $k \geqslant 0$.

If the point ξ is the left end-point of the interval I, the proof is analogous.

LEMMA 4. *Let the function $f(x)$ fulfil Hypothesis* 1 *and let the point ξ be an end-point of the interval I; moreover, let x_0 be an arbitrarily fixed point of the interval I. If the point ξ is the right end-point of the interval I and $\xi \in I$, then the relation*

(20) $x_{k+1} = \inf \{x : x \in I, m(x) > x_k\}$,

where

(21) $$m(x) = \inf_{\langle x, \xi\rangle} f(t),$$

defines for $k \geqslant 0$ an infinite or finite, strictly decreasing sequence x_k with elements belonging to the interval I,

(22) $f\big(\langle x_{k+1}, x_k\rangle\big) \subset \langle x_k, \xi\rangle$ *for* $k \geqslant 0$;

if the sequence x_k is infinite, then

$$(23) \qquad I = \lim_{k \to \infty} \langle x_k, \xi \rangle,$$

and if the sequence x_k is finite and x_{k_0}, where $k_0 \geqslant 0$, is its last existing element, then $x_{k_0} \epsilon I$ or $x_{k_0} \overline{\epsilon} I$ and

$$(24) \qquad I = \langle x_{k_0}, \xi \rangle \quad or \quad I = (x_{k_0}, \xi);$$

if the point ξ is the left end-point of the interval I and $\xi \epsilon I$, then the relation

$$(25) \qquad x_{k+1} = \sup\{x: x \epsilon I, m(x) < x_k\},$$

where

$$(26) \qquad m(x) = \sup_{\langle \xi, x \rangle} f(t),$$

defines for $k \geqslant 0$ an infinite or finite, strictly increasing sequence x_k with elements belonging to the interval I,

$$f\big((x_k, x_{k+1})\big) \subset \langle \xi, x_k \rangle \quad for \quad k \geqslant 0;$$

if the sequence x_k is infinite, then

$$I = \lim_{k \to \infty} \langle \xi, x_k \rangle,$$

and if the sequence x_k is finite and x_{k_0}, where $k_0 \geqslant 0$, is its last existing element, then $x_{k_0} \epsilon I$ or $x_{k_0} \overline{\epsilon} I$ and

$$I = \langle \xi, x_{k_0} \rangle \quad or \quad I = \langle \xi, x_{k_0});$$

finally, if $\xi \overline{\epsilon} I$, then all the above statements remain true when all the intervals occurring in them are replaced by intervals which are analogous but open on the side of the point ξ.

Proof. Let the point ξ be the right end-point of the interval I, let $\xi \epsilon I$, let the function $m(x)$ be defined by relation (21) and let the sequence x_k for $k > 0$ be defined by relation (20). In virtue of the continuity of the function $f(x)$ in the interval I the function $m(x)$ is continuous and increasing in I; moreover, $m(x) \leqslant f(x)$ for $x \epsilon I$. The monotonicity and continuity of the function $m(x)$ imply that $m(x) > x_k$ for $x > x_{k+1}$ and $m(x_{k+1}) = x_k$ for $k \geqslant 0$; moreover, $m(x) > x$ for $x \epsilon I$. Consequently, the sequence x_k is strictly decreasing. Since for $x \geqslant x_{k+1}$ we have $f(x) \geqslant m(x) \geqslant m(x_{k+1}) = x_k$, relation (22) holds. If for every $k \geqslant 0$ there exists an $x \epsilon I$ such that $m(x) < x_k$, then the sequence x_k is infinite and relation (23) holds; in the other case the sequence x_k is finite and relation (24) holds.

If the point ξ is the left end-point of the interval I, the proof of Lemma 4 is analogous.

3. Regarding the function $\hat{h}(x, \hat{y})$, we shall assume that it fulfils the following

HYPOTHESIS 2. *The function* $\hat{h}(x, \hat{y})$ *is defined and continuous in a domain* (^{1}) Ω *of the* $(n+1)$*-dimensional space of the variables* $x, y_1, \ldots, y_n$ *such that for every* $x \epsilon I$ *the set*

$$\varGamma_x = \{\hat{y}: \ (x, \hat{y}) \epsilon \Omega\}$$

is a domain of the n*-dimensional space of the variables* $y_1, \ldots, y_n$, *and the range of the function* $\hat{h}(x, \hat{y})$ *for* $\hat{y} \epsilon \varGamma_x$:

$$\varLambda_x = \hat{h}_x(\varGamma_x), \quad \textit{where } \hat{h}_x(\hat{y}) = \hat{h}(x, \hat{y}),$$

is a domain of the n*-dimensional space of the variables* $y_1, \ldots, y_n$.

We shall also assume the following

HYPOTHESIS 3. *For every* $x \epsilon I$

$$\varLambda_{f(x)} \subset \varGamma_x.$$

We shall prove the following

LEMMA 5. *Let the function* $f(x)$ *fulfil Hypothesis* 1, *let* $\xi \notin I$, *let the function* $f(x)$ *be strictly increasing in the interval* I *and let Hypotheses* 2 *and* 3 *be fulfilled; furthermore, let* $I_0 = \langle x_0, f(x_0)\rangle$ *and* $J_0 = \langle x_0, \xi\rangle$ *if the point* ξ *is the right end-point of the interval* I, *whereas* $I_0 = (f(x_0), x_0\rangle$ *and* $J_0 = (\xi, x_0\rangle$ *if the point* ξ *is the left end-point of the interval* I, x_0 *being an arbitrarily fixed point of the interval* I. *Then, for an arbitrary function* $\hat{\varphi}_0(x)$ *defined and continuous in the interval* I_0 *and fulfilling the conditions*

(27) $\qquad\qquad \hat{\varphi}_0[f(x)] \epsilon \varGamma_x \qquad \textit{for every } f(x) \epsilon I_0 \cap f(I)$

and

(28) $\qquad\qquad \lim_{x \to x_0, f(x) \epsilon I_0} \hat{h}\big(x, \hat{\varphi}_0[f(x)]\big) = \hat{\varphi}_0(x_0),$

there exists exactly one function $\hat{\varphi}(x)$ *defined in the interval* $(I - J_0) \cup I_0$, *fulfilling in the interval* $I - J_0$ *the condition*

(29) $\qquad\qquad\qquad\qquad \hat{\varphi}(x) \epsilon \varLambda_x,$

satisfying in the interval $I - J_0$ *the system of equations* (2) *and such that*

(30) $\qquad\qquad\qquad \hat{\varphi}(x) = \hat{\varphi}_0(x) \qquad \textit{for } x \epsilon I_0;$

moreover, the function $\hat{\varphi}(x)$ *is continuous in the interval* $(I - J_0) \cup I_0$.

Proof. Let the point ξ be the right end-point of the interval I so that $I_0 = \langle x_0, f(x_0)\rangle$ and $J_0 = \langle x_0, \xi\rangle$. (If the point ξ is the left end-point of the interval I, the proof is analogous.) Let the sequence x_k be defined

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(^{1}) Domain means here an open and connected set.

for $k > 0$ by relation (13). According to Lemma 2 the sequence x_k is strictly decreasing, $x_k \epsilon I$, and relation (14) holds. Let us assume that the sequence x_k is infinite, whence, by Lemma 2, relation (15) follows and, in virtue of the monotonicity of the sequence x_k, we have

$$(31) \qquad I - J_0 = \bigcup_0^\infty \langle x_{k+1}, x_k \rangle.$$

(If the sequence x_k is finite and relation (16) holds, the proof of the lemma is analogous, with the only difference that we must take into account the inequality $k \leqslant k_0$ and the last inductive step may refer to the interval Λ instead of $\langle x_{m+2}, x_{m+1} \rangle$.)

Let $\hat{\varphi}_0(x)$ be an arbitrary function defined and continuous in the interval I_0 and fulfilling relations (27) and (28). (The existence of such a function is guaranteed by Hypotheses 2 and 3; moreover, there exist infinitely many such functions.) We are going to show that the formula

$$(32) \qquad \hat{\varphi}(x) = \begin{cases} \hat{\varphi}_0(x) & \text{for } x \epsilon \langle x_0, f(x_0) \rangle, \\ \hat{h}(x, \hat{\varphi}[f(x)]) & \text{for } x \epsilon \langle x_{k+1}, x_k \rangle, \ k = 0, 1, 2, \ldots, \end{cases}$$

defines a continuous function $\hat{\varphi}(x)$ in the interval $(I - J_0) \cup I_0$ and that the function $\hat{\varphi}(x)$ fulfils in the interval $I - J_0$ relation (29).

It follows from the definition of the function $\hat{\varphi}_0(x)$ that formula (32) defines the function $\hat{\varphi}(x)$ in the interval $\langle x_0, f(x_0) \rangle$ and that the function $\hat{\varphi}(x)$ is continuous in this interval. Further, the proof runs by induction. Proving for $m \geqslant 0$ that formula (32) defines a continuous function $\hat{\varphi}(x)$ in the interval $\langle x_{m+1}, x_m \rangle$, we shall show also that the function $\hat{\varphi}(x)$ fulfils in the interval $\langle x_{m+1}, x_m \rangle$ condition (29) and that it is continuous at the point x_m.

For $x \epsilon \langle x_1, x_0 \rangle$ we have by (14) $f(x) \epsilon \langle x_0, f(x_0) \rangle$, whence by (32) and (27)

$$\hat{\varphi}[f(x)] = \hat{\varphi}_0[f(x)] \epsilon \Gamma_x.$$

Consequently, the expression $\hat{h}(x, \hat{\varphi}[f(x)])$ is meaningful and formula (32) defines in the interval $\langle x_1, x_0 \rangle$ the function $\hat{\varphi}(x)$ and this function $\hat{\varphi}(x)$ fulfils in the interval $\langle x_1, x_0 \rangle$ condition (29). The continuity of the function $\hat{\varphi}(x)$ in the interval $\langle x_1, x_0 \rangle$ results from the continuity of the functions $f(x)$ and $\hat{h}(x, \hat{y})$ and from the continuity — already proved — of the function $\hat{\varphi}(x)$ in the interval $\langle x_0, f(x_0) \rangle$; moreover, in virtue of the continuity of the function $\hat{\varphi}(x)$ in the interval $\langle x_0, f(x_0) \rangle$, we have

$$\lim_{x \to x_0+} \hat{\varphi}(x) = \hat{\varphi}(x_0),$$

whereas by (28) and (32) we have

$$\lim_{x \to x_0-} \hat{\varphi}(x) = \lim_{x \to x_0-} \hat{h}(x, \hat{\varphi}[f(x)]) = \lim_{x \to x_0-} \hat{h}(x, \hat{\varphi}_0[f(x)]) = \hat{\varphi}_0(x_0) = \hat{\varphi}(x_0);$$

consequently, the function $\hat{\varphi}(x)$ is continuous also at the point x_0.

Now let us suppose that for an $m \geqslant 0$ formula (32) defines in the interval $\langle x_{m+1}, x_m \rangle$ a continuous function $\hat{\varphi}(x)$, the function $\hat{\varphi}(x)$ fulfils in the interval $\langle x_{m+1}, x_m \rangle$ relation (29) and it is continuous also at the point x_m.

For $x \in \langle x_{m+2}, x_{m+1} \rangle$ we have by (14) $f(x) \in \langle x_{m+1}, x_m \rangle$, whence, in virtue of the induction hypothesis and Hypothesis 3,

$$\hat{\varphi}[f(x)] \in \Lambda_{f(x)} \subset \Gamma_x;$$

consequently, the expression $\hat{h}(x, \hat{\varphi}[f(x)])$ is meaningful and formula (32) defines the function $\hat{\varphi}(x)$ in the interval $\langle x_{m+2}, x_{m+1} \rangle$ and this function $\hat{\varphi}(x)$ fulfils condition (29) in the interval $\langle x_{m+2}, x_{m+1} \rangle$. The continuity of the function $\hat{\varphi}(x)$ in the interval $\langle x_{m+2}, x_{m+1} \rangle$ results from the continuity of the functions $f(x)$ and $\hat{h}(x, \hat{y})$ and from the assumed continuity of the function $\hat{\varphi}(x)$ in the interval $\langle x_{m+1}, x_m \rangle$; moreover, in virtue of the assumed continuity of the function $\hat{\varphi}(x)$ at the point x_m, the function $\hat{\varphi}(x)$ is continuous also at the point x_{m+1}.

Hence — on account of the induction principle and in view of relation (31) — we infer that the function $\hat{\varphi}(x)$ is defined and continuous in the whole interval $(I - J_0) \cup I_0$ and that the function $\hat{\varphi}(x)$ fulfils relation (29) in the interval $I - J_0$. The fact that the function $\hat{\varphi}(x)$ satisfies the system of equations (2) in the interval $I - J_0$ and fulfils condition (30) is a direct consequention of formula (32). It is also obvious from this formula (because of the way of constructing the function $\hat{\varphi}(x)$) that the function $\hat{\varphi}(x)$ obtained is the only function which is defined in the interval $(I - J_0) \cup I_0$, fulfils relation (29) in the interval $I - J_0$, satisfies the system of equations (2) in the interval $I - J_0$, and fulfils condition (30).

This completes the proof of Lemma 5.

Further, we assume about the function $\hat{h}(x, \hat{y})$ that it fulfils also the following

HYPOTHESIS 4. *For every fixed* $x \in I$ *the function* $\hat{h}(x, \hat{y})$ *is a homeomorphism of the domain* Γ_x *onto the domain* Λ_x, *i.e., there exists a function* $\hat{g}(x, \hat{y}) = (g_1(x, \hat{y}), \ldots, g_n(x, \hat{y}))$ *such that*

$$(33) \qquad \hat{g}(x, \hat{h}(x, \hat{y})) = \hat{y} \qquad \text{for every } (x, \hat{y}) \in \Omega,$$

and the function $\hat{g}(x, \hat{y})$ *is defined and continuous in a domain* Ω' *of the* $(n+1)$-*dimentional space of the variables* $x, y_1, \ldots, y_n$ *such that for every* $x \in I$

$$\{\hat{y} : (x, \hat{y}) \in \Omega'\} = \Lambda_x,$$

and Γ_x *is the range of the function* $\hat{g}(x, \hat{y})$ *for* $\hat{y} \in \Lambda_x$, *i.e.,*

$$\Gamma_x = \hat{g}_x(\Lambda_x), \qquad where \quad \hat{g}_x(\hat{y}) = \hat{g}(x, \hat{y}).$$

We assume also the following

HYPOTHESIS 5. *For every* $x \in I$

$$\Gamma_x \subset \Lambda_{f(x)}.$$

Now we prove

LEMMA 6. *Let the function* $f(x)$ *fulfil Hypothesis* 1, *let* $\xi \notin I$, *let the function* $f(x)$ *be strictly increasing in the interval* I *and let Hypotheses* 2, 4 *and* 5 *be fulfilled; further, we put* $I_0 = \langle x_0, f(x_0) \rangle$ *and* $J_0 = \langle x_0, \xi \rangle$ *if the point* ξ *is the right end-point of the interval* I, *and* $I_0 = (f(x_0), x_0\rangle$ *and* $J_0 = (\xi, x_0\rangle$ *if the point* ξ *is the left end-point of the interval* I, x_0 *being an arbitrarily fixed point of the interval* I. *Then, for an arbitrary function* $\hat{\varphi}_0(x)$ *defined and continuous in the interval* I_0 *and fulfilling the relations*

(34) $$\hat{\varphi}_0(x) \in \Lambda_x \qquad \text{for every } x \in I_0$$

and (28), *there exists exactly one function* $\hat{\varphi}(x)$ *defined in the interval* J_0, *fulfilling in the interval* $J_0 - I_0$ *the relation*

(35) $$\hat{\varphi}(x) \in \Gamma_{f^{-1}(x)},$$

satisfying the system of equations (2) *in the interval* J_0 *and fulfilling condition* (30); *moreover, the function* $\hat{\varphi}(x)$ *is continuous in the interval* J_0.

Proof. Let the point ξ be the right end-point of the interval I so that $I_0 = \langle x_0, f(x_0) \rangle$ and $J_0 = \langle x_0, \xi \rangle$. (If the point ξ is the left end-point of the interval I, the proof is analogous). Let the sequence x_k be defined for $k > 0$ by relation (17). According to Lemma 3 the sequence x_k is infinite, strictly increasing, $x_k \in I$, and relations (18) and (19) hold.

Let $\hat{\varphi}_0(x)$ be an arbitrary function defined and continuous in the interval I_0 and fulfilling relations (34) and (28). (The existence of such a function is guaranteed by Hypotheses 2, 4 and 5; moreover, there exist infinitely many such functions.) In view of (33) relation (28) is equivalent to

(36) $$\lim_{x \to x_0, f(x) \in I_0} \hat{\varphi}_0[f(x)] = \hat{g}(x_0, \hat{\varphi}_0(x_0)).$$

We shall show that the formula

(37) $$\hat{\varphi}(x) = \begin{cases} \hat{\varphi}_0(x) & \text{for } x \in \langle x_0, x_1 \rangle, \\ \hat{g}(f^{-1}(x), \hat{\varphi}[f^{-1}(x)]) & \text{for } x \in \langle x_k, x_{k+1} \rangle, \quad k = 1, 2, \ldots, \end{cases}$$

defines a continuous function $\hat{\varphi}(x)$ in the interval J_0 and the function $\hat{\varphi}(x)$ fulfils relation (35) in the interval $J_0 - I_0$.

It follows from the definition of the function $\hat{\varphi}_0(x)$ that formula (37) defines the function $\hat{\varphi}(x)$ in the interval $\langle x_0, x_1 \rangle$ and that the function $\hat{\varphi}(x)$ is continuous in this interval. Further the proof runs by induction. Proving for $m \geqslant 1$ that formula (37) defines a continuous function $\hat{\varphi}(x)$ in the interval $\langle x_m, x_{m+1} \rangle$, we shall show also that the function $\hat{\varphi}(x)$ ful-

fils relation (35) in the interval $\langle x_m, x_{m+1})$ and that it is continuous also at the point x_m.

For $x \in \langle x_1, x_2)$ we have, by (18), $f^{-1}(x) \in \langle x_0, x_1)$, whence on account of relations (37) and (34)

$$\hat{\varphi}[f^{-1}(x)] = \hat{\varphi}_0[f^{-1}(x)] \in \Lambda_{j-1(x)}.$$

Consequently, the expression $\hat{g}(f^{-1}(x), \hat{\varphi}[f^{-1}(x)])$ is meaningful and thus relation (37) defines the function $\hat{\varphi}(x)$ in the interval $\langle x_1, x_2)$ and this function $\hat{\varphi}(x)$ fulfils condition (35) in the interval $\langle x_1, x_2)$. The continuity of the function $\hat{\varphi}(x)$ in the interval $\langle x_1, x_2)$ results from the continuity of the functions $f(x)$ and $\hat{g}(x, \hat{y})$ and from the continuity, already proved, of the function $\hat{\varphi}(x)$ in the interval $\langle x_0, x_1)$; moreover, in virtue of the continuity of the function $\hat{\varphi}(x)$ in the interval $\langle x_1, x_2)$, we have

$$\lim_{x \to x_1+} \hat{\varphi}(x) = \hat{\varphi}(x_1),$$

whereas by (36) and (37) we have

$$\lim_{x \to x_1-} \hat{\varphi}(x) = \lim_{x \to x_1-} \hat{\varphi}_0(x) = \lim_{x \to x_0-} \hat{\varphi}_0[f(x)] = \hat{g}(x_0, \hat{\varphi}_0(x_0))$$
$$= \hat{g}(f^{-1}(x_1), \hat{\varphi}[f^{-1}(x_1)]) = \hat{\varphi}(x_1);$$

consequently, the function $\hat{\varphi}(x)$ is continuous also at the point x_1.

Now let us suppose that for an $m \geqslant 1$ formula (37) defines a continuous function $\hat{\varphi}(x)$ in the interval $\langle x_m, x_{m+1})$, the function $\hat{\varphi}(x)$ fulfils relation (35) in the interval $\langle x_m, x_{m+1})$ and it is continuous also at the point x_m.

For $x \in \langle x_{m+1}, x_{m+2})$ we have, by (18), $f^{-1}(x) \in \langle x_m, x_{m+1})$, whence, in virtue of the induction hypothesis and Hypothesis 5,

$$\hat{\varphi}[f^{-1}(x)] \in \Gamma_{j-2(x)} \subset \Lambda_{j-1(x)};$$

consequently, the expression $\hat{g}(f^{-1}(x), \hat{\varphi}[f^{-1}(x)])$ is meaningful and formula (37) defines the function $\hat{\varphi}(x)$ in the interval $\langle x_{m+1}, x_{m+2})$ and the function $\hat{\varphi}(x)$ fulfils relation (35) in this interval. The continuity of the function $\hat{\varphi}(x)$ results from the continuity of the functions $f(x)$ and $\hat{g}(x, \hat{y})$ and from the continuity assumed of the function $\hat{\varphi}(x)$ in the interval $\langle x_m, x_{m+1})$; moreover, in virtue of the continuity assumed of the function $\hat{\varphi}(x)$ at the point x_m, the function $\hat{\varphi}(x)$ is continuous also at the point x_{m+1}.

Hence — on account of the induction principle and in view of relation (19) — we infer that the function $\hat{\varphi}(x)$ is defined and continuous in the whole interval J_0 and that the function $\hat{\varphi}(x)$ fulfils relation (35) in the interval $J_0 - I_0$. The fact that the function $\hat{\varphi}(x)$ satisfies the system of equations (2) in the interval J_0 and fulfils condition (30) is a direct consequence of formulae (37) and (33). It is also obvious from those for-

mulae (because of the way of constructing the function $\hat{\varphi}(x)$) that the function $\hat{\varphi}(x)$ obtained is the only function which is defined in the interval J_0, fulfils relation (35) in the interval $J_0 - I_0$, satisfies the system of equations (2) in the interval J_0 and fulfils condition (30).

This completes the proof of Lemma 6.

The following theorem is a consequence of Lemmas 5 and 6.

THEOREM 2. *Let the function* $f(x)$ *fulfil Hypothesis* 1, *let* $\xi \notin I$, *let the function* $f(x)$ *be strictly increasing in the interval* I *and let Hypotheses* 2, 3, 4 *and* 5 *be fulfilled; further, we put* $I_0 = \langle x_0, f(x_0)\rangle$ *and* $J_0 = \langle x_0, \xi\rangle$ *if the point* ξ *is the right end-point of the interval* I, *and* $I_0 = (f(x_0), x_0\rangle$ *and* $J_0 = (\xi, x_0\rangle$ *if the point* ξ *is the left end-point of the interval* I, x_0 *being and arbitrarily fixed point of the interval* I. *Then, for an arbitrary function* $\hat{\varphi}_0(x)$ *defined and continuous in the interval* I_0 *and fulfilling relations* (34) *an* (28), *there exists exactly one function* $\hat{\varphi}(x)$ *defined in the interval* I, *fulfilling relation* (29) *in the interval* I, *satisfying the system of equations* (2) *in the interval* I *and fulfilling condition* (30); *moreover, the function* $\hat{\varphi}(x)$ *is continuous in the interval* I.

Proof. It is enough to notice that the simultaneous fulfilment of Hypotheses 3 and 5 yields

$$\Lambda_{f(x)} = \Gamma_x$$

for every $x \in I$; that $I_0 \cap f(I) \subset I_0$, $[(I-J_0) \cup I_0] \cup J_0 = I$, $(I-J_0) \cup J_0 = I$, $(I-J_0) \cup (J_0-I_0) \cup I_0 = I$, $[(I-J_0) \cup I_0] \cap J_0 = I_0$; and that relation (29) results from (35) and relation (27) results from (34).

Theorem 2 says that a function defined in a not very large interval can be uniquely extended (under suitable assumptions) onto the whole interval considered to a solution of system (2). Now we shall show that a solution of the system of equations (2) defined in a somewhat larger interval can be uniquely extended — under weaker assumptions — onto the whole interval considered. This is expressed by the following

THEOREM 3. *Let the function* $f(x)$ *fulfil Hypothesis* 1 $(\xi \in I$ *or* $\xi \notin I)$ *and let Hypotheses* 2 *and* 3 *be fulfilled; further, let* $I_0 = I \cap \langle \xi - \delta, \xi + \delta\rangle$, *where* δ *is a positive number. Then, for an arbitrary function* $\hat{\varphi}_0(x)$ *defined and continuous in the interval* I_0, *fulfilling relation* (29) *in the interval* I_0 *and satisfying the system of equations* (2) *in the interval* I_0, *there exists exactly one function* $\hat{\varphi}(x)$ *defined in the interval* I, *fulfilling relation* (29) *in the interval* I, *satisfying the system of equations* (2) *in the interval* I *and fulfilling condition* (30); *moreover, the function* $\hat{\varphi}(x)$ *is continuous in the interval* I.

Proof. Let the point ξ be the right end-point of the interval I, let $\xi \in I$, and let $\delta > 0$ be such that $\xi - \delta \in I$; we write $x_0 = \xi - \delta$ so that $I_0 = \langle x_0, \xi\rangle$ and let the sequence x_k be defined for $k > 0$ by relations (20) and (21). If the point ξ is the left end-point of the interval I, the proof is analogous with the only difference that the sequence x_k is de-

fined by relations (25) and (26); if the point ξ is an inner point of the interval I, the construction given below of the solution of system (2) must be carried out independently in every part of the interval I obtained by the division of the latter by the point ξ; finally, if the point ξ does not belong to I, the proof is also analogous.

According to Lemma 4 the sequence x_k is strictly decreasing, $x_k \epsilon I$ and relation (22) holds. Let us assume that the sequence x_k is infinite and thus, by Lemma 4, relation (23) holds. If the sequence x_k is finite, the proof is analogous with the only exception that the inequality $k \leqslant k_0$ must be taken into account.

Let $\hat{\varphi}_0(x)$ be an arbitrary function defined and continuous in the interval I_0, fulfilling relation (29) in the interval I_0 and satisfying the system of equations (2) in the interval I_0. Let us note that in view of Hypothesis 3 the function $\hat{\varphi}_0(x)$ fulfils also relation (27). We shall show that the formula

$$(38) \qquad \hat{\varphi}(x) = \begin{cases} \hat{\varphi}_0(x) & \text{for } x \epsilon \langle x_0, \xi \rangle, \\ \hat{h}\big(x, \hat{\varphi}[f(x)]\big) & \text{for } x \epsilon \langle x_{k+1}, x_k \rangle, \ k = 0, 1, 2, \ldots, \end{cases}$$

defines a continuous function $\hat{\varphi}(x)$ in the interval I and that the function $\hat{\varphi}(x)$ fulfils relation (29) in the interval I.

It follows from the definition of the function $\hat{\varphi}_0(x)$ that formula (38) defines the function $\hat{\varphi}(x)$ in the interval $\langle x_0, \xi \rangle$ and that the function $\hat{\varphi}(x)$ is continuous in this interval and fulfils relation (29) there. Further the proof runs by induction.

For $x \epsilon \langle x_1, x_0 \rangle$ we have, by (22), $f(x) \epsilon \langle x_0, \xi \rangle$, whence by (38) and (27)

$$\hat{\varphi}[f(x)] = \hat{\varphi}_0[f(x)] \epsilon \Gamma_x.$$

Consequently, the expression $\hat{h}\big(x, \hat{\varphi}[f(x)]\big)$ is meaningful and formula (38) defines a function $\hat{\varphi}(x)$ in the interval $\langle x_1, x_0 \rangle$ and the function $\hat{\varphi}(x)$ fulfils relation (29) in the interval $\langle x_1, x_0 \rangle$. The continuity of the function $\hat{\varphi}(x)$ in the interval $\langle x_1, x_0 \rangle$ results from the continuity of the functions $f(x)$ and $\hat{h}(x, \hat{y})$ and from the continuity, already proved, of the function $\hat{\varphi}(x)$ in the interval $\langle x_0, \xi \rangle$; moreover, in virtue of the continuity of the function $\hat{\varphi}(x)$ in the interval $\langle x_0, \xi \rangle$, we have

$$\lim_{x \to x_0+} \hat{\varphi}(x) = \hat{\varphi}(x_0),$$

whereas, by (38) and in view of the fact that $\hat{\varphi}(x)$ satisfies the system of equations (2) at the point x_0, we have

$$\lim_{x \to x_0-} \hat{\varphi}(x) = \lim_{x \to x_0-} \hat{h}\big(x, \hat{\varphi}[f(x)]\big) = \hat{h}\big(x_0, \hat{\varphi}[f(x_0)]\big) = \hat{\varphi}(x_0);$$

consequently, the function $\hat{\varphi}(x)$ is continuous also at the point x_0. Thus

the function $\hat{\varphi}(x)$ is defined and continuous in the interval $\langle x_1, \xi \rangle$ and fulfils relation (29) in the interval $\langle x_1, \xi \rangle$.

Now let us suppose that for an $m \geqslant 1$ formula (38) defines a continuous function $\hat{\varphi}(x)$ in the interval $\langle x_m, \xi \rangle$ and that the function $\hat{\varphi}(x)$ fulfils relation (29) in the interval $\langle x_m, \xi \rangle$.

For $x \in \langle x_{m+1}, x_m \rangle$ we have, by (22), $f(x) \in \langle x_m, \xi \rangle$, whence in virtue of the induction hypothesis and Hypothesis 3

$$\hat{\varphi}[f(x)] \in \Lambda_{f(x)} \subset \Gamma_x;$$

consequently, the expression $h(x, \hat{\varphi}[f(x)])$ is meaningful and formula (38) defines the function $\hat{\varphi}(x)$ in the interval $\langle x_{m+1}, x_m \rangle$ and the function $\hat{\varphi}(x)$ fulfils relation (29) in this interval. The continuity of the function $\hat{\varphi}(x)$ in the interval $\langle x_{m+1}, x_m \rangle$ results from the continuity of the functions $f(x)$ and $h(x, \hat{y})$ and from the assumed continuity of the function $\hat{\varphi}(x)$ in the interval $\langle x_m, \xi \rangle$; moreover, in virtue of the assumed continuity of the function $\hat{\varphi}(x)$ in the interval $\langle x_m, \xi \rangle$, the function $\hat{\varphi}(x)$ is continuous also at the point x_m. Consequently, the function $\hat{\varphi}(x)$ is defined and continuous in the interval $\langle x_{m+1}, \xi \rangle$ and fulfils relation (29) in this interval.

Hence — on account of the induction principle and in view of relation (23) — we infer that the function $\hat{\varphi}(x)$ is defined and continuous in the whole interval I and that the function $\hat{\varphi}(x)$ fulfils relation (29) in the interval I. The fact that the function $\hat{\varphi}(x)$ satisfies the system of equations (2) in I is a direct consequence of formula (38) and of the fact that $\hat{\varphi}_0(x)$ satisfies the system of equations (2) in I_0. It is also obvious from formula (38) (because of the way of constructing the function $\hat{\varphi}(x)$) that the function $\hat{\varphi}(x)$ obtained is the only function which is defined in the interval I, fulfils relation (29) in the interval I, satisfies the system of equations (2) in the interval I and fulfils condition (30).

This completes the proof of Theorem 3.

4. In the sequel for any vector $\hat{y} = (y_1, \ldots, y_n)$ we shall denote by $|\hat{y}|$ its length defined as

$$|\hat{y}| = \sqrt{y_1^2 + \ldots + y_n^2},$$

and for any real square matrix $\mathfrak{A}$ we shall denote by $\|\mathfrak{A}\|$ its norm defined as

$$\|\mathfrak{A}\| = \sup_{y \neq 0} \frac{|\mathfrak{A}\hat{y}|}{|\hat{y}|}$$

(cf. [5], p. 110).

In order to prove further theorems we shall need two lemmas.

LEMMA 7. *For an arbitrary number* $\varepsilon > 0$ *and an arbitrary real square matrix* $\mathfrak{A}$ *there exists a real square matrix* $\mathfrak{T}$ *such that*

$$\|\mathfrak{T}\mathfrak{A}\mathfrak{T}^{-1}\| \leqslant \lambda_0 + \varepsilon,$$

where

$$\lambda_0 \;=\; \max_{\varkappa} |\lambda_\varkappa|,$$

and $\lambda_\varkappa$ are the characteristic roots of the matrix $\mathfrak{A}$.

This lemma is a modification for the case of real matrices of a theorem given for complex matrices by A. Ostrowski (cf. [15], p. 120). The proof given below follows the same basic idea as that given by A. Ostrowski.

Proof. Let $\mathfrak{A}$ be an arbitrary square real matrix. Then there exists (cf. [14], p. 86) a real matrix $\mathfrak{T}$ such that

$$\mathfrak{A}^* \;=\; \mathfrak{T}\mathfrak{A}\mathfrak{T}^{-1} \;=\; \begin{bmatrix} \mathfrak{A}_1 & & \\ & \mathfrak{A}_2 & \\ & & \ddots \end{bmatrix},$$

where $\mathfrak{A}_\sigma$ are real suqare matrices and have one of the forms

$$[t], \quad \begin{bmatrix} t & & & \\ 1 & t & & \\ & & t & \\ & & 1 & t \end{bmatrix}, \quad \mathfrak{B}, \quad \text{or} \quad \begin{bmatrix} \mathfrak{B} & & & \\ \mathfrak{C} & \mathfrak{B} & & \\ & & \mathfrak{B} & \\ & & \mathfrak{C} & \mathfrak{B} \end{bmatrix},$$

where

$$\mathfrak{B} \;=\; \begin{bmatrix} a & b \\ -b & a \end{bmatrix}, \quad \mathfrak{C} \;=\; \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Let ε be an arbitrary positive real number. Applying the above argument to the matrix $\mathfrak{A}/\varepsilon$ and multiplying the resulting matrix by ε, we obtain

$$\mathfrak{A}^* \;=\; \mathfrak{T}\mathfrak{A}\mathfrak{T}^{-1} \;=\; \begin{bmatrix} \mathfrak{B}_1 & & \\ & \mathfrak{B}_2 & \\ & & \ddots \end{bmatrix},$$

where $\mathfrak{B}_\sigma$ are real square matrices and have one of the forms

$$[t], \quad \begin{bmatrix} t & & & \\ \varepsilon & t & & \\ & & t & \\ & & \varepsilon & t \end{bmatrix}, \quad \mathfrak{B}, \quad \text{or} \quad \begin{bmatrix} \mathfrak{B} & & & \\ \mathfrak{F} & \mathfrak{B} & & \\ & & \mathfrak{B} & \\ & & \mathfrak{F} & \mathfrak{B} \end{bmatrix},$$

where

$$\mathfrak{B} = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}, \quad \mathfrak{F} = \begin{bmatrix} \varepsilon & 0 \\ 0 & \varepsilon \end{bmatrix}.$$

We represent the matrix $\mathfrak{A}^*$ as the sum of two matrices, $\mathfrak{G}$ and $\mathfrak{H}$, such that

$$\mathfrak{G} = \begin{bmatrix} \mathfrak{G}_1 & & \\ & \mathfrak{G}_2 & \\ & & \cdot \\ & & & \cdot \end{bmatrix}, \quad \mathfrak{H} = \begin{bmatrix} \mathfrak{H}_1 & & \\ & \mathfrak{H}_2 & \\ & & \cdot \\ & & & \cdot \end{bmatrix},$$

where $\mathfrak{G}_\sigma$ are real square matrices and have one of the forms

$$[t], \quad \begin{bmatrix} t & & & \\ & t & & \\ & & \cdot \; t & \\ & & & t \end{bmatrix}, \quad \mathfrak{B}, \quad \text{or} \quad \begin{bmatrix} \mathfrak{B} & & & \\ & \mathfrak{B} & & \\ & & \mathfrak{B} & \\ & & & \mathfrak{B} \end{bmatrix},$$

and $\mathfrak{H}_\sigma$ are real square matrices and have one of the forms

$$[0], \quad \begin{bmatrix} 0 & & & \\ \varepsilon & 0 & & \\ & & 0 & \\ & & \varepsilon & 0 \end{bmatrix}, \quad \mathfrak{D}, \quad \text{or} \quad \begin{bmatrix} \mathfrak{D} & & & \\ \mathfrak{F} & \mathfrak{D} & & \\ & & \mathfrak{D} & \\ & & \mathfrak{F} & \mathfrak{D} \end{bmatrix},$$

where

$$\mathfrak{D} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathfrak{F} = \begin{bmatrix} \varepsilon & 0 \\ 0 & \varepsilon \end{bmatrix}.$$

Let $\hat{y} = (y_1, \ldots, y_n)$, where n denotes the order of the matrix $\mathfrak{A}$, be an arbitrary vector. Then either the $\varkappa$-th component of the vector $\mathfrak{G}\hat{y}$ is equal to $ty_\varkappa$, or two consecutive components, $\varkappa$-th and $(\varkappa+1)$-st, of the vector $\mathfrak{G}\hat{y}$ are equal to $ay_\varkappa + by_{\varkappa+1}$ and $ay_{\varkappa+1} - by_\varkappa$, and similarly, the $\varkappa$-th component of the vector $\mathfrak{H}\hat{y}$ is equal either to 0, or to $\varepsilon y_{\varkappa-1}$, or to $\varepsilon y_{\varkappa-2}$. Consequently,

$$\|\mathfrak{G}\| = \sup_{\hat{y} \neq 0} \frac{|\mathfrak{G}\hat{y}|}{|\hat{y}|} \leqslant \max_{t,(a,b)} (|t|, \sqrt{a^2 + b^2}),$$

and

$$\|\mathfrak{H}\| = \sup_{\hat{y} \neq 0} \frac{|\mathfrak{H}\hat{y}|}{|\hat{y}|} \leqslant \varepsilon.$$

Hence

$$\|\mathfrak{A}^*\| \leqslant \|\mathfrak{G}\| + \|\mathfrak{H}\| \leqslant \max_{t,(a,b)} (|t|, \sqrt{a^2+b^2}) + \varepsilon.$$

Since t, $a+ib$ $(i = \sqrt{-1})$ are characteristic roots of the matrices $\mathfrak{B}_\sigma$, and the characteristic polynomial of the matrix $\mathfrak{A}^*$ equals the product of the characteristic polynomials of the matrices $\mathfrak{B}_\sigma$, they are also characteristic roots of the matrix $\mathfrak{A}^*$, and hence also of the similar matrix $\mathfrak{A}$. Consequently,

$$\|\mathfrak{A}^*\| \leqslant \max_\varkappa |\lambda_\varkappa| + \varepsilon.$$

This completes the proof of Lemma 7.

LEMMA 8. Let the function $f(x)$ fulfil Hypothesis 1 ($\xi \epsilon I$ or $\xi \notin I$) and let Hypotheses 2 and 3 be fulfilled; further, let $I_0 = I \cap \langle \xi - \delta, \xi + \delta \rangle$, where δ is a positive number, and let $\hat{\varphi}_0(x)$ be an arbitrary function defined in the interval I and fulfilling the relation

(39) $\hat{\varphi}_0[f(x)] \epsilon \Gamma_x$ for every $x \epsilon I$.

Then, if the sequence $\hat{\varphi}_k(x)$ defined in the interval I by the formula

(40) $\hat{\varphi}_{k+1}(x) = \hat{h}(x, \hat{\varphi}_k[f(x)])$ for $k = 0, 1, 2, \ldots,$

converges in the interval I_0 to a function $\hat{\varphi}(x)$ defined and continuous in the interval I_0 and fulfilling relation (29) in the interval I_0, then the sequence $\hat{\varphi}_k(x)$ converges in the whole interval I and its limit $\hat{\varphi}(x)$ is defined and continuous in the interval I, fulfils relation (29) in I and satisfies the system of equations (2) in I.

Proof. In virtue of relation (39) and Hypotheses 2 and 3 the functions $\hat{\varphi}_k(x)$ for $k > 0$ are defined by relation (40) in the interval I and fulfil relation (29) in I. Let us suppose that the sequence $\hat{\varphi}_k(x)$ converges in the interval I_0 to a function $\hat{\varphi}(x)$ defined and continuous in the interval I_0 and fulfilling condition (29) in I_0. It follows from relation (40) that if $x_0 \epsilon I$ and the sequence $\hat{\varphi}_k(x)$ converges for $x = f(x_0)$, then it converges also for $x = x_0$. Thus, since for every $x_0 \epsilon I$ and for m sufficiently large we have $f^m(x_0) \epsilon I_0$, the sequence of the functions $\hat{\varphi}_k(x)$ converges in the interval I to a function $\hat{\varphi}(x)$ defined in the interval I. Furthermore, the function $\hat{\varphi}(x)$ fulfils relation (29) in the interval I. The function $\hat{\varphi}(x)$ restricted to the interval I_0 is, by our assumptions, a continuous solution of the system of equations (2) in the interval I_0 and fulfils relation (29) in I_0. On account of Theorem 3 it can be uniquely extended onto the interval I to a solution of the system of equations (2) and this solution fulfils relation (29) in the interval I and is a continuous function in this interval. But the function $\hat{\varphi}(x)$ as a whole — defined in the interval I — is just an extension of the continuous solution of the system of equa-

tions (2) in the interval I_0 which fulfils relation (29) in I. Thus the function $\hat{\varphi}(x)$ must be continuous in the interval I.

This completes the proof of Lemma 8.

Theorem 2 has given sufficient conditions in order that the system of equations (2) have a continuous solution depending on an arbitrary function in an interval that does not contain the point ξ. That theorem implies the existence of infinitely many such solutions. Now we are going to show that if the point ξ belongs to the interval I, then in some cases there exists at most one continuous solution of the system of equations (2) in the interval I.

Since $f(\xi) = \xi$, for every solution $\hat{\varphi}(x)$ of the system of equations (2) defined at the point ξ the value

$$(41) \qquad \hat{\eta} = \hat{\varphi}(\xi)$$

must fulfil the equation

$$(42) \qquad \hat{\eta} = h(\xi, \hat{\eta}).$$

Sufficient conditions for the uniqueness of continuous solutions of the system of equations (2) are given by the following

Theorem 4. *Let the function $f(x)$ fulfil Hypothesis* 1, *let $\xi \in I$, and let Hypotheses* 2 *and* 3 *be fulfilled; further, let $\hat{\eta}$ be a root of equation* (42) *such that $\hat{\eta} \in \Lambda_\xi$, and let the function $h(x, \hat{y})$ fulfil with respect to the variable $\hat{y}$ a Lipschitz condition with a constant less than* 1, *in a neighbourhood V of the point $(\xi, \hat{\eta})$ contained in the domain Ω, i.e.*

$$(43) \qquad |h(x, \hat{y}_1) - h(x, \hat{y}_2)| \leqslant \vartheta |\hat{y}_1 - \hat{y}_2|$$

$$\textit{for every } (x, \hat{y}_1) \in V \textit{ and } (x, \hat{y}_2) \in V,$$

where $0 < \vartheta < 1$. Then there exists exactly one function $\hat{\varphi}(x)$ which is defined and continuous in the interval I, fulfils relation (29) *in I, fulfils condition* (41) *and satisfies the system of equations* (2) *in the interval I. This function is given by the formula*

$$(44) \qquad \hat{\varphi}(x) = \lim_{k \to \infty} \hat{\varphi}_k(x),$$

where the sequence of the functions $\hat{\varphi}_k(x)$ is defined for $k > 0$ in the interval I by relation (40) *and $\hat{\varphi}_0(x)$ is an arbitrary function defined and continuous in the interval I and fulfilling condition* (39) *and*

$$(45) \qquad \hat{\varphi}_0(\xi) = \hat{\eta}.$$

Proof. Let the point ξ be the right end-point of the interval I. If the point ξ is the left end-point of the interval I, the proof is analogous, and if ξ is an inner point of the interval I, then, in view of relation (41), the existence and uniqueness of the solution of system (2) in the left part

of the interval I and its existence and uniqueness in the right part of the interval I (ξ being the point of division) yield its existence and uniqueness in the whole interval I.

Let us fix numbers $c > 0$ and $d > 0$ in such a manner that $\xi - c \epsilon I$ and

(46) $\{(x, \hat{y}): \xi - c \leqslant x \leqslant \xi, |\hat{y} - \hat{\eta}| \leqslant d\} \subset V$,

and

(47) $|\hat{h}(x, \hat{\eta}) - \hat{h}(\xi, \hat{\eta})| \leqslant (1 - \vartheta)d$ for $x \epsilon \langle \xi - c, \xi \rangle$.

Condition (47) can be realized in virtue of the continuity of the function $\hat{h}(x, \hat{\eta})$ at the point ξ.

Let Φ be the space of functions $\hat{\varphi}$ that are defined and continuous in the interval $\langle \xi - c, \xi \rangle$, fulfil condition (29) in the interval $\langle \xi - c, \xi \rangle$, and fulfil relation (41) and

(48) $|\hat{\varphi}(x) - \hat{\eta}| \leqslant d$ for $x \epsilon \langle \xi - c, \xi \rangle$.

Further, let Φ be endowed with the metric

(49) $\varrho(\hat{\varphi}_1, \hat{\varphi}_2) = \sup_{\langle \xi - c, \xi \rangle} |\hat{\varphi}_1(x) - \hat{\varphi}_2(x)|$.

Since for $x \epsilon \langle \xi - c, \xi \rangle$ we have $f(x) \epsilon \langle \xi - c, \xi \rangle$, relation (48) implies

(50) $|\hat{\varphi}[f(x)] - \hat{\eta}| \leqslant d$ for $x \epsilon \langle \xi - c, \xi \rangle$.

Hence it follows according to relations (46) and (43) that

(51) $|\hat{h}(x, \hat{\varphi}[f(x)]) - \hat{h}(x, \hat{\eta})| \leqslant \vartheta |\hat{\varphi}[f(x)] - \hat{\eta}|$ for $x \epsilon \langle \xi - c, \xi \rangle$ and $\hat{\varphi} \epsilon \Phi$.

In the space Φ we define the transform

(52) $\hat{\psi}(x) = \hat{h}(x, \hat{\varphi}[f(x)])$.

By relations (50) and (46) we have $(x, \hat{\varphi}[f(x)]) \epsilon \Omega$ for $x \epsilon \langle \xi - c, \xi \rangle$; consequently, the functions $\hat{\psi}$ are defined in the interval $\langle \xi - c, \xi \rangle$ and fulfil relation (29) in $\langle \xi - c, \xi \rangle$. The continuity of the functions $\hat{\psi}$ in this interval results from the continuity of the functions $f(x)$ and $\hat{h}(x, \hat{y})$ and of the function $\hat{\varphi}$. Moreover, we have by (52), (42), (51), (47) and (50)

$|\hat{\psi}(x) - \hat{\eta}| = |\hat{h}(x, \hat{\varphi}[f(x)]) - \hat{\eta}| = |\hat{h}(x, \hat{\varphi}[f(x)]) - \hat{h}(\xi, \hat{\eta})|$

$\leqslant |\hat{h}(x, \hat{\varphi}[f(x)]) - \hat{h}(x, \hat{\eta})| + |\hat{h}(x, \hat{\eta}) - \hat{h}(\xi, \hat{\eta})|$

$\leqslant \vartheta |\hat{\varphi}[f(x)] - \hat{\eta}| + (1 - \vartheta)d \leqslant \vartheta d + (1 - \vartheta)d = d$,

and by (52), (41) and (42)

$\hat{\psi}(\xi) = \hat{h}(\xi, \hat{\varphi}[f(\xi)]) = \hat{h}(\xi, \hat{\varphi}(\xi)) = \hat{h}(\xi, \hat{\eta}) = \eta$.

This means that transform (52) maps the space Φ into itself. Furthermore, we have by (50), (46) and (43) for $\hat{\psi}_1(x) = \hat{h}(x, \hat{\varphi}_1[f(x)])$, $\hat{\psi}_2(x) = \hat{h}(x, \hat{\varphi}_2[f(x)])$, $\hat{\varphi}_1 \epsilon \Phi$, $\hat{\varphi}_2 \epsilon \Phi$:

$$|\hat{\psi}_1(x) - \hat{\psi}_2(x)| = |\hat{h}(x, \hat{\varphi}_1[f(x)]) - \hat{h}(x, \hat{\varphi}_2[f(x)])|$$

$$\leqslant \vartheta |\hat{\varphi}_1[f(x)] - \hat{\varphi}_2[f(x)]| \quad \text{for } x\epsilon\langle \xi - c, \xi \rangle,$$

whence, in virtue of definition (49),

$$\varrho(\hat{\psi}_1, \hat{\psi}_2) \leqslant \vartheta\varrho(\hat{\varphi}_1, \hat{\varphi}_2).$$

This means that transform (52) is a contraction.

According to Banach's theorem there exists a unique fixed point of transform (52) in the space Φ, given as the limit of successive approximations. In other words, there exists a unique function $\hat{\varphi}(x)$ which is defined and continuous in the interval $\langle \xi - c, \xi \rangle$, fulfils relation (29) in the interval $\langle \xi - c, \xi \rangle$, fulfils relations (41) and (48) and satisfies the system of equations (2) in the interval $\langle \xi - c, \xi \rangle$; this function $\hat{\varphi}(x)$ is given by formula (44). Taking $\delta = c$ and applying Lemma 8, we obtain the existence of the required solution of the system of equations (2), given by formula (44). Its uniqueness results from the uniqueness of the continuous solutions of system (2) in the interval $\langle \xi - c, \xi \rangle$ and from the fact that every function $\hat{\varphi}(x)$ which is defined and continuous in the interval I and fulfils condition (41), restricted to the interval $\langle \xi - c, \xi \rangle$, belongs to the space Φ provided that c is sufficiently small.

This completes the proof of Theorem 4.

Now we shall prove another theorem giving sufficient conditions for the uniqueness of continuous solutions of the system of equations (2), namely

THEOREM 5. *Let the function* $f(x)$ *fulfil Hypothesis* 1, *let* $\xi \epsilon I$, *and let Hypotheses* 2 *and* 3 *be fulfilled; further, let* $\hat{\eta}$ *be a root of equation* (42) *such that* $\hat{\eta} \epsilon \Lambda_\xi$, *and suppose that the function* $\hat{h}(x, \hat{y})$ *has continuous partial derivatives* $\dfrac{\partial h_a}{\partial y_\beta}(x, \hat{y})$ *of the first order, with respect to the variables* y_β, *for* $a = 1, \ldots, n$ *and* $\beta = 1, \ldots, n$, *in a neighbourhood* V *of the point* $(\xi, \hat{\eta})$ *contained in the domain* Ω. *Finally, let all the characteristic roots* $\lambda_\varkappa$, $\varkappa = 1, \ldots$

$\ldots, n$, *of the Jacobi matrix* $\mathfrak{A} = \left[\dfrac{\partial h_a}{\partial y_\beta}(\xi, \hat{\eta})\right]$ *of the function* $\hat{h}(x, \hat{y})$ *at the point* $(\xi, \hat{\eta})$ *be in modulus less than* 1. *Then there exists exactly one function* $\hat{\varphi}(x)$ *which is defined and continuous in the interval* I, *fulfils relation* (29) *in the interval* I, *fulfils condition* (41), *and satisfies the system of equations* (2) *in the interval* I. *This function is given by relation* (44), *where the sequence of the functions* $\hat{\varphi}_k(x)$ *is defined for* $k > 0$ *in the interval* I *by rela-*

tion (40), *and* $\hat{\varphi}_0(x)$ *is an arbitrary function defined and continuous in the interval I and fulfilling conditions* (39) *and* (45).

Proof. Let us take an arbitrary, constant (independent of the variable x) and non-singular real square matrix $\mathfrak{X}$ with n rows and n columns, and let us write:

$$\hat{y}^* = \mathfrak{X}\hat{y},$$

$$\hat{h}^*(x, \hat{y}^*) = \mathfrak{X}\hat{h}(x, \hat{y}) \quad \text{for } x\epsilon I,$$

$$\hat{\eta}^* = \mathfrak{X}\hat{\eta},$$

$$\Omega^* = \{(x, \hat{y}^*): \ (x, \hat{y})\epsilon\Omega\},$$

$$\Gamma_x^* = \{\hat{y}^*: \ \hat{y}\epsilon\Gamma_x\} \quad \text{for } x\epsilon I,$$

$$\Lambda_x^* = \{\hat{y}^*: \ \hat{y}\epsilon\Lambda_x\} \quad \text{for } x\epsilon I.$$

On account of the assumptions of the present theorem, the function $\hat{h}^*(x, \hat{y}^*)$ and the sets Ω^*, Γ_x^* and Λ_x^* fulfil conditions analogous to those in Hypotheses 2 and 3 for the function $\hat{h}(x, \hat{y})$ and the sets Ω, Γ_x and Λ_x; $\hat{\eta}^*$ is a root of the equation corresponding to equation (42) such that $\hat{\eta}^*\epsilon\Lambda_\xi^*$; and there exists a convex neighbourhood V' of the point $(\xi, \hat{\eta}^*)$ contained in the domain Ω^* in which the function $\hat{h}^*(x, \hat{y}^*)$ has continuous partial derivatives $\dfrac{\partial h_a^*}{\partial y_\beta^*}(x, \hat{y}^*)$ of the first order, with respect to the variables y_β^*, for $a = 1, \ldots, n$ and $\beta = 1, \ldots, n$. Moreover, the system of equations

(53) $$\hat{\varphi}^*(x) = \hat{h}^*\big(x, \hat{\varphi}^*[f(x)]\big),$$

where

(54) $$\hat{\varphi}^*(x) = \mathfrak{X}\hat{\varphi}(x),$$

is equivalent to the system of equations (2), i.e., if a function $\hat{\varphi}(x)$ satisfies the system of equations (2), then the function $\hat{\varphi}^*(x)$, defined by (54) as $\mathfrak{X}\hat{\varphi}(x)$, satisfies the system of equations (53), and if a function $\hat{\varphi}^*(x)$ satisfies the system of equations (53), then the function $\hat{\varphi}(x)$, defined by (54) as $\mathfrak{X}^{-1}\hat{\varphi}^*(x)$, satisfies the system of equations (2). Furthermore, if $\hat{\varphi}^*(x) = \lim_{k\to\infty}\hat{\varphi}_k^*(x)$, then $\hat{\varphi}(x) = \lim_{k\to\infty}\hat{\varphi}_k(x)$, where $\hat{\varphi}_k^*(x) = \mathfrak{X}\hat{\varphi}_k(x)$, and relations (29), (41), (40), (39) and (45) are equivalent to analogous relations for quantities with the asterisk. Therefore, in order to prove the theorem, it is enough — in view of Theorem 4 — to show that it is possible to choose the transform $\mathfrak{X}$ and a neighbourhood V^* of the point $(\xi, \hat{\eta}^*)$, contained in the domain Ω^*, in such a manner that the function $\hat{h}^*(x, \hat{y}^*)$ will fulfil in this neighbourhood a Lipschitz condition with a constant less than 1 with respect to the variable $\hat{y}^*$.

To this end let us note that the Jacobi matrix of the function $h^*(x, \hat{y}^*)$ at the point $(\xi, \hat{\eta}^*)$

$$\mathfrak{A}^* = \left[\frac{\partial h_a^*}{\partial y_\beta^*}(\xi, \hat{\eta}^*)\right]$$

fulfils the relation

$$\mathfrak{A}^* = \mathfrak{T}\mathfrak{A}\mathfrak{T}^{-1},$$

i.e., it is similar to the Jacobi matrix of the function $h(x, \hat{y})$ at the point $(\xi, \hat{\eta})$, and consequently has the same characteristic roots $\lambda_\varkappa$, $\varkappa = 1, ..., n$, as the matrix $\mathfrak{A}$. Let $\lambda_0 = \max_\varkappa |\lambda_\varkappa|$. According to Lemma 7 the transform $\mathfrak{T}$ may be chosen so that $\|\mathfrak{A}^*\| \leqslant \lambda_0 + \varepsilon'$, where ε' is an arbitrarily fixed positive number. By the hypotheses of the theorem, λ_0 is less than 1; therefore, for a sufficiently small positive number ε, we have $1 - \lambda_0 - 2\varepsilon > 0$ and we can choose the transform $\mathfrak{T}$ in such a manner that

(55) $$\|\mathfrak{A}^*\| \leqslant \lambda_0 + 1 - \lambda_0 - 2\varepsilon = 1 - 2\varepsilon.$$

In view of the continuity of the partial derivatives $\dfrac{\partial h_a^*}{\partial y_\beta^*}(x, \hat{y}^*)$ in the neighbourhood V', we have for arbitrary points $(x, \hat{y}_1^*)$ and $(x, \hat{y}_2^*)$ belonging to V'

(56) $$h^*(x, \hat{y}_1^*) - h^*(x, \hat{y}_2^*) = \mathfrak{M}(\hat{y}_1^* - \hat{y}_2^*),$$

where $\mathfrak{M}$ is a matrix whose elements are equal to the partial derivatives of the first order $\dfrac{\partial h_a^*}{\partial y_\beta^*}(x, \hat{y}^*)$ calculated at some points belonging to the neighbourhood V' Since the norm of a matrix is a continuous function of its elements (the continuity of the norm results e.g. from the inequality $\|\mathfrak{M}\| \leqslant n\max|m_{a\beta}|$, where n denotes the order of the matrix $\mathfrak{M} = [m_{a\beta}]$; cf. [2], p. 45), there exists a neighbourhood $V^* \subset V'$ of the point $(\xi, \hat{\eta}^*)$ such that for arbitrary points $(x, \hat{y}_1^*)$ and $(x, \hat{y}_2^*)$ belonging to the neighbourhood V^* we have

$$\|\mathfrak{M}\| \leqslant \|\mathfrak{A}^*\| + \varepsilon.$$

Hence it follows by (55) that

$$\|\mathfrak{M}\| \leqslant 1 - \varepsilon,$$

whence, in view of (56) and by the definition of the norm of a matrix, we obtain

$$|h^*(x, \hat{y}_1^*) - h^*(x, \hat{y}_2^*)| \leqslant \|\mathfrak{M}\| |\hat{y}_1^* - \hat{y}_2^*| \leqslant (1 - \varepsilon)|\hat{y}_1^* - \hat{y}_2^*|$$

$$\text{for } (x, \hat{y}_1^*) \in V^* \text{ and } (x, \hat{y}_2^*) \in V^*,$$

i.e., the function $h^*(x, \hat{y}^*)$ fulfils in the neighbourhood V^* of the point

$(\xi, \hat{\eta}*)$, contained in the domain Ω^*, the Lipschitz condition (43) with respect to the variable $\hat{y}^*$, with the constant $\vartheta = 1 - \varepsilon < 1$.

This completes the proof of Theorem 5.

5. Now we are going to formulate theorems concerning the properties of the system of equations (1) resulting from Theorem 1 and from the theorems concerning the properties of the system of equations (2) proved in the previous sections. For this purpose we introduce some hypotheses regarding the function $\hat{H}(x, \hat{y})$.

HYPOTHESIS 6. *The function* $\hat{H}(x, \hat{y})$ *is defined and continuous in a domain* Ω *of the* $(pN+1)$-*dimensional space of the variables* $x, y_1, \ldots, y_{pN}$ *such that for every* $x \in I$ *the set*

$$\Gamma_x = \{\hat{y}: (x, \hat{y}) \in \Omega\}$$

is a domain of the pN-*dimensional space of the variables* $y_1, \ldots, y_{pN}$; *the set of values of the function* $\hat{H}(x, \hat{y})$ *for* $\hat{y} \in \Gamma_x$:

$$\Lambda'_x = \hat{H}_x(\Gamma_x), \quad \text{where } \hat{H}_x(\hat{y}) = \hat{H}(x, \hat{y}),$$

is a domain of the N-*dimensional space of the variables* $y_1, \ldots, y_N$; *moreover, the domain* Λ_x *of the* pN-*dimensional space of the variables* $y_1, \ldots, y_{pN}$ *is defined by the formula*

$$\Lambda_x = \{(\hat{Y}_1, \hat{Y}_2): \hat{Y}_1 \in \Lambda'_x, \; \exists \hat{Y}_3 \in \Gamma'_x, \; (\hat{Y}_2, \hat{Y}_3) \in \Gamma_x, \; \hat{Y}_1 = \hat{H}(x, (\hat{Y}_2, \hat{Y}_3))\},$$

and the domain Γ'_x *of the* N-*dimensional space of the variables* $y_{(p-1)N+1}, \ldots, y_{pN}$ *is defined by the formula*

$$\Gamma'_x = \{\hat{Y}_3: \; \exists \hat{Y}_2, \; (\hat{Y}_2, \hat{Y}_3) \in \Gamma_x\}.$$

HYPOTHESIS 7. *For every fixed* $x \in I$ *and every fixed* $\hat{Y}_2$ *such that there exists a* $\hat{Y}_3 \in \Gamma'_x$ *and* $(\hat{Y}_2, \hat{Y}_3) \in \Gamma_x$, *the function* $\hat{H}(x, \hat{y})$ *is a homeomorphism of the domain* Γ'_x *onto the domain* Λ'_x, *i.e., there exists a function*

$$\hat{G}(x, \hat{y}) = (g_{(p-1)N+1}(x, \hat{y}), \ldots, g_{pN}(x, \hat{y}))$$

such that

(57) $\hat{G}(x, (\hat{H}(x, (\hat{Y}_2, \hat{Y}_3)), \hat{Y}_2)) = \hat{Y}_3$ *for every* $(x, (\hat{Y}_2, \hat{Y}_3)) \in \Omega$,

and the function $\hat{G}(x, \hat{y})$ *is defined and continuous in a domain* Ω' *of the* $(pN+1)$-*dimensional space of the variables* $x, y_1, \ldots, y_{pN}$ *such that for every* $x \in I$

$$\{\hat{y}: (x, \hat{y}) \in \Omega'\} = \Lambda_x,$$

and the function $\hat{G}(x, \hat{y})$ *has* Γ'_x *as its range for* $\hat{y} \in \Lambda_x$, *i.e.*

$$\Gamma'_x = \hat{G}_x(\Lambda_x), \quad \text{where } \hat{G}_x(\hat{y}) = \hat{G}(x, \hat{y});$$

moreover, the following relations hold:

$$\Gamma_x = \{(\hat{Y}_2, \hat{Y}_3): \hat{Y}_3 \epsilon \Gamma_x', \exists \hat{Y}_1 \epsilon \Lambda_x', (\hat{Y}_1, \hat{Y}_2) \epsilon \Lambda_x, \hat{Y}_3 = \hat{G}(x, (\hat{Y}_1, \hat{Y}_2))\}$$

and

$$\Lambda_x' = \{\hat{Y}_1: \exists \hat{Y}_2, (\hat{Y}_1, \hat{Y}_2) \epsilon \Lambda_x\}.$$

As a consequence of Theorem 2, we get the following

THEOREM 6. *Let the function* $f(x)$ *fulfil Hypothesis 1, let* $\xi \notin I$, *let the function* $f(x)$ *be strictly increasing in the interval* I *and let Hypotheses* 6, 7, 3 *and* 5 *be fulfilled; further, we put* $I_0' = \langle x_0, f^p(x_0) \rangle$ *and* $J_0 = \langle x_0, \xi \rangle$ *if the point* ξ *is the right end-point of the interval* I, *and* $I_0' = \langle f^p(x_0), x_0 \rangle$ *and* $J_0 = \langle \xi, x_0 \rangle$ *if the point* ξ *is the left end-point of the interval* I, x_0 *being an arbitrarily fixed point of the interval* I. *Then, for an arbitrary function* $\hat{\Phi}_0(x)$ *defined and continuous in the interval* I_0' *and fulfilling the relations*

$$(58) \qquad \hat{\Phi}_0(x) \epsilon \Lambda_x' \quad \text{for every } x \epsilon I_0'$$

and

$$(59) \qquad \lim_{x \to x_0, f^p(x) \epsilon I_0'} \hat{H}(x, \hat{\Phi}_0[f(x)], \hat{\Phi}_0[f^2(x)], ..., \hat{\Phi}_0[f^p(x)]) = \hat{\Phi}_0(x_0),$$

there exists exactly one function $\hat{\Phi}(x)$ *defined in the interval* I, *fulfilling relation*

$$(60) \qquad \hat{\Phi}(x) \epsilon \Lambda_x'$$

in the interval I, *satisfying the system of equations* (1) *in the interval* I, *and fulfilling the condition*

$$(61) \qquad \hat{\Phi}(x) = \hat{\Phi}_0(x) \quad \text{for } x \epsilon I_0';$$

moreover, the function $\hat{\Phi}(x)$ *is continuous in the interval* I.

Proof. With the notation from Theorem 1 we consider the system of equations (2) under the hypotheses of the present theorem. After having written $n = pN$, Hypothesis 2 results from Hypothesis 6 and relation (3). Writing

$$(62) \qquad g_{(r-1)N+\mu}(x, \hat{y}) = y_{\nu N + \mu} \quad \text{for } \nu = 1, ..., p-1 \text{ and } \mu = 1, ..., N,$$

we get by (3)

$$(63) \qquad g_{(\nu-1)N+\mu}(x, \hat{h}(x, \hat{y})) = h_{\nu N+\mu}(x, \hat{y}) = y_{(\nu-1)N+\mu}$$

$$\text{for } \nu = 1, ..., p-1 \text{ and } \mu = 1, ..., N,$$

whereas in virtue of Hypothesis 7 and of relations (3) and (57) we have

$$(64) \qquad g_{(p-1)N+\mu}(x, \hat{h}(x, \hat{y})) = y_{(p-1)N+\mu} \quad \text{for } \mu = 1, ..., N,$$

and relations (63) and (64) imply relation (33) and Hypothesis 4. Consequently, the hypotheses of Theorem 2 are fulfilled.

Let $\hat{\Phi}_0(x)$ be an arbitrary function defined and continuous in the interval I_0' and fulfilling conditions (58) and (59). Let us define the function $\hat{\varphi}_0(x)$ in the interval I_0 by the relation

$$(65) \qquad \hat{\varphi}_0(x) = \left(\Phi_0(x),\ \hat{\Phi}_0[f(x)],\ \ldots,\ \hat{\Phi}_0[f^{p-1}(x)]\right).$$

Since $\hat{\Phi}_0(x)$ is defined and continuous in the interval I_0', the function $\hat{\varphi}_0(x)$ is defined and continuous in the interval I_0. Relations (65) and (58) and Hypotheses 3 and 5 imply relation (34). Relations (59), (65) and (3) and the continuity of the function $\hat{\Phi}_0(x)$ in the interval I_0' imply relation (28). By Theorem 2 there exists exactly one function $\hat{\varphi}(x)$ defined in the interval I, fulfilling condition (29) in the interval I, satisfying the system of equations (2) in the interval I, and fulfilling condition (30). This function is continuous in the interval I. On account of the second part of Theorem 1 the function $\Phi(x) = \left(\varphi_1(x), \ldots, \varphi_N(x)\right)$ satisfies the system of equations (1) in the interval I. Relation (60) follows from (29), and condition (61) results from (30) and (65). The continuity of the function $\hat{\Phi}(x)$ results directly from that of the function $\hat{\varphi}(x)$.

The uniqueness of the solution of the system of equations (1) follows from the uniqueness of the solution of the system of equations (2) and from the mutual correspondence between the solutions of the systems of equations (1) and (2).

This completes the proof of Theorem 6.

As a consequence of Theorem 3, we obtain the following

THEOREM 7. *Let the function $f(x)$ fulfil Hypothesis 1 ($\xi \epsilon I$ or $\xi \notin I$) and let Hypotheses 6 and 3 be fulfilled; further, let $I_0 = I \cap \langle \xi - \delta, \xi + \delta \rangle$, where δ is a positive number. Then, for an arbitrary function $\hat{\Phi}_0(x)$ defined and continuous in the interval I_0, fulfilling relation (60) in the interval I_0 and satisfying the system of equations (1) in the interval I_0, there exists exactly one function $\hat{\Phi}(x)$ defined in the interval I, fulfilling relation (60) in the interval I, satisfying the system of equations (1) in the interval I and fulfilling the condition*

$$(66) \qquad \hat{\Phi}(x) = \hat{\Phi}_0(x) \qquad for \ x \epsilon I_0;$$

moreover, the function $\hat{\Phi}(x)$ is continuous in the interval I.

Proof. With the notation from Theorem 1, we consider the system of equations (2) under the hypotheses of the present theorem. After having written $n = pN$, we infer Hypothesis 2 from Hypothesis 6 and relation (3). Consequently, the hypotheses of Theorem 3 are fulfilled.

Let $\hat{\varPhi}_0(x)$ be an arbitrary function defined and continuous in the interval I_0, fulfilling relation (60) in the interval I_0, and satisfying the system of equations (1) in the interval I_0. Let the function $\hat{\varphi}_0(x)$ be defined in the interval I_0 by relation (65). Since the function $\hat{\varPhi}_0(x)$ is defined and continuous in the interval I_0, the function $\hat{\varphi}_0(x)$ is also defined and continuous in the interval I_0. According to Hypothesis 3 and relations (65) and (60), the function $\hat{\varphi}_0(x)$ fulfils relation (29) in the interval I_0. On account of the fact that the function $\hat{\varPhi}_0(x)$ satisfies the system of equations (1) in the interval I_0 and in view of relations (3) and (4), the function $\hat{\varphi}_0(x)$ satisfies the system of equations (2) in the interval I_0. By Theorem 3 there exists exactly one function $\hat{\varphi}(x)$ defined in the interval I, fulfilling relation (29) in the interval I, satisfying the system of equations (2) in the interval I and fulfilling condition (30). This function is continuous in the interval I. On account of the second part of Theorem 1 the function $\hat{\varPhi}(x) = \big(\varphi_1(x), \ldots, \varphi_N(x)\big)$ satisfies the system of equations (1) in the interval I. Relation (60) follows from (29), and condition (66) results from (30) and (65). The continuity of the function $\hat{\varPhi}(x)$ results directly from that of the function $\hat{\varphi}(x)$.

The uniqueness of the solution of the system of equations (1) follows from the uniqueness of the solution of the system of equations (2) and from the mutual correspondence between the solutions of the systems of equations (1) and (2).

This completes the proof of Theorem 7.

It is formally possible to formulate a theorem corresponding to Theorem 4, but it is pointless, for then we obtain a theorem whose assumptions cannot be fulfilled. A function $\hat{h}(x, \hat{y})$ such that (3) holds cannot fulfil a Lipschitz condition with respect to the variable $\hat{y}$ with a constant less than one.

But we have the following theorem, resulting from Theorem 5.

THEOREM 8. *Let the function* $f(x)$ *fulfil Hypothesis* 1, *let* $\xi \in I$, *and let Hypotheses* 6 *and* 3 *be fulfilled; moreover, let* $\hat{\varXi}$ *be a root of the equation*

$$(67) \qquad \hat{\varXi} = \hat{H}(\xi, \hat{\varXi}, \ldots, \hat{\varXi})$$

such that $\hat{\varXi} \in A'_\xi$, *and let the function* $\hat{H}(x, \hat{y})$ *have the continuous partial derivatives* $\dfrac{\partial h_\mu}{\partial y_\beta}(x, \hat{y})$ *of the first order with respect to the variables* y_β *for* $\mu = 1, \ldots, N$ *and* $\beta = 1, \ldots, pN$, *in a neighbourhood* $V \subset \varOmega$ *of the point* $(\xi, \hat{\varXi}, \ldots, \hat{\varXi})$. *Further, let all the characteristic roots* $\lambda_\varkappa$ *for* $\varkappa = 1, \ldots, pN$, *of the matrix*

$$\mathfrak{A}'' = \begin{bmatrix} \mathfrak{A}' \\ \mathfrak{E} \quad \mathfrak{D} \end{bmatrix},$$

be less than one in absolute value, where $\mathfrak{A}'$ is the Jacobi matrix

$$\left[\frac{\partial h_\mu}{\partial y_\beta}(\xi, \hat{\Xi}, \ldots, \hat{\Xi})\right] \text{ of the function } \hat{H}(x, \hat{y}) \text{ at the point } (\xi, \hat{\Xi}, \ldots, \hat{\Xi}), \mathfrak{E}$$

denotes the unit matrix and $\mathfrak{O}$ denotes the zero matrix. Then there exists exactly one function $\hat{\Phi}(x)$, defined and continuous in the interval I, fulfilling relation (60) in the interval I, fulfilling the condition

$$(68) \qquad\qquad\qquad \hat{\Phi}(\xi) = \hat{\Xi},$$

and satisfying the system of equations (1) in the interval I. This function is given the formula

$$(69) \qquad\qquad\qquad \hat{\Phi}(x) = \lim_{k\to\infty} \hat{\Phi}_k(x),$$

where the sequence of the functions $\hat{\Phi}_k(x)$ is defined in the interval I for $k > 0$ by the relation

$$(70) \qquad \hat{\Phi}_{k+1}(x) = \hat{H}\big(x, \hat{\Phi}_k[f(x)], \hat{\Phi}_k[f^2(x)], \ldots, \hat{\Phi}_k[f^p(x)]\big)$$
$$\text{for } k = 0, 1, 2, \ldots,$$

$\Phi_0(x)$ being an arbitrary function which is defined and continuous in the interval I and fulfils the conditions

$$(71) \qquad\qquad\qquad \hat{\Phi}_0[f(x)] \epsilon \Gamma'_x \quad \text{for every } x \epsilon I$$

and

$$(72) \qquad\qquad\qquad \hat{\Phi}_0(\xi) = \hat{\Xi}.$$

Proof. With the notation from Theorem 1 we consider the system of equations (2) under the hypotheses of the present theorem. After having written $n = pN$, we infer Hypothesis 2 from Hypothesis 6 and relation (3). By (3) and (67) $\hat{\eta} = (\hat{\Xi}, \ldots, \hat{\Xi})$ fulfils equation (42). It follows from Hypothesis 6 that if $\hat{\Xi} \epsilon \Lambda'_\xi$, then $\hat{\eta} \epsilon \Lambda_\xi$. Since the matrix $\mathfrak{A}''$ is equal to the matrix $\mathfrak{A}$, all the characteristic roots of the latter have a modulus less than one. Consequently, the hypotheses of Theorem 5 are fulfilled.

Let $\hat{\Phi}_0(x)$ be an arbitrary function defined and continuous in the interval I and fulfilling relations (71) and (72). Let the function $\hat{\varphi}_0(x)$ be defined in the interval I by relation (65). Since the function $\hat{\Phi}_0(x)$ is defined and continuous in the interval I, the function $\hat{\varphi}_0(x)$ is also defined and continuous in the interval I. In virtue of Hypothesis 3 and by relations (71) and (72) the function $\hat{\varphi}_0(x)$ fulfils conditions (39) and (45). Thus Theorem 5 yields the existence of exactly one function $\hat{\varphi}(x)$ which is defined and continuous in the interval I, fulfils relation (29) in the interval I, fulfils condition (41) and satisfies the system of equations (2) in the interval I. This function is given by formula (44), where

the sequence of the functions $\hat{\varphi}_k(x)$ is defined in the interval I for $k > 0$ by relation (40). On account of the second part of Theorem 1, the function $\hat{\Phi}(x) = (\varphi_1(x), \ldots, \varphi_N(x))$ satisfies the system of equations (1) in the interval I. The continuity of the function $\hat{\Phi}(x)$ results directly from that of the function $\hat{\varphi}(x)$. Relations (60), (68) and (69) with (70) follow from (29), (41), (44) and (40).

The uniqueness of the solution of the system of equations (1) follows from the uniqueness of the solution of the system of equations (2) and from the mutual correspondence between the solutions of the systems of equations (1) and (2).

This completes the proof of Theorem 8.

Remark. In the case of a single functional equation of order p (for $N = 1$)

$$\varphi(x) = h\left(x, \varphi[f(x)], \varphi[f^2(x)], \ldots, \varphi[f^p(x)]\right),$$

the characteristic equation of the matrix $\mathfrak{A}''$ reduces to

$$\lambda^p - \sum_{\nu=1}^{p} a_\nu \lambda^{p-\nu} = 0,$$

where $a_\nu = \dfrac{\partial h}{\partial y_\nu}(\xi, \eta, \ldots, \eta)$, and $\eta = \varphi(\xi)$.

6. The system of equations (1) is a particular case of the system of N functional equations of order p

$$(73) \qquad \hat{\Phi}(x) = \hat{H}\left(x, \hat{\Phi}[f_1(x)], \hat{\Phi}[f_2(x)], \ldots, \hat{\Phi}[f_p(x)]\right),$$

in which $f_\nu(x)$ for $\nu = 1, \ldots, p$ are given functions. The system of equations (73) is dealt with in [13], where also a theorem on the uniqueness of continuous solutions (Theorem 3 from section 3) is given; however, it is given under different conditions, independent of the hypotheses of Theorem 8 above. In particular, in [13] the author postulates the existence of constants $c_{\gamma\varkappa}$ such that

$$|h_\gamma(x, \hat{y}_1) - h_\gamma(x, \hat{y}_2)| \leqslant \sum_{\varkappa=1}^{pN} c_{\gamma\varkappa} |y_{1\varkappa} - y_{2\varkappa}| \quad \text{for } \gamma = 1, \ldots, N,$$

and

$$\sum_{\gamma=1}^{N} \max_{\mu=1,\ldots,N} \sum_{\nu=0}^{p-1} c_{\gamma, \nu N + \mu} < 1.$$

These conditions are not fulfilled e.g. for a single functional equation of the second order

$$(74) \qquad \varphi(x) = \tfrac{5}{4}\varphi[f(x)] - \tfrac{3}{8}\varphi[f^2(x)],$$

for which the uniqueness of continuous solutions results from Theorem 8, since the characteristic roots, equal here to $\frac{1}{2}$ and to $\frac{3}{4}$, have moduli less than one.

A particular case of the system of equations (73) is the single equation of order p

$$(75) \qquad \varphi(x) = h\big(x, \varphi[f_1(x)], \varphi[f_2(x)], \ldots, \varphi[f_p(x)]\big),$$

dealt with in [9], where a theorem on the existence of continuous solutions is given, and in [3], where a theorem on the uniqueness of continuous solutions (Theorem 1 from section 2) is to be found. The theorem on the existence of continuous solutions of equation (75) is an analogue of Theorem 6 concerning the system of equations (1), but the theorem on the uniqueness of continuous solutions of equation (75) is given in [3] under conditions different from those occurring in Theorem 8 for the system of equations (1). In particular, in [3] the author postulates the existence of constants c_ν such that

$$|h(x, Y_1) - h(x, \hat{Y}_2)| \leqslant \sum_{\nu=1}^{p} c_\nu |y_{1\nu} - y_{2\nu}|$$

and

$$\sum_{\nu=1}^{p} c_\nu < 1.$$

Again equation (74) yields an example where the above condition is not fulfilled, but the uniqueness of continuous solutions results from Theorem 8.

The equation

$$\varphi(x) = h\big(x, \varphi[f(x)], \varphi[f^2(x)], \ldots, \varphi[f^p(x)]\big)$$

is a particular case of equation (75) as well as of the system of equations (1). This equation is dealt with in [7]. The only theorem given there, concerning the existence of continuous solutions, results from Theorem 6 for the system of equations (1).

The results obtained in the present paper for the systems of equations (1) and (2) contain the results obtained so far for the single equation of the first order

$$\varphi(x) = h\big(x, \varphi[f(x)]\big).$$

This equation has been dealt with in [8] and [11], but in the present paper the assumptions regarding the function $f(x)$ are weaker.

The condition occurring in Theorem 8 that all the roots of the Jacobi matrix have moduli less than 1 is essential. In the case where at least one characteristic root has a modulus greater than 1, the system of

equations (1) may have infinitely continuous solutions in an interval I containing the point ξ. E.g., the system of equations

$$\varphi_\varkappa(x) = h_\varkappa(x)\varphi_\varkappa[f(x)], \qquad \varkappa = 1, 2, \ldots, n,$$

for which the characteristic roots are the values $\lambda_\varkappa = h_\varkappa(\xi)$, has infinitely many continuous solutions in an interval I containing the point ξ provided that $|h_\varkappa(\xi)| \neq 1$ for $\varkappa = 1, \ldots, n$, and for at least one $\varkappa = \varkappa_0$ we have $|h_{\varkappa_0}(\xi)| > 1$ (cf. [10]). The case where the modulus of some characteristic roots equals one is the indeterminate case and only for a single linear equation of the first order $(N = 1, p = 1)$

$$\varphi(x) = h_1(x)\varphi[f(x)] + h_2(x)$$

some results have been obtained (cf. [4]).

Systems of functional equations of form (2) are also dealt with in [6], but from another point of view, without assuming continuity. Such equations are considered also in [1]. A more complete bibliography of this subject may be found in [12].

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