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Radicals, crossed products, and flows

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Abstract. Associated with a homeomorphism τ of a locally compact Hausdorff space X is a subalgebra $\mathfrak{A}(\tau, X)$ of the transformation group C^*-algebra determined by τ. As shown by Arveson and Josephson in [1], $\mathfrak{A}(\tau, X)$ frequently constitutes a complete set of conjugacy invariants for τ. Addressing a question raised by them, we give conditions under which $\mathfrak{A}(\tau, X)$ will be semi-simple. We also show that $\mathfrak{A}(\tau, X)$ need not be semi-simple and we compute the radical in a special case.

1. Introduction. Our objective in this note is to address a problem posed by Arveson and Josephson in [1], p. 120. They showed how a discrete flow, i.e., a homeomorphism τ of a locally compact Hausdorff space X may be used to construct an operator algebra $\mathfrak{A}(\tau, X)$ which, under suitable technical hypotheses, determines the flow up to conjugacy. They also identified all of the *continuous* automorphisms of $\mathfrak{A}(\tau, X)$. Their arguments were complicated by the fact that they did not know if $\mathfrak{A}(\tau, X)$ admits discontinuous automorphisms, and consequently they asked, "When is $\mathfrak{A}(\tau, X)$ semi-simple?" For, semi-simplicity guarantees continuity. We shall show that for those flows which seem to be of most interest in topological dynamics, the corresponding algebra is semi-simple. We then show that for the flow consisting of translation by 1 on the integers Z, the radical of $\mathfrak{A}(\tau, X)$ is non-zero and we discribe it explicitly. Using this result, we then give a very general sufficient condition for $\mathfrak{A}(\tau, X)$ to have a non-zero radical.

2. The definition of $\mathfrak{A}(\tau, X)$. The algebra $\mathfrak{A}(\tau, X)$ is a subalgebra of the transformation group C^*-algebra or crossed product, $C^*(\tau, X)$, determined by τ and X and we begin by recalling its definition. Let $K(\tau, X)$ denote the space of all complex, compactly supported, continuous functions on $Z \times X$,

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where $\boldsymbol{Z}$ is the group of integers. For f and g in $K(\tau, X)$, $f*g$ is defined by the formula

$$f*g(n, x) = \sum_{k=-\infty}^{\infty} f(k, x)g(n-k, \tau^{-k}x)$$

and f^* is defined by the formula

$$f^*(n, x) = \overline{f(-n, \tau^{-n}x)}.$$

With respect to these operations, pointwise addition and scalar multiplication, and norm $\| \ \|_0$ defined by the formula

$$\|f\|_0 = \sum_k \|f(k, \cdot)\|_\infty,$$

where $\|\cdot\|_\infty$ denotes the ordinary sup-norm, $K(\tau, X)$ becomes a normed, $*$-algebra whose enveloping C^*-algebra is $C^*(\tau, X)$. (See [2] for a detailed analysis of $C^*(\tau, X)$.)

The algebra $\mathfrak{A}(\tau, X)$ is, by definition, the closure in $C^*(\tau, X)$ of the set of functions f in $K(\tau, X)$ such that $f(k, x) = 0$ when $k < 0$, for all x in X. Thus we have this analogy which should be kept in mind throughout: $K(\tau, X)$ should be viewed as the algebra of trigonometric polynomials on the circle $\boldsymbol{T}$, $C^*(\tau, X)$ should be viewed as $C(\boldsymbol{T})$, and $\mathfrak{A}(\tau, X)$ should be viewed as the disc algebra $A(\boldsymbol{T})$. Indeed, when X is a one point space, this is exactly what $K(\tau, X)$, $C^*(\tau, X)$, and $\mathfrak{A}(\tau, X)$ are up to isomorphism. Specifically, if $X = \{x\}$, then the map from $K(\tau, X)$ to the trigonometric polynomials on $\boldsymbol{T}$, defined by the formula $f \to \sum_n f(n, x)z^n$, is an isomorphism which extends to a C^*-isomorphism from $C^*(\tau, X)$ onto $C(\boldsymbol{T})$ and which carries $\mathfrak{A}(\tau, X)$ onto the disc algebra. It should be noted, too, that our definition of $\mathfrak{A}(\tau, X)$ is slightly different from the one given by Arveson and Josephson in the beginning of their paper [1]. However, as they show in Section 5, the two are equivalent.

For z in $\boldsymbol{T}$, define α_z on $K(\tau, X)$ by the formula $(\alpha_z(f))(n, X) = f(n, x)z^n$. Then α_z extends to a $*$-automorphism of $C^*(\tau, X)$ and $\{\alpha_z\}_{z\in\boldsymbol{T}}$ is a strongly continuous automorphism group of $C^*(\tau, X)$ called the *dual automorphism group* of τ, [14]. For f in $C^*(\tau, X)$, set $f_n = \int_{\boldsymbol{T}} \bar{z}^n \alpha_z(f)dz$, $n \in \boldsymbol{Z}$. (The integral converges in the norm of $C^*(\tau, X)$.) Then f_n is in $K(\tau, X)$ and, in fact, $f_n(k, x) = 0$, $k \neq n$. The f_n may be viewed as the Fourier coefficients of f and formally we write $f \sim \sum_{n\in\boldsymbol{Z}} f_n$. The series $\int_{n\in\boldsymbol{Z}} f_n$ does not converge to f in $C^*(\tau, X)$, generally, but it is Cesaro summable to f. This is because the nth arithmetic mean of the series can be written as $\int_{\boldsymbol{T}} k_n(z)\alpha_z(f)dz$, where $\{k_n\}_{n=0}^{\infty}$

is the Fejer kernel – an approximate identity for $L^1(T)$, [4], Lemma 1. It follows that we may view elements in $C^*(\tau, X)$ as functions on $Z \times X$; for f in $C^*(\tau, X)$, simply set $f(n, x) = f_n(n, x)$.

We note in passing that while it is difficult to compute the norm of a general element of $C^*(\tau, X)$, we do have this useful fact. If $f = f_n$ then the norm of f in $C^*(\tau, X)$, $\|f\|$, is $\|f(n, \cdot)\|_\infty$. To see this, observe that $\|f\| = \|f * f^*\|^{1/2}$, and that $f * f^*(k, x) = |f(n, x)|^2$, $k = 0$, and is zero otherwise. For elements of this form, the assertion follows from Corollary 4.10 of [2].

Recall that the *radical* J of an algebra A is the intersection of all of its maximal, modular, left ideals. Equivalently, J is the intersection of all of its maximal, modular, right ideals. Thus J is a two-sided ideal invariant under all automorphisms of A. One says that A is semi-simple when $J = \{0\}$. When A is a Banach algebra, J is norm closed and may be described as the collection of all a such that ab is quasi-nilpotent for every b in A. For these things see [9].

The discussion to this point comes together to provide a proof of the following proposition which lies at the heart of our analysis.

PROPOSITION 2.1. *The algebra* $\mathfrak{A}(\tau, X)$ *is* $\{f \in C^*(\tau, X) \mid f_n = 0,\ n < 0\}$ *and* f *lies in its radical* J *if and only if* f_n *is in* J *for each* $n \geqslant 0$.

In the language of [6], this proposition describes $\mathfrak{A}(\tau, X)$ as the collection of f in $C^*(\tau, X)$ such that the spectrum of f with respect to $\{\alpha_z\}_{z \in T}$, $sp_\alpha(f)$, is non-negative.

COROLLARY 2.1.1. *The radical* J *of* $\mathfrak{A}(\tau, X)$ *is contained in* $\{f \mid f_0 = 0\}$.

Proof. The collection $\{f \in C^*(\tau, X) \mid f = f_0\}$ is a subalgebra of $\mathfrak{A}(\tau, X)$ isomorphic to $C_0(X)$, the continuous functions on X vanishing at infinity and no such element is quasi-nilpotent.

Thus we see in particular that $\mathfrak{A}(\tau, X)$ is never a radical algebra. We note, too, that J is frequently zero as we show below, while $\{f \in \mathfrak{A}(\tau, X) \mid f_0 = 0\}$ is never zero; this shows that the inclusion in Corollary 2.1.1 may be proper.

3. A sufficient condition for semi-simplicity. While we lack a necessary and sufficient condition for deciding when $\mathfrak{A}(\tau, X)$ is semi-simple, the following theorem identifies a very usable criterion ensuring semi-simplicity. In it, we write $\mathcal{O}_x$ for the orbit determined by x, i.e., $\mathcal{O}_x = \{\tau^n x\}_{n=-\infty}^{\infty}$, and we write $\mathcal{O}_x^{cl}$ for its closure.

THEOREM 3.1. *Suppose that* τ *is a homeomorphism of a locally compact Hausdorff space* X *satisfying these two hypotheses:*

(a) *For each* $x \in X$, $\mathcal{O}_x^{cl}$ *is compact; and*

(b) *For each pair of points* x *and* y *in* X, $\mathcal{O}_x^{cl}$ *and* $\mathcal{O}_y^{cl}$ *are either disjoint or equal.*

Then $\mathfrak{A}(\tau, X)$ *is semi-simple.*

Proof. Suppose f is a non-zero element in the radical J of $\mathfrak{A}(\tau, X)$. Then by Proposition 2.1, we may assume $f = f_n$ for some n. That is, we may assume that $f(k, x) \equiv 0$ when $k \neq n$. Define $g(k, x) = \overline{f(n, x)}$ when $k = 0$ and set $g(k, x) = 0$, $k \neq 0$. Then g is in $\mathfrak{A}(\tau, X)$ and so $g * f$ lies in J. Since $g * f(k, x) = |f(n, x)|^2$ when $k = n$, and is zero otherwise, we may assume without loss of generality that $f(n, x) \geqslant 0$. Let U be the open set $\{x \in X \mid f(n, x) > 0\}$ and fix an x in U. Then $\mathcal{O}_x \subseteq \bigcup_{k=-\infty}^{\infty} \tau^k(U)$ and hypothesis (b) implies that $\mathcal{O}_x^{\mathrm{cl}}$ is contained in $\bigcup_{k=-\infty}^{\infty} \tau^k(U)$. Since $\mathcal{O}_x^{\mathrm{cl}}$ is compact by hypothesis (a), we may select a finite subcover $\tau^{k_1}(U), \ldots, \tau^{k_l}(U)$ of $\mathcal{O}_x^{\mathrm{cl}}$. Observe that the map Φ defined by the formula $(\Phi f)(k, x) = f(k, \tau^{-1} x)$, $f \in K(\tau, X)$ extends to a $*$-automorphism of $C^*(\tau, X)$ mapping $\mathfrak{A}(\tau, X)$ onto $\mathfrak{A}(\tau, X)$. Thus $\Phi(J) = J$. We set $h = \sum_{j=1}^{l} \Phi^{k_j} f$. Then h is a non-zero element in J, $h(k, x) = 0$, $k \neq n$, and there is an $\varepsilon > 0$ such that $h(n, y) \geqslant \varepsilon$, $y \in \mathcal{O}_x^{\mathrm{cl}}$. Now calculate to see that for each $j > 0$, $h^j = h * h * \ldots * h$, j times, has this form:

$$h^j(k, z) = \begin{cases} h(n, z)\, h(n, \tau^{-n} z) \ldots h(n, \tau^{-(j-1)n} z), & k = jn, \\ 0, & k \neq jn. \end{cases}$$

It follows that $\|h^j\| = \sup_{z \in X} |h^j(jn, z)| \geqslant \sup_{z \in \mathcal{O}_x^{\mathrm{cl}}} |h^j(jn, z)| \geqslant \varepsilon^j$ and, consequently, that $\lim_{j \to \infty} \|h^j\|^{1/j} \geqslant \varepsilon$. This is a contradiction, so $J = \{0\}$, and the proof is complete.

The hypotheses of Theorem 3.1 are together equivalent to the condition known as *pointwise almost periodicity*, [3], Proposition 2.6. Most flows appearing in topological dynamics are pointwise almost periodic. (Caution: Pointwise almost periodicity is a much weaker notion than the more familiar, but restricted, concept of uniform almost periodicity or equicontinuity.) Recall that τ is called *minimal* if τ and τ^{-1} have no common, proper, closed, invariant sets.

Corollary 3.1.1. (a) *If X is compact and τ is minimal, then $\mathfrak{A}(\tau, X)$ is semi-simple.*

(b) *If X is compact and τ is distal* [3], *then $\mathfrak{A}(\tau, X)$ is semi-simple.*

Proof. Under the hypothesis of (a), there is only one orbit closure while (b) follows from Proposition 2.6 and Corollary 5.5 of [3].

We note, too, that if each orbit in X is finite, then $\mathfrak{A}(\tau, X)$ is semi-simple. In particular, this is the case when X itself is finite and τ is a cyclic permutation. This should be compared with Theorem 4.1 below. Incidentally, arguments of McAsey [8] go to show that when X is a finite set of cardinality n and τ is a cyclic permutation, then $\mathfrak{A}(\tau, X)$ is isomorphic to the subalgebra of $n \times n$ matrices (a_{ij}) over the disc algebra $A(T)$ such that $a_{ij}(0) = 0$ when $j > i$.

4. A sufficient condition for non-semi-simplicity. While it is difficult to identify the radical of $\mathfrak{A}(\tau, X)$ explicitly for every choice of X and τ, one particularly revealing case stands out where the radical can be described completely. Our general sufficient condition for non-semi-simplicity depends heavily upon it.

THEOREM 4.1. *Let $X = Z$ with τ defined by $\tau(n) = n+1$. Then there is a faithful representation π of $C^*(\tau, X)$ on $l^2(Z)$, whose image is the full algebra of compact operators, $\mathcal{K}(l^2(Z))$, such that $\pi(\mathfrak{A}(\tau, X))$ consists of those operators A in $\mathcal{K}(l^2(Z))$ such that the matrix of A with respect to the usual orthonormal basis for $l^2(Z)$ is lower triangular. The radical of $\mathfrak{A}(\tau, X)$ consists of those f in $\mathfrak{A}(\tau, X)$ such that the matrix of $\pi(f)$ is strictly lower triangular; and this is precisely $\{f \in \mathfrak{A}(\tau, X) \mid f_0 = 0\}$.*

Proof. The existence of the faithful representation π of $C^*(\tau, X)$ whose image is $\mathcal{K}(l^2(Z))$ (and the fact that it is unique up to unitary equivalence) is of course a celebrated result, going back essentially to Mackey [7], if not to Weyl and von Neumann. For a proof more in the spirit of the present study, see Rieffel's paper [10] or Proposition 3.3 of Takai's paper [14]. It is given by the formula

$$(\pi(f)g)(n) = \sum_{k \in Z} f(k, n) g(n-k),$$

$f \in K(\tau, x)$, $g \in l^2(Z)$, and is extended to all of $C^*(\tau, X)$ by continuity. For $z \in T$, define W_z on $l^2(Z)$ by the formula $(W_z f)(n) = z^n f(n)$. Then a calculation shows that $\{W_z\}_{z \in T}$ is a strongly continuous unitary representation of T on $l^2(Z)$ such that

$$W_z \pi(f) W_z^* = \pi(\alpha_z(f)) \tag{4.1}$$

for all $f \in C^*(\tau, X)$ and $z \in T$. One may write W_z in its spectral form

$$W_z = \sum_{n=-\infty}^{\infty} z^n E_n, \quad \text{where} \quad (E_n f)(k) = \begin{cases} f(n), & k = n, \\ 0, & k \neq n. \end{cases}$$

Then on account of equation (4.1), together with the fact that $\mathfrak{A}(\tau, X) = \{f \in C^*(\tau, X) \mid f_n = 0,\ n < 0\}$ (Proposition 2.1), we may appeal to Corollary 2.14 of [6] to conclude that $\pi(\mathfrak{A}(\tau, X))$ consists of those operators A in $\pi(C^*(\tau, X)) = \mathcal{K}(l^2(Z))$ such that A leaves invariant the spaces $F_n l^2(Z)$, $n \in Z$, where $F_n = \sum_{k \geq n} E_n$; i.e., the matrix of A is lower triangular. Note, too, that the matrix of $\pi(f)$, $f \in \mathfrak{A}(\tau, X)$, is strictly lower triangular if and only if $f_0 = 0$. Indeed, quite generally, if $f = f_n$ $n \geq 0$, then the matrix of $\pi(f)$ is zero everywhere but on the nth subdiagonal and the entry in the kth row is $f(k, n)$. As pointed out earlier in Corollary 2.1.1, the radical J of $\mathfrak{A}(\tau, X)$ is contained in $\{f \in \mathfrak{A}(\tau, X) \mid f_0 = 0\}$. Thus $\pi(J)$ is contained in the set of operators in $\mathcal{K}(l^2(\boldsymbol{Z}))$ which have strictly lower triangular matrices with respect to the usual basis. On the other hand, $\pi(\mathfrak{A}(\tau, X))$ is contained in the full algebra $\mathfrak{B}$ of operators on $l^2(\boldsymbol{Z})$ which leave the spaces $F_n l^2(\boldsymbol{Z})$, $n \in \boldsymbol{Z}$, invariant. By Ringrose's theorem [1], Theorem 5.4, it is easy to see that the radical of $\mathfrak{B}$ is the collection of all compact operators which have strictly lower triangular matrices with respect to the usual basis for $l^2(\boldsymbol{Z})$ and, with this, the proof is complete.

Recall that a set $U \subseteq X$ is called *wandering* under τ if $\{n \mid \tau^n(U) \cap U \neq \emptyset\}$ is finite. It is evident that if a point has a wandering neighbourhood, then it has a neigbourhood U such that $\tau^n U \cap U = \emptyset$ for all $n \neq 0$.

THEOREM 4.2. *If there is a point in X with a wandering neighbourhood, then the radical of $\mathfrak{A}(\tau, X)$ is non-zero.*

Proof. Choose an open set U in X so that $\tau^n U \cap U = \emptyset$ for all n and set $Y = \bigcup_{n \in \boldsymbol{Z}} \tau^n U$. Then Y is an open, translation invariant subset of X and so $C^*(\tau, Y)$ is isomorphic to a non-zero ideal in $C^*(\tau, X)$, [5], Lemma 1. Thus $\mathfrak{A}(\tau, Y)$ may be viewed as an ideal in $\mathfrak{A}(\tau, X)$ and it suffices to prove that the radical of $\mathfrak{A}(\tau, Y)$ is non-zero. But the map α from $U \times \boldsymbol{Z}$ to Y defined by the equation $\alpha(u, n) = \tau^n u$ is a homeomorphism which implements a $*$-isomorphism between $C^*(\tau, Y)$ and $C_0(U) \otimes (\tau', \boldsymbol{Z})$, where τ' is translation by 1 on $\boldsymbol{Z}$. This isomorphism carries $\mathfrak{A}(\tau, Y)$ onto $C_0(U) \otimes \mathfrak{A}(\tau', \boldsymbol{Z})$, of course. While examples show that the radical of $C_0(U) \otimes \mathfrak{A}(\tau', \boldsymbol{Z})$ may be properly contained in $C_0(U) \otimes J$, where J is the radical of $\mathfrak{A}(\tau', \boldsymbol{Z})$, nevertheless, from the form of J, we may argue that the radical of $C_0(U) \otimes \mathfrak{A}(\tau', \boldsymbol{Z})$ is non-zero as follows. Use the representation π of Theorem 4.1 to identify $C_0(U) \otimes C^*(\tau', \boldsymbol{Z})$ with the space $C_0\big(U, \mathcal{K}(l^2(\boldsymbol{Z}))\big)$ consisting of all continuous functions (which vanish at infinity if U is not compact) from U to $\mathcal{K}(l^2(\boldsymbol{Z}))$ in such a way that $C_0(U) \otimes \mathfrak{A}(\tau', \boldsymbol{Z})$ becomes identified with the space of those function which take their values in the algebra consisting of the compact operators having lower triangular matrices with respect to the usual basis in $l^2(\boldsymbol{Z})$. Let e be an element in $\mathfrak{A}(\tau', \boldsymbol{Z})$ such that $\pi(e)$ is strictly lower triangular and has but one non-zero matrix entry, let g be a non-zero function in $C_0(U)$, and set $f(u) = g(u)e$, $u \in U$. Then f is a non-zero element in $C_0(U) \otimes \mathfrak{A}(\tau', \boldsymbol{Z})$ and $f \times (C_0(U) \otimes \mathfrak{A}(\tau', \boldsymbol{Z}))$ consists entirely of nilpotent elements (of index at most 2). It follows that the radical of $C_0(U) \otimes \mathfrak{A}(\tau', \boldsymbol{Z})$ is non-zero. This completes the proof.

Recall that τ is called *freely acting* if there are no periodic orbits; i.e., if for no k does τ^k have a fixed point. The following corollary is an immediate consequence of Theorem 4.2 and Corollary 18 of [5].

COROLLARY 4.2.1. *If X is second countable, if τ is freely acting, and if $C^*(\tau, X)$ is type I, then the radical of $\mathfrak{A}(\tau, X)$ is non-zero.*

Remarks 4.3. It is attractive to conjecture that the converse of Theorem 4.2 is true – yielding a necessary and sufficient condition that $\mathfrak{A}(\tau, X)$ be semi-simple. That is, all our evidence leads us to believe that if $\mathfrak{A}(\tau, X)$ is *not* semi-simple, then some point in X has a wandering neighbourhood. In this connection it is worth noting that there is a condition weaker than having a wandering neighbourhood which implies that $\mathfrak{A}(\tau, X)$ has a *quotient* with non-zero radical. It is the condition that there is a point x such that the map $k \to \tau^k x$ is a homeomorphism of Z onto $\mathcal{O}_x$ (with its relative topology). We do not know if $\mathfrak{A}(\tau, X)$ can be semi-mimple, but have a non-semi-simple quotient. We note, too, that when X is Z and τ is translation by 1, the radical of $\mathfrak{A}(\tau, X)$ is complemented as a Banach space; i.e., an analogue of Wedderburn's principal theorem is true. For what other choices of X and τ is the radical of $\mathfrak{A}(\tau, X)$ complemented? Finally, consider the question which led to the present investigation: Are all isomorphisms between two algebras of the type we have been discussing necessarily continuous? Of course when the algebras are semi-simple, the answer is yes.

On the other hand the answer may be yes even in the non-semi-simple case. Indeed, the arguments of Ringrose in [11] or [13] coupled with Theorem 4.1 show that when $X = Z$ and τ is translation by 1, then every automorphism of $\mathfrak{A}(\tau, X)$ is continuous. What the situation is in general, we do not know.

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