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Multivalued contraction with parameter

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Abstract. Let X be a paracompact perfectly normal topological space, Y be a Banach space, and let $\mathrm{CCl}(Y)$ be the class of nonempty closed convex subsets of Y. We show that if a multivalued mapping $G\colon X \times Y \to \mathrm{CCl}(Y)$ is continuous in the first variable and is a generalized contraction in the second variable, then there exists a continuous mapping $g\colon X \mapsto Y$ such that $g(x) \in G(x, g(x))$ and we state a Krasnoselskii type fixed point theorem.

1. Preliminaries. Throughout the paper, if it is not assumed otherwise, X is a paracompact perfectly normal topological space, Y is a nonempty closed convex subset of a Banach space $(Z, |\cdot|)$. Let $\mathrm{Cl}(Y)$ be the collection of all nonempty closed subsets of Y endowed with the generalized Hausdorff metric defined by

$$D(A, B) = \begin{cases} \max(\sup\{d(a, B)\colon a \in A\},\ \sup\{d(b, A)\colon b \in B\}) & \\ & \text{if the supremum exists,} \\ +\infty & \text{otherwise,} \end{cases}$$

where $d(e, C) = \inf\{|e-c|\colon c \in C\}$. $\mathrm{Cl}(Y)$ is a generalized metric space (see [12]).

Let $\mathrm{CCl}(Y)$ be the family of convex elements of $\mathrm{Cl}(Y)$.

In this paper a multivalued mapping $F\colon X \mapsto \mathrm{Cl}(Y)$ is called *continuous* if it is continuous in the sense of the generalized Hausdorff metric and is called *lower semicontinuous* if

$$\{x \in X\colon F(x) \cap G \neq \emptyset\} \quad \text{is open for any open} \quad G \subset Y. \tag{1}$$

Clearly, if F is continuous, then it is lower semicontinuous (see [12], Lemma 1.4).

The symbols $\boldsymbol{R}$, $\boldsymbol{R}^+$, $C(\boldsymbol{R})$ denote the real axis, the positive halfaxis, the class of all nonempty convex subsets of $\boldsymbol{R}$, respectively. A multivalued mapping $K\colon X \mapsto C(\boldsymbol{R})$ is said to be lower semicontinuous if (1) holds.

Let us assume that $\varphi\colon X \times \boldsymbol{R}^+ \mapsto \boldsymbol{R}^+$ is a function such that:

1° φ is continuous,

2° $\varphi(x, \cdot)\colon \boldsymbol{R}^+ \mapsto \boldsymbol{R}^+$ is nondecreasing for $x \in X$, 3° $\sum_{n=0}^{\infty} \varphi^n(x, t) < \infty$ for $x \in X$, $t \in \boldsymbol{R}^+$, where $\varphi^0(x, t) = t$, $\varphi^n(x, t) = \varphi(x, \varphi^{n-1}(x, t))$,

4° for any continuous function δ: $X \mapsto \boldsymbol{R}^+$ the function S: $X \mapsto \boldsymbol{R}^+$ defined by $S(x) = \sum_{n=0}^{\infty} \varphi^n(x, \delta(x))$ is continuous.

We will prove the following theorems.

THEOREM 1. *Suppose that a multivalued mapping* G: $X \times Y \mapsto \mathrm{CCl}(Y)$ *is such that*

(i) $G(\cdot, u)$: $X \mapsto \mathrm{CCl}(Y)$ *is continuous for* $u \in Y$,

(ii) $D(G(x, u), G(x, v)) \leqslant \varphi(x, |u-v|)$ *for any function* φ *satisfying* 1°–4°.

Then there exists a continuous function g: $X \mapsto Y$ *such that* $g(x) \in G(x, g(x))$ *for* $x \in X$.

THEOREM 2. *Let* Γ: $Y \mapsto X$ *be an operator,* G: $\Gamma(Y) \times Y \mapsto \mathrm{CCl}(Y)$ *be as in Theorem* 1. *Suppose that for any continuous function* g: $\Gamma(Y) \mapsto Y$ *satisfying* $g(x) \in G(x, g(x))$ *for* $x \in \Gamma(Y)$ *the superposition* $g \circ \Gamma$: $Y \mapsto Y$ *has a fixed point. Then there exists an element* $a \in Y$ *such that* $a \in G(\Gamma(a), a)$.

In particular, if Y *is bounded and* Γ *is completely continuous, then* G *needs only fulfil* (i) *and* (ii) *with any function satisfying* 1°–3° *and*

4'° *for* $t \in \boldsymbol{R}^+$ *the function* S_t: $X \mapsto \boldsymbol{R}^+$ *is continuous on* $\Gamma(Y)$, *where* $S_t(x) = \sum_{n=0}^{\infty} \varphi^n(x, t)$.

Similar theorems were given by Melvin for singlevalued mappings [6], by M. Kisielewicz [2], and also by M. Kisielewicz and the author for multivalued mappings in a uniformly convex Banach space [4]. Theorems presented here generalize those results. Other results of this type can be found in the papers of Rzepecki [9], [10].

2. Lemmas.

LEMMA 1. *Suppose that* φ: $X \times \boldsymbol{R}^+ \mapsto \boldsymbol{R}^+$ *satisfies* 1°, 2° *and* $\varphi(x, t) < t$ *for* $x \in X$, $t > 0$. *Let* δ: $X \mapsto \boldsymbol{R}^+$ *be a continuous function. Define the function* k: $X \mapsto \boldsymbol{R}^+$ *as follows*:

(2) *if* $\delta(x) > 0$ *then*

$$k(x) = \begin{cases} \sup\{l > 0:\ \varphi(x, \delta(x)+l) < \delta(x)\} & \text{if it exists,} \\ 1 & \text{otherwise,} \end{cases}$$

(3) *if* $\delta(x) = 0$, *then* $k(x) = 0$.

Then k *is lower semicontinuous and* $k(x) > 0$ *whenever* $\delta(x) > 0$.

Proof. Since $\varphi(x, \cdot)$ is continuous and $\varphi(x, t) < t$ for $t > 0$, we have $\lim_{s \to t^+} \varphi(x, s) < t$ for $t > 0$. Thus k is well defined and $\delta(x) > 0$ implies

$k(x) > 0$. If for any x we have $\delta(x) = 0$, then obviously $\liminf_{z \to x} k(z) \geqslant k(x) = 0$.

Let x be any element with $\delta(x) > 0$. Choose an arbitrary $l \in [0, k(x))$. From the continuity of $\delta(\cdot)$ and $\varphi(\cdot, \delta(\cdot)+l)$ it follows that there is an open neighbourhood $V_l(x)$ such that $z \in V_l(x)$ implies

$$\delta(x) - \delta(z) < \tfrac{1}{2}\big(\delta(x) - \varphi(x, \delta(x)+l)\big), \tag{4}$$

$$\varphi(z, \delta(z)+l) - \varphi(x, \delta(x)+l) < \tfrac{1}{2}\big(\delta(x) - \varphi(x, \delta(x)+l)\big). \tag{5}$$

Hence, from (4) we have

$$\tfrac{1}{2}\big(\delta(x) + \varphi(x, \delta(x)+l)\big) < \delta(z) \quad \text{for } z \in V_l(x),$$

and from (5) we have

$$\varphi(z, \delta(z)+l) < \tfrac{1}{2}\big(\delta(x) + \varphi(x, \delta(x)+l)\big).$$

Thus $\varphi(z, \delta(z)+l) < \delta(z)$ whenever $z \in V_l(x)$. This means that $l \leqslant k(z)$ for $z \in V_l(x)$. Therefore k is lower semicontinuous.

LEMMA 2. *Suppose that* $\varphi\colon X \times \boldsymbol{R}^+ \mapsto \boldsymbol{R}^+$ *satisfies* 1°–4°. *Then*

1'° $\varphi(x, 0) = 0$, $\varphi(x, t) < t$ *for* $x \in X$, $t > 0$,

2'° *for any continuous function* $\delta\colon X \mapsto \boldsymbol{R}^+$ *there is a sequence of continuous functions* $h_n\colon X \mapsto \boldsymbol{R}^+$, $n = 0, 1, \ldots$, *such that* $h_n(x) \leqslant 2^{-n}$ *for* $x \in X$ *and the sequence of continuous functions* $\psi_n\colon X \mapsto \boldsymbol{R}^+$ *defined by* $\psi_0(x) = \delta(x)$, $\psi_n(x) = \varphi\big(x, \psi_{n-1}(x) + h_{n-1}(x)\big)$ *satisfies the inequalities*

$$\psi_{2k}(x) \leqslant \varphi^k(x, \delta(x)), \quad \psi_{2k+1}(x) \leqslant \varphi^k(x, \delta(x)), \tag{6}$$

$$\textit{if } h_n(x) = 0, \textit{ then } \psi_n(x) = 0 \textit{ for } x \in X. \tag{7}$$

Proof. If for any $t > 0$ we have $\varphi(x, t) \geqslant t$, then $t \leqslant \varphi(x, t) \leqslant \varphi(x, \varphi(x, t)) = \varphi^2(x, t) \leqslant \varphi(x, \varphi^2(x, t)) \leqslant \ldots$, by 2°, so $\varphi^n(x, t) \geqslant t$. But 4° implies $\varphi^n(x, t) \to 0$ as $n \to \infty$. Therefore 1'° holds.

2'° Define $k_0\colon X \mapsto \boldsymbol{R}^+$ as k in Lemma 1. Let $K_0\colon X \mapsto C(R)$ be defined by

$$K_0(x) = \begin{cases} (0, k_0(x)) & \text{if } \delta(x) > 0, \\ \{0\} & \text{if } \delta(x) = 0. \end{cases}$$

Since k_0 is lower semicontinuous, K_0 is lower semicontinuous (use Proposition 2.1 of Michael [7]). Then K_0 has a continuous selection ([7], Theorem 3.1‴). Thus there is a continuous function $\tilde{k}_0\colon X \mapsto \boldsymbol{R}^+$ such that $0 < \tilde{k}_0(x) < k_0(x)$ if $\delta(x) > 0$, $\tilde{k}_0(x) = 0$ if $\delta(x) = 0$. Let $h_0\colon X \mapsto \boldsymbol{R}^+$ be defined by

$$h_0(x) = \min(2, \tilde{k}_0(x)). \tag{8}$$

It is clear that h_0 is continuous, $h_0(x) = 0$ iff $\delta(x) = 0$ and $h_0(x) \leqslant 2^0$ for $x \in X$. Now,

(9) $$\psi_1(x) = \varphi(x, \delta(x) + h_0(x)) \leqslant \delta(x) = \varphi^0(x, \delta(x)).$$

Moreover, $\psi_1(x) < \varphi^0(x, \delta(x))$ whenever $\delta(x) > 0$. Evidently ψ_1 is continuous. Define $k_1\colon X \mapsto \boldsymbol{R}^+$ by $k_1(x) = \varphi^0(x, \delta(x)) - \psi_1(x)$ and then $h_1\colon X \mapsto \boldsymbol{R}^+$ by

(10) $$h_1(x) = \min(2^{-1}, k_1(x)).$$

Observe that if $\psi_1(x) > 0$, then $\delta(x) > 0$, but in this case $k_1(x) > 0$. Hence (10) defines a continuous function such that $h_1(x) \leqslant 2^{-1}$ and if $\psi_1(x) > 0$, then $h_1(x) > 0$. Now, $\psi_2(x) = \varphi(x, \psi_1(x) + h_1(x))$ and we have

(11) $$\psi_2(x) \leqslant \varphi(x, \psi_1(x) + k_1(x)) = \varphi^1(x, \delta(x)).$$

Clearly, ψ_2 is continuous. Define $k_2\colon X \mapsto \boldsymbol{R}^+$ as k in Lemma 1 taking ψ_2 instead of δ. Then k_2 is lower semicontinuous and $k_2(x) = 0$ iff $\psi_2(x) = 0$. As above for k_0, choose for k_2 any continuous function $\tilde{k}_2\colon X \mapsto \boldsymbol{R}^+$ such that $0 < \tilde{k}_2(x) < k_2(x)$ if $\psi_2(x) > 0$ and $\tilde{k}_2(x) = 0$ if $\psi_2(x) = 0$. Let $h_2\colon X \mapsto \boldsymbol{R}^+$ be defined by

(12) $$h_2(x) = \min(2^{-2}, \tilde{k}_2(x)).$$

Then h_2 is continuous, $h_2(x) = 0$ iff $\psi_2(x) = 0$ and $h_2(x) \leqslant 2^{-2}$. Now,

(13) $$\psi_3(x) = \varphi(x, \psi_2(x) + h_2(x)) \leqslant \psi_2(x) \leqslant \varphi^1(x, \delta(x)).$$

Moreover, $\psi_3(x) < \varphi^1(x, \delta(x))$ whenever $\psi_2(x) > 0$. Put $k_3(x) = \varphi^1(x, \delta(x)) - \psi_3(x)$ for $x \in X$ and then

(14) $$h_3(x) = \min(2^{-3}, k_3(x)).$$

Thus $h_3\colon X \mapsto \boldsymbol{R}^+$ is continuous, $h_3(x) \leqslant 2^{-3}$ and if $\psi_3(x) > 0$, then $h_3(x) > 0$. Now, $\psi_4(x) = \varphi(x, \psi_3(x) + h_3(x))$ and we have

(15) $$\psi_4(x) \leqslant \varphi(x, \psi_3(x) + k_3(x)) = \varphi(x, \varphi^1(x, \delta(x))) = \varphi^2(x, \delta(x)).$$

Continuing this procedure we obtain the sequences (h_n), (ψ_n) satisfying assertion 2'°.

Lemma 3. *Denote* $B = \{z \in Z\colon |z| \leqslant 1\}$, $B^0 = \{z \in Z\colon |z| < 1\}$, $\eta(x)C = \{\eta(x)z\colon z \in C\}$. *Suppose that* $F\colon X \mapsto \mathrm{CCl}(Z)$ *and* $\eta\colon X \mapsto \boldsymbol{R}^+$ *are lower semicontinuous. If* $F(x) \cap \eta(x)B^0 \neq \emptyset$ *for* $x \in X$, *then the multivalued mapping* $\mathscr{F}\colon X \mapsto \mathrm{CCl}(Z)$ *defined by* $\mathscr{F}(x) = F(x) \cap \eta(x)B$ *is lower semicontinuous.*

Proof. We will show that for every $x_0 \in X$, $y_0 \in \mathscr{F}(x_0)$, $\varepsilon > 0$ there is a neighbourhood U of x_0 in X such that for every $x \in U$ we have $d(y_0, \mathscr{F}(x)) < \varepsilon$.

First assume that $\eta(x_0) = 0$. Then $F(x_0) \cap \eta(x_0)B = \{0\}$. It suffices to show that there is a neighbourhood U of x_0 such that for every $x \in U$ there exists a $z \in F(x) \cap \eta(x)B$, $|z| < \varepsilon$. From the lower semicontinuity of F it follows that there is a neighbourhood U_0 of x_0 such that for $x \in U_0$ there exists a $z \in F(x)$ with $|z| < \varepsilon$. Now, if $\eta(x) \geqslant \varepsilon$, then $z \in \eta(x)B$. In the case $\eta(x) < \varepsilon$ there exists a $v \in F(x) \cap \eta(x)B$ with $|v| < \varepsilon$, since $F(x) \cap \eta(x)B \neq \emptyset$.

Assume that $\eta(x_0) > 0$. Observe that $F(x) \cap \eta(x)B = \overline{F(x) \cap \eta(x)B^0} = \overline{F(x) \cap \operatorname{int} \eta(x)B}$ ($\bar{C}$ denotes the closure of C). Now, the proof of Proposition 2 in [5] can be repeated with slight changes to show that $\mathscr{F}$ is lower semicontinuous in x_0.

LEMMA 4. *Let F: $X \mapsto \mathrm{CCl}(Y)$, g: $X \mapsto Y$ be continuous, h: $X \mapsto \boldsymbol{R}^+$ be lower semicontinuous. Suppose that for any $x \in X$, $d(g(x), F(x)) > 0$ implies $h(x) > 0$. Then there exists a continuous selection f of F such that*

$$|f(x) - g(x)| \leqslant d(g(x), F(x)) + h(x). \tag{16}$$

Proof. Define F_g: $X \mapsto \mathrm{CCl}(Z)$ by $F_g(x) = F(x) - g(x)$. It is easy to check that F_g is continuous. Let $\eta(x) = d(0, F_g(x)) + h(x)$ for $x \in X$. Clearly, η: $X \mapsto \boldsymbol{R}^+$ is lower semicontinuous and we have $F_g(x) \cap \eta(x)B^0 \neq \emptyset$ for $x \in X$. In virtue of Lemma 3, the multivalued mapping $\mathscr{F}$: $X \mapsto \mathrm{CCl}(Z)$ defined by $\mathscr{F}(x) = F_g(x) \cap \eta(x)B$ is lower semicontinuous. Thus by the Michael Selection Theorem ([7], Theorem 3.2) it follows that there exists a continuous function f_g: $X \mapsto Y$ such that $f_g(x) \in \mathscr{F}(x)$ for $x \in X$. Let $f(x) = f_g(x) + g(x)$ for $x \in X$. Then f: $X \mapsto Y$ is continuous and $f(x) \in F(x)$ for $x \in X$. Moreover,

$$|f(x) - g(x)| = |f_g(x)| \leqslant \eta(x) = d(g(x), F(x)) + h(x).$$

3. Proof of Theorem 1. Let us fix an arbitrary $u \in Y$. Since the mapping $G(\cdot, u)$ is continuous, the function δ: $X \mapsto \boldsymbol{R}^+$ defined by $\delta(x) = d(u, G(x, u))$ is continuous. Choose the sequences of continuous functions (h_n), (ψ_n) as in Lemma 2. By Lemma 3 we can find a continuous selection of $G(\cdot, u)$, call it g_1, such that

$$|g_1(x) - u| \leqslant \delta(x) + h_0(x). \tag{17}$$

The function δ_1: $X \mapsto \boldsymbol{R}^+$ defined by $\delta_1(x) = d(g_1(x), G(x, g_1(x)))$ is continuous, since g_1 and $G(\cdot, g_1(\cdot))$ are continuous. Since $g_1(x) \in G(x, u)$, we have $\delta_1(x) \leqslant D(G(x, u), G(x, g_1(x)))$. Thus

$$\delta_1(x) \leqslant \varphi(x, |g_1(x) - u|) \leqslant \varphi(x, \delta(x) + h_0(x)) = \psi_1(x).$$

Then we can find a continuous selection g_2 of $G(\cdot, g_1(\cdot))$ such that

$$|g_2(x) - g_1(x)| \leqslant \delta_1(x) + h_1(x) \leqslant \psi_1(x) + h_1(x). \tag{18}$$

Define δ_2: $X \mapsto \boldsymbol{R}^+$ by $\delta_2(x) = d(g_2(x), G(x, g_2(x)))$. Clearly, δ_2 is continuous and $\delta_2(x) \leqslant D(G(x, g_1(x)), G(x, g_2(x)))$. Thus

$$\delta_2(x) \leqslant \varphi(x, |g_2(x) - g_1(x)|) \leqslant \varphi(x, \psi_1(x) + h_1(x)) = \psi_2(x).$$

Continuing this procedure we obtain a sequence of continuous functions g_n: $X \mapsto Y$, $n = 0, 1, 2, \ldots$, such that $g_0(x) = u$, $g_{n+1}(x) \in G(x, g_n(x))$ and

(19) $$|g_{n+1}(x) - g_n(x)| \leqslant \psi_n(x) + h_n(x).$$

Let us write $\gamma_n(x) = \varphi^k(x, \delta(x))$ for $n = 2k$ or $n = 2k+1$, $n, k = 0, 1, 2, \ldots$, $x \in X$. Obviously $\sum_{n=0}^{\infty} \gamma_n(x) = 2S(x)$. Thus, by Lemma 2, from (19) we get

$$|g_{n+p}(x) - g_n(x)| \leqslant \sum_{i=n}^{n+p-1} \gamma_i(x) + 2^{-n+1}.$$

Hence $(g_n(x))$ is a Cauchy sequence, and therefore there exists $\lim_{n\to\infty} g_n(x)$ for $x \in X$. Denote $g(x) = \lim_{n\to\infty} g_n(x)$, $R_n(x) = 2S(x) + 2^{-n+1} - \sum_{i=0}^{n-1} \gamma_i(x)$ for $x \in X$. By 1° and 4°, R_n: $X \mapsto \boldsymbol{R}^+$ is continuous for $n = 1, 2, \ldots$, obviously, $R_n(x)$ decrease to 0 as $n \to \infty$.

We have

$$d\big(g(x), G(x, g(x))\big) \leqslant |g(x) - g_n(x)| + d\big(g_n(x), G(x, g(x))\big)$$

for $n = 1, 2, \ldots$, $x \in X$. But $g_n(x) \in G(x, g_{n-1}(x))$ implies

$$d\big(g_n(x), G(x, g(x))\big) \leqslant \varphi\big(x, |g(x) - g_{n-1}(x)|\big) < |g(x) - g_{n-1}(x)|.$$

Since $|g(x) - g_n(x)| \leqslant R_n(x)$, we have

$$d\big(g(x), G(x, g(x))\big) \leqslant R_n(x) + R_{n-1}(x), \quad n = 1, 2, \ldots, x \in X.$$

Therefore $g(x) \in G(x, g(x))$ for $x \in X$.

It remains to show that g is continuous on X. Fix an arbitrary $x_0 \in X$. For $\varepsilon > 0$ let m be a positive integer such that $R_m(x_0) < \frac{1}{4}\varepsilon$. Choose an open neighbourhood U_m of x_0 such that $|g_m(x) - g_m(x_0)| < \frac{1}{4}\varepsilon$, $|R_m(x) - R_m(x_0)| < \frac{1}{4}\varepsilon$ whenever $x \in U_m$. Since

$$\begin{aligned} |g(x) - g(x_0)| &\leqslant |g(x) - g_m(x)| + |g_m(x) - g_m(x_0)| + |g_m(x_0) - g(x_0)| \\ &\leqslant R_m(x) + |g_m(x) - g_m(x_0)| + R_m(x_0) \end{aligned}$$

for $x \in U_m$ we have

$$|g(x) - g(x_0)| \leqslant R_m(x_0) + \tfrac{1}{4}\varepsilon + \tfrac{1}{4}\varepsilon + R_m(x_0) < \varepsilon.$$

Therefore g is continuous. The proof is complete.

Remark 1. One can see that for multivalued contraction with no parameter the iterative technique used in this paper leads to the following principle covering the results of Węgrzyk ([12], Theorem 2.1) and Turinici ([11], Theorem 2.1).

THEOREM 3 ([8]). *Let (Y, ϱ) be a complete generalized metric space, $G\colon Y \mapsto \mathrm{Cl}(Y)$ be a multivalued mapping, $\mu\colon \boldsymbol{R}^+ \mapsto \boldsymbol{R}^+$ be a nondecreasing function such that $\lim_{s \to t^+} \mu(s) < t$ and $\sum_{n=0}^{\infty} \mu^n(t) < \infty$ for $t > 0$, where $\mu^0(t) = t$, $\mu^n(t) = \mu(\mu^{n-1}(t))$ for $n = 1, 2, \ldots$ Suppose that for any $\varepsilon > 0$ and for every $u \in Y$ with $d(u, G(u)) < \infty$ there exists $v \in G(u)$ such that $\varrho(v, u) < d(u, G(u)) + \varepsilon$, $d(v, G(v)) \leqslant \mu(\varrho(v, u))$. If for $u_0 \in Y$ we have $d(u_0, G(u_0)) < \infty$, then there exists a sequence (u_n), $n = 0, 1, 2, \ldots$, such that $u_{n+1} \in G(u_n)$ and u_n converges to some point $z \in G(z)$.*

Remark 2. Assumptions (i), (ii) in Theorem 1 can be replaced by:

(i′) $G\colon X \times Y \mapsto \mathrm{CCl}(Y)$ is continuous,

(ii′) for every pair of continuous functions $g\colon X \mapsto Y$, $h\colon X \mapsto \boldsymbol{R}^+$, with the property that $d(g(x), G(x, g(x))) > 0$ implies $h(x) > 0$, there exists a continuous function $g_1\colon X \mapsto Y$ such that $|g_1(x) - g(x)| \leqslant d(g(x), G(x, g(x))) + h(x)$ and $d(g_1(x), G(x, g_1(x))) \leqslant \varphi(x, |g_1(x) - g(x)|)$ for any function φ satisfying 1°–4°.

Remark 3. A particular case of Theorem 1 is the case with

(ii″) $D(G(x, u), G(x, v)) \leqslant k(x)|u - v|$ for any continuous function $k\colon X \mapsto [0, 1)$, $u, v \in Y$,

instead of (ii). Obviously, φ defined by $\varphi(x, t) = k(x)t$ satisfies 1°–4°.

Proof of Theorem 2. By Theorem 1 (with $\Gamma(Y)$ instead of X), there exists a continuous function $g\colon \Gamma(Y) \mapsto Y$ such that $g(x) \in G(x, g(x))$ for $x \in \Gamma(Y)$. Then there exists an $a \in Y$ with $a = g(\Gamma(a))$. But $g(\Gamma(a)) \in G(\Gamma(a), g(\Gamma(a)))$. Thus $a \in G(\Gamma(a), a)$.

If Y is bounded and Γ is completely continuous, then $\overline{\Gamma(Y)}$ is compact. In this case we can repeat the proofs of Lemma 1, Lemma 2 and Theorem 1 with the constant function $\delta_u\colon X \to \boldsymbol{R}^+$ defined by $\delta_u(x) = t_u = \max\{d(u, G(x, u))\colon x \in \Gamma(Y)\}$. Therefore 4° can be replaced by 4′°. Now, for any continuous function $g\colon \Gamma(Y) \mapsto Y$ the superposition $g \circ \Gamma$ is completely continuous, hence it has a fixed point in virtue of the Schauder Theorem.

Remark 4. One can get another version of Theorem 2 without the assumption that Γ is completely continuous, provided some condition is assumed wich makes it possible to apply to $g \circ \Gamma$ a fixed point theorem of the Darbo type (see e.g. [1]).

Remark 5. An application of Theorem 2 to the generalized functional-differential equation $\dot{x}(t) \in F(t, x(\cdot), \dot{x}(\cdot))$ with $F\colon I \times C \times L^p \mapsto \mathrm{CCl}(\boldsymbol{R})$, $p \geqslant 1$, can be given in the same way as in [3] for $p > 1$.

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