

Generalized Hopf bifurcations and applications to planar quadratic systems

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Abstract. We give here first a proof of a generalized Hopf bifurcation theorem, proof which is both simple and practical for applications. The method uses the Poincaré normal form of a vector field around a singular point. We then apply this method to the study of the maximal number and position of limit cycles for a quadratic differential system. We show that by performing a *succession* of two Hopf bifurcations we can obtain up to four limit cycles necessarily having the configuration (1, 3).

1. Introduction. The Hopf bifurcation theorem is a result of local nature which deals with the birth of limit cycles from a singular point, in a one-parameter perturbation of a differential system

$$(1.1) \quad \dot{x} = f(x, y), \quad \dot{y} = g(x, y),$$

when the two eigenvalues cross the imaginary axis [11]. The generic case is the **birth (or death) of a unique limit cycle**. If, however, the perturbation of (1.1) depends on more than one parameter, then, generically, the simultaneous birth of several limit cycles from a singular point can occur. If (1.1) depends on k parameters, then in a structurally stable way, at most k limit cycles can arise from the singular point ([4], [5], [10], [19]), in what is called a *generalized Hopf bifurcation*. We give here a short proof of the generalized Hopf bifurcation result: If the origin is a singular point of (1.1), such that the linearized system at the origin has a pair of pure imaginary eigenvalues, and if the system (1.1) is degenerate around the origin to order k (to be defined later), then for any n -parameters perturbation of (1.1) there are at most k limit cycles around the origin. Also, for any $i \leq k$, there exists perturbations having exactly i limit cycles. The proof we give is close to that of [4]. This proof has several existential statements. We are essentially interested in showing how one can use the proof to deal with examples, in particular how one determines in practice regions where a given number of cycles occur. The proof uses the fact that the vector field can be brought to

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the Poincaré normal form around the singular point [2]. This reduction is completely algorithmic. We show that this reduction contains essentially all the practical information we need to work with particular examples. We discuss by an example the relation between the system in Poincaré normal form and the truncated system to a finite order.

In the second part of this paper we apply this method for producing limit cycles of planar quadratic differential systems. The determination of limit cycles for quadratic differential systems is part of Hilbert's 16-th problem which can be stated as follows: *Find the maximal number and positions of limit cycles for a differential system (1.1), with f and g polynomials in x and y .* The question is still open, even for f and g quadratic. Poincaré showed that, if the system has a center (weak focus), then limit cycles can be made to appear by varying slightly the coefficients of f and g . Frommer [9], Bautin [3], Chin and Pu [7], Tung [20], and Shi Songling [16], [17], used this method for producing limit cycles. Bautin showed that, by the variation of the coefficients of a quadratic differential system, a singular point cannot give rise to more than three limit cycles [3]. Tung showed [20] that each limit cycle contains exactly one singular point and that at most two singular points can have limit cycles surrounding them. Petrovskii and Landis "proved" that a quadratic system has at most three limit cycles [12], [13], but, in 1967, S. Novikov found a mistake in the proof of the main lemma and the authors acknowledged it [14]. A counter example with four limit cycles was given by Shi Songling [16], and another by Chen and Wang [6].

The following system is the counter example of Shi Songling [16]:

$$(1.2) \quad \dot{x} = \lambda x - y - 10x^2 + (5 + \delta)xy + y^2, \quad \dot{y} = x + x^2 + (8\varepsilon - 25 - 9\delta)xy$$

for $\lambda = -10^{-200}$, $\delta = -10^{-13}$, $\varepsilon = -10^{-52}$. The system (1.2) has two singular points $(0, 0)$ and $(0, 1)$, three limit cycles around the origin, and one around $(0, 1)$. Shi Songling proved his result [16] by constructing several Poincaré-Bendixson domains. A first look at this example prompts the questions:

(a) Why such a choice for λ , δ , ε ? Why such a choice for the coefficients of xy in the expression of the vector field (1.3)?

(b) Can one construct an example without the need of using such small numbers? This would make it easier to draw the phase portraits with a computer, and perhaps the limit cycles could even be glimpsed on the computer drawing.

In this paper we explain the existence of four limit cycles by a succession of two Hopf bifurcations, a first one, non-degenerate, which gives rise to the cycle around $(0, 1)$, followed by a second one, degenerate to third order (which we call generalized Hopf bifurcation of order 3), which gives birth simultaneously to the three cycles around $(0, 0)$. The use of the Hopf bifurcations provides a conceptual understanding of this example. It explains why λ , δ , ε are small numbers, why they are all negative, why $|\lambda|$ need be

much smaller than $|\varepsilon\delta|$, why the coefficients of xy in (1.2) are taken as they are.

Next we use the same method to provide the means for constructing in general, examples of quadratic systems with several limit cycles. We show that we cannot obtain more than four limit cycles in this way. Also we show that the only possible configuration for the four limit cycles thus obtained is: three cycles around one singular point and one around the other singular point.

The results of this paper are mainly of a local nature: We consider the birth of limit cycles around two singular points via simultaneous or successive Hopf bifurcations. In the case of the successive Hopf bifurcations for the Shi Songling example, the first one guarantees the existence of the limit cycle around $(0, 1)$ in a small neighbourhood of this point for values of the parameter β positive and small. This cycle, however, stays on and grows larger when β increases from 0 to 5. Arguments of a global nature assure us of this.

The Hopf bifurcation is of course not the only method by which limit cycles can be produced: They can also appear from homoclinic or heteroclinic loops joining saddle points (finite or infinite), or saddle-node points. The analysis for these methods is far more complex than for the Hopf bifurcation. The results are incomplete even for one homoclinic loop, and the difficulties appear to be substantial in case one tries to do successive homoclinic loop bifurcations. Finally, limit cycles can also arise via multiple limit cycles: for example when one semistable limit cycle (attractive on one side, repulsive on the other) bifurcates into two limit cycles, one stable one unstable. As both computer calculations and understanding improve, one may be able to handle a succession of bifurcations of the various types mentioned above.

2. Degenerate Hopf bifurcations of order k . The degenerate Hopf bifurcation theorem of order k deals with the birth of a limit cycle from a singular point. It is a local phenomenon. The theorem does not give any information about limit cycles that may exist far from the singular point.

We consider here a C^∞ -system defined in a neighbourhood of the origin and depending on parameters $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{R}^n$

$$(2.1) \quad \dot{x} = f_\mu(x, y), \quad \dot{y} = g_\mu(x, y).$$

We suppose that the origin is a singular point, and that for $\mu = 0$, the linear part of (2.1) around the origin has pure imaginary eigenvalues $\pm i\omega$. We now consider the system for $\mu = 0$. By a suitable linear change of coordinates and passing to complex coordinates $z = x + iy$, the system (2.1) becomes

$$(2.2) \quad \dot{z} = i\omega z + F(z, \bar{z}), \quad \dot{\bar{z}} = -i\omega \bar{z} + \overline{F(z, \bar{z})},$$

F can be developed to arbitrary order as a power series

$$(2.3) \quad F(z, \bar{z}) = \sum_{2 \leq i+j \leq r} a_{ij} z^i \bar{z}^j + O(|z|^{r+1}),$$

We make a polynomial change of variables, locally around the origin

$$(2.4) \quad z = w + \sum_{2 \leq i+j \leq r} b_{ij} w^i \bar{w}^j$$

and we determine the b_{ij} in order to get rid of all the terms in $z^i \bar{z}^j$ in (2.3), except the terms in $z^{i+1} \bar{z}^i$. In the new variable w , equation (2.2) becomes

$$(2.5) \quad \dot{w} = i\omega w + c_1 w^2 \bar{w} + c_2 w^3 \bar{w}^2 + \dots + O(|w|^{2k+3}),$$

(2.5) is called the *Poincaré normal form* of (2.2) (cf. [2]). We say that (2.1) has a *Hopf bifurcation of order k* at the origin if

$$(2.6) \quad \operatorname{Re}(c_1) = \dots = \operatorname{Re}(c_{k-1}) = 0 \quad \text{and} \quad \operatorname{Re}(c_k) \neq 0,$$

The whole system (2.1) can also be brought to Poincaré normal form

$$(2.7) \quad \dot{w} = c_{0,\mu} w + i\omega_\mu w + c_{1,\mu} w^2 \bar{w} + \dots + c_{k,\mu} w^{k+1} \bar{w}^k + O(|w|^{2k+3})$$

via a change of variables depending smoothly on the parameter μ , with $c_{i,\mu} = c_i$ and $c_{0,\mu} = 0$ for $\mu = 0$. In polar coordinates (2.7) becomes

$$(2.8) \quad \begin{aligned} \dot{r} &= \operatorname{Re}(c_{0,\mu}) r + \operatorname{Re}(c_{1,\mu}) r^3 + \dots + \operatorname{Re}(c_{k,\mu}) r^{2k+1} + O(r^{2k+3}), \\ \dot{\theta} &= \omega + O(r^2) = H(r, \theta). \end{aligned}$$

We divide the vector field (2.8) by the function $H(r, \theta)$ which is invertible in a neighbourhood of $(0, 0)$, as in [19]. Then we obtain a unique equation

$$(2.9) \quad dr/d\theta = d_{0,\mu} r + d_{1,\mu} r^3 + \dots + d_{k,\mu} r^{2k+1} + O(r^{2k+3}).$$

To illustrate how (2.9) settles the problem of the number of limit cycles of (2.1) around the origin, we assume first, for the sake of simplicity, that the rest $O(r^{2k+3})$ in (2.9) does not depend on θ , and contains only odd powers of r (this last hypothesis can be assumed at least to arbitrary high order). In this case, since the birth of limit cycles around the origin is equivalent to the birth of positive zeros of the equation $\dot{r} = 0$, by an application of the Malgrange-Weierstrass preparation theorem [15] $\dot{r} = 0$ would be equivalent to a polynomial equation

$$(2.10) \quad r(x_{k,\mu} \varrho^k + \dots + x_{1,\mu}) = 0, \quad \text{with } \varrho = r^2.$$

The number of positive roots would give the number of limit cycles arising from the origin. The parameter space could then be partitioned around the origin, into regions with a given number of limit cycles around the origin. This number is at most k .

In the case where the rest $O(r^{2k+3})$ depends on θ in (2.9), this argument no longer works. This is the case covered by the generalized Hopf bifurcation theorem. Although the argument in this case is longer, essentially the ideas are the same, θ is eliminated and we are reduced to a 1-dimensional problem by introducing the Poincaré map $P(x, \mu)$ on the x -axis [11], and then the displacement map $V(x, \mu) = P(x, \mu) - x$. Since limit cycles around the origin intersect the x -axis twice, once for $x > 0$, once for $x < 0$, we are interested in positive roots of $V(x, \mu) = 0$. The result is obtained by applying the Malgrange-Weierstrass preparation theorem to V around $(0, 0)$.

THEOREM. *Let*

$$(2.11) \quad \dot{x} = f(x, y), \quad \dot{y} = g(x, y)$$

be a C^k -system with a singular point at $(0, 0)$, and such that the system can be brought around the origin to the Poincaré normal form (2.5) with conditions (2.6). Then:

(i) *For any C^k -perturbation*

$$(2.12) \quad \dot{x} = f_\mu(x, y), \quad \dot{y} = g_\mu(x, y),$$

where $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{R}^n$, there exists a C^k -mapping $\Phi(\mu_1, \dots, \mu_n) = (\alpha_1, \dots, \alpha_{2k+1})$ such that the limit cycles of the perturbation around the origin intersect the x -axis at the real roots of the polynomial equation

$$(2.13) \quad Q(x, \mu) = \alpha_{2k+1} x^{2k} + \alpha_{2k} x^{2k-1} + \dots + \alpha_1 = 0.$$

These roots come in pairs, one positive, one negative. Thus there are at most k limit cycles. The stability (instability) of these cycles is determined by the sign of $Q(x, \mu)$ in a neighbourhood of such a positive root. In particular, the cycle is stable (unstable) if Q crosses the x -axis passing from positive to negative (negative to positive) values.

(ii) *For any $i \leq k$, there is a perturbation with exactly i limit cycles.*

Proof. (i) We can suppose that our system is in Poincaré normal form (2.8) with $\text{Re}(c_{i,0}) = 0$ for $i < k$, $\omega_\mu = 1$, $\dot{\theta} = 1$. We define $P(x, \mu)$ to be the return map along the x -axis, as in [11] and [4] and $V(x, \mu) = P(x, \mu) - x$ to be the displacement map. We calculate the $\partial^i V / \partial x^i(0, 0)$. For this we note that $x = r$ on the positive x -axis. We write

$$(2.14) \quad R(r, \theta, \mu) = u_1(\theta, \mu)r + u_2(\theta, \mu)r^2 + \dots + u_{2k+1}(\theta, \mu)r^{2k+1} + \dots$$

for the solution of (2.9) such that $R(r, 0, \mu) = r$. We have at $\mu = 0$

$$(2.15) \quad dr/d\theta = c_k r^{2k+1} + O(r^{2k+3}).$$

If $\psi(r_0, \theta)$ is the solution of (2.15), with $\psi(r_0, 0) = r_0$, then the return map at

$\mu = 0$ is $P(r_0, 0) = \psi(r_0, 2\pi) = R(r_0, 2\pi, 0)$. We compute $\partial V^i / \partial x^i(0, 0)$ for $1 \leq i \leq 2k+1$.

We have $(\partial V / \partial x)(0, 0) = (\partial P / \partial x)(0, 0) - 1 = (\partial / \partial r)|_{r=0} \psi(r, 2\pi) - 1$.

For $i > 1$, we have

$$(2.16) \quad (\partial V^i / \partial x^i)(0, 0) = (\partial P^i / \partial x^i)(0, 0) = (\partial^i / \partial r^i)|_{r=0} \psi(r, 2\pi).$$

To calculate $(\partial^i / \partial r^i)|_{r=0} \psi(r, 2\pi)$ for $1 \leq i \leq 2k+1$, we note that, since $\psi(r, \theta)$ is a solution of (2.15), we have

$$(2.17) \quad (\partial / \partial \theta)(\partial^i / \partial r^i)|_{r=0} \psi(r, \theta) = \begin{cases} 0 & \text{for } i < 2k+1, \\ (2k+1)! \operatorname{Re}(c_k) & \text{for } i = 2k+1. \end{cases}$$

So $(\partial^i / \partial r^i)|_{r=0} \psi(r, \theta)$ is constant for $i < 2k+1$ and is equal to $(2k+1)! \theta \operatorname{Re}(c_k)$ for $i = 2k+1$. So for $i < 2k+1$, $(\partial^i / \partial r^i)|_{r=0} \psi(r, \theta) = (\partial^i / \partial r^i)|_{r=0} \psi(r, 0)$, and, since $\psi(r, 0) = r$, $(\partial^i V / \partial x^i)(0, 0)$ is 0 for $i < 2k+1$ and is equal to $2\pi(2k+1)! \operatorname{Re}(c_k)$ for $i = 2k+1$. So $(\partial^{2k+1} V / \partial x^{2k+1})(0, 0) \neq 0$ and the previous derivatives vanish. We are exactly in the hypothesis of the Malgrange–Weierstrass preparation theorem [15] for $V(r, \mu)$ (or $V(x, \mu)$) and so we have

$$(2.18) \quad V(x, \mu) = Q(x, \mu) h(x, \mu)$$

with Q a polynomial of degree $2k+1$ and $h(x, \mu)$ invertible in a neighbourhood of $(0, 0)$. We can suppose $h(x, \mu) > 0$. We remark that Q is divisible by x , and that other roots of Q come in pairs, one positive, one negative, each pair corresponding to a limit cycle. In particular, there are at most k limit cycles. Their stability (instability) is decided by the sign of Q in a neighbourhood of the root.

(ii) This part is inspired by [4] and [17]. We suppose the system in normal form

$$(2.19) \quad \dot{z} = iz + c_k z^{k+1} \bar{z}^k + O(|z|^{2k+3}) = F(z, \bar{z}).$$

We take a perturbation of the form

$$(2.20) \quad \dot{z} = F(z, \bar{z}) + \beta_0 z + \beta_1 z^2 \bar{z} + \dots + \beta_{k-1} z^k \bar{z}^{k-1}, \quad \beta_i \in \mathbb{R}.$$

In polar coordinates (2.20) gives

$$(2.21) \quad \dot{r} = \beta_0 r + \beta_1 r^3 + \dots + \beta_{k-1} r^{2k-1} + \operatorname{Re}(c_k) r^{2k+1} + O(r^{2k+3}).$$

We can suppose $\operatorname{Re}(c_k) > 0$. In order to obtain i cycles for the system (2.21) we take $\beta_0, \dots, \beta_{k-1}$ in the following way. In a neighbourhood U of the origin, there exists $M > 0$ such that $|O(r^{2k+3})| < Mr^{2k+3}$. We choose $0 < r_k < 1$ such that the ball of radius r_k is included in U and

$$(2.22) \quad Mr_k^{2k+3} < \operatorname{Re}(c_k) r_k^{2k+1},$$

β_{k-1} is chosen negative, with $|\beta_{k-1}| < \operatorname{Re}(c_k)$, small enough so that

$$(2.23) \quad \beta_{k-1} r_k^{2k-1} + \operatorname{Re}(c_k) r_k^{2k+1} + O(r_k^{2k+3}) > 0.$$

Then $r_{k-1} < r_k$ is chosen small enough so that

$$(2.24) \quad \beta_{k-1} r_{k-1}^{2k-1} + \operatorname{Re}(c_k) r_{k-1}^{2k+1} + O(r_{k-1}^{2k+3}) < 0,$$

β_{k-2} is chosen positive with $\beta_{k-2} < |\beta_{k-1}|$ small enough so that

$$(2.25) \quad \beta_{k-2} r_k^{2k-3} + \beta_{k-1} r_k^{2k-1} + \operatorname{Re}(c_k) r_k^{2k+1} + O(r_k^{2k+3}) > 0,$$

$$(2.26) \quad \beta_{k-2} r_{k-1}^{2k-3} + \beta_{k-1} r_{k-1}^{2k-1} + \operatorname{Re}(c_k) r_{k-1}^{2k+1} + O(r_{k-1}^{2k+3}) < 0.$$

Then there exists $0 < r_{k-2} < r_{k-1}$, with

$$(2.27) \quad \beta_{k-2} r_{k-2}^{2k-3} + \beta_{k-1} r_{k-2}^{2k-1} + \operatorname{Re}(c_k) r_{k-2}^{2k+1} + O(r_{k-2}^{2k+3}) > 0;$$

$\beta_{k-3}, r_{k-3}, \dots, \beta_{k-i}, r_{k-i}$ are chosen similarly with $\operatorname{Re}(c_k)$, $\beta_{k-1}, \dots, \beta_{k-i}$ of alternate sign, $0 < |\beta_{k-i}| < \dots < |\beta_{k-1}| < \operatorname{Re}(c_k)$, $0 < r_{k-i} < \dots < r_{k-1} < r_k$, and we have finally

$$(2.28) \quad \begin{aligned} \dot{r} &> 0 && \text{on } r = r_k, r_{k-2}, \dots, \\ \dot{r} &< 0 && \text{on } r = r_{k-1}, r_{k-3}, \dots \end{aligned}$$

This gives i Poincaré–Bendixson domains, each containing a limit cycle ($\dot{\theta} > 0$ implies that the origin is the only singular point).

Remark 1. The Hopf bifurcation theorem only ensures us of the presence of limit cycles for small values of the parameters. The limit cycles may still exist for larger values of the parameters, but this has to be checked by other methods.

Remark 2. Our theorem is a consequence of Takens' result [19], and our proof is an alternative proof. The proof of Takens is merely existential. We find our proof simpler and more practical when one has to deal with particular examples, and wants to know the parameter region where the limit cycles are present for a given family of vector fields. The algorithmic aspect of the proof will make itself felt in the calculations on Macsyma in Sections 3 and 4 and in particular in formula (4.12). This algorithmic aspect is also developed in the rest of this section.

PROPOSITION. *Under the hypothesis of the theorem, for small μ , the power series of $V(r, \mu)$ is very close to*

$$2\pi(\operatorname{Re}(c_{0,\mu})r + \operatorname{Re}(c_{1,\mu})r^3 + \dots + \operatorname{Re}(c_{k,\mu})r^{2k+1}),$$

up to higher order terms.

Proof. We calculate explicitly the $u_i(\mu, 2\pi)$ in (2.14). For the rest of the

calculation we write only c_i for $c_{i,\mu}$ and u_i for $u_{i,\mu}$. We have

$$(2.29) \quad \partial/\partial\theta u_1(\theta) = \operatorname{Re}(c_0) u_1(\theta)$$

$$\text{and} \quad u_1(0) = 1 \Rightarrow u_1(\theta) = e^{\theta \operatorname{Re}(c_0)} \Rightarrow u_1(2\pi) = e^{2\pi \operatorname{Re}(c_0)}.$$

Similarly

$$(2.30) \quad \partial/\partial\theta u_2(\theta) = 2\operatorname{Re}(c_0) u_2(\theta)$$

$$\text{and} \quad u_2(0) = 0 \Rightarrow u_2(\theta) \equiv 0 = u_2(2\pi),$$

$$(2.31) \quad \partial/\partial\theta u_3(\theta) = (\operatorname{Re}(c_0) u_3 + \operatorname{Re}(c_1) u_1^3)$$

$$\Rightarrow u_3(\theta) = (e^{3\theta \operatorname{Re}(c_0)} - e^{\theta \operatorname{Re}(c_0)}) \operatorname{Re}(c_1)/2\operatorname{Re}(c_0)$$

$$\Rightarrow u_3(2\pi) = (e^{6\pi \operatorname{Re}(c_0)} - e^{2\pi \operatorname{Re}(c_0)}) \operatorname{Re}(c_1)/2\operatorname{Re}(c_0),$$

$$(2.32) \quad u_4(\theta) \equiv 0 = u_4(2\pi),$$

$$(2.33) \quad u_5(\theta) = (e^{4\theta \operatorname{Re}(c_0)} - 3e^{2\theta \operatorname{Re}(c_0)} + 2e^{\theta \operatorname{Re}(c_0)}) (\operatorname{Re}(c_1))^2/2(\operatorname{Re}(c_0))^2 + \\ + (e^{5\theta \operatorname{Re}(c_0)} - e^{\theta \operatorname{Re}(c_0)}) \operatorname{Re}(c_2)/4\operatorname{Re}(c_0),$$

etc. For small μ we get (if we take the linear approximations)

$$(2.34) \quad u_1 \cong 1 + 2\pi \operatorname{Re}(c_0), \quad u_{2i+1}(2\pi) \cong 2\pi \operatorname{Re}(c_i), \quad u_{2i}(2\pi) = 0$$

for $0 < i \leq k$.

We get the result by remarking that $V(x, \mu) = P(x, \mu) - x$.

We now want to discuss in which sense the information on the positive zeros of $Q(x, \mu)$ given in (2.18) is “almost contained” in the equation

$$(2.35) \quad \operatorname{Re}(c_0)r + \operatorname{Re}(c_1)r^3 + \dots + \operatorname{Re}(c_k)r^{2k+1} = 0$$

in the case where $Q(x, \mu)$ has exactly k positive zeros, which is the case which interests us here:

(1) The displacement function is very close to the left-hand side of (2.35) (see proposition).

(2) In the second part of the theorem we choose $\operatorname{Re}(c_k), \beta_{k-1}, \dots, \beta_1, \beta_0$, of alternate sign. We can remark here that a polynomial of degree k with k positive roots will have coefficients of alternate sign.

(3) For a polynomial as in (2.35) to have k positive roots when the c_i depend on μ , when $\operatorname{Re}(c_k)$ is different from 0 and $\operatorname{Re}(c_0), \dots, \operatorname{Re}(c_{k-1})$ are null for $\mu = 0$, for μ sufficiently small, we must have

$$(2.36) \quad |\operatorname{Re}(c_0)| \ll |\operatorname{Re}(c_1)| \ll \dots \ll |\operatorname{Re}(c_k)|.$$

This comes from the fact that the $\operatorname{Re}(c_i)$ are, up to sign, the symmetric functions in the roots of (2.35) which are arbitrarily close to 0 for μ small enough.

(4) We now consider the reduction of (2.8) to (2.9). This can be achieved in two steps. The first one is a division by ω , which is just a change of scale in time. It is easy to check that the division by $H(r, \theta)/\omega$, in (2.8) gives the $d_{i,\mu}$ to be very close to the $\operatorname{Re}(c_{i,\mu})$, as long as (2.36) is verified.

One can also notice that the proof of (ii) in the theorem works directly with the first equation of (2.8), without the reduction to (2.9).

(5) One remarks also that the presence of the $O(r^{2k+3})$ term does not change anything in the proof of (ii) in the theorem. Since we can start with a r_k small enough so that it is negligible in front of the $\operatorname{Re}(c_k)r_k^{2k+1}$, and that it becomes even more so in front of the other terms when we take the $r_i < r_k$, the second part of the proof works exactly as if the term $O(r^{2k+3})$ did not exist and we were only dealing with a polynomial.

(6) We consider (2.18) and we compare the first terms of the power series of V with the polynomial Q . We have

$$(2.37) \quad V(x, \mu) = a_{1,\mu}x + a_{3,\mu}x^3 + \dots + a_{2k+1,\mu}x^{2k+1} + O(|x|^{2k+3})$$

with $a_{i,\mu} = u_i(2\pi, \mu)$ for $i > 1$ and $a_{1,\mu} = u_1(2\pi, \mu) - 1$ (cf. (2.14)). Also

$$(2.38) \quad Q(x, \mu) = \alpha_{1,\mu}x + \alpha_{2,\mu}x^2 + \dots + \alpha_{2k+1,\mu}x^{2k+1}.$$

The function $h(x, \mu)$ and all its derivatives are bounded for x and μ in a neighbourhood of 0. We identify the two series, term by term. We get

$$(2.39) \quad a_{1,\mu} = \alpha_{1,\mu}h(0, \mu),$$

$$(2.40) \quad 0 = \alpha_{2,\mu}h(0, \mu) + \alpha_{1,\mu}h'(0, \mu),$$

$$(2.41) \quad a_{3,\mu} = \alpha_{3,\mu}h(0, \mu) + \alpha_{2,\mu}h'(0, \mu) + \alpha_{1,\mu}h''(0, \mu)/2$$

etc. (Where all derivatives of h are taken with respect to x .) We can suppose that $h(0, \mu) = 1$. Assuming (2.36) for μ sufficiently small, we have

$$(2.42) \quad a_{1,\mu} \ll a_{3,\mu} \ll \dots \ll a_{2k+1,\mu}$$

(cf. Proposition). So for μ small, we have

$$(2.43) \quad a_{2i+1,\mu} \cong \alpha_{2i+1,\mu}, \quad \backslash$$

We finally illustrate these remarks on an example.

EXAMPLE. We consider the two systems

$$(2.44) \quad \dot{r} = r^5 + \varepsilon_1 r^3 + \varepsilon_0 r, \quad \dot{\theta} = 1,$$

$$(2.45) \quad \dot{r} = -r^7 + r^5 + \varepsilon_1 r^3 + \varepsilon_0 r, \quad \dot{\theta} = 1,$$

The first system is already in truncated normal form and has

$$(2.46) \quad 2 \text{ limit cycles for } \varepsilon_1^2 - 4\varepsilon_0 > 0, \varepsilon_1 < 0, \varepsilon_0 > 0,$$

$$(2.47) \quad 1 \text{ limit cycle for } \varepsilon_0 < 0, \text{ for } \varepsilon_1^2 - 4\varepsilon_0 = 0, \varepsilon_1 < 0, \varepsilon_0 > 0 \text{ and for } \varepsilon_0 = 0, \varepsilon_1 < 0 \text{ no limit cycles elsewhere.}$$

In the equation $\dot{r} = 0$ for the second system there are three roots whose sum is 1, so at least one root is far from zero. The product of the roots is ε_0 which is small, so at least one root is small, in absolute value, and the sum of products of roots two at a time is also small, so two of the roots are small in absolute value. System (2.45) has therefore one real root close to 1 and two small roots, in absolute value, for small $\varepsilon_0, \varepsilon_1$. We are interested only in the limit cycles which are close to the origin. Since one limit cycle is far from the origin, the system has:

$$(2.48) \quad 2 \text{ limit cycles close to the origin for}$$

$$\Delta = \varepsilon_1^2 - 4\varepsilon_0 + 4\varepsilon_1^3 - 18\varepsilon_1\varepsilon_0 - 27\varepsilon_0^2 > 0, \quad \varepsilon_0 > 0, \varepsilon_1 < 0;$$

$$(2.49) \quad 1 \text{ limit cycle for } \varepsilon_0 < 0 \text{ for } \varepsilon_0 = 0, \varepsilon_1 < 0 \text{ and for } \Delta = 0, \varepsilon_1 < 0, \varepsilon_0 > 0, \text{ no limit cycles close to the origin elsewhere.}$$

The curve $\Delta = 0$ is represented in Fig. 1 together with the parabola $\varepsilon_1^2 - 4\varepsilon_0 = 0$ and the two curves can be seen to be very close in the neighbourhood of the origin. The regions where (2.44) and (2.45) have two (resp. one, zero) limit cycles close to the origin are nearly the same but not exactly the same. Since system (2.44) can be viewed as a truncation of system (2.45), this

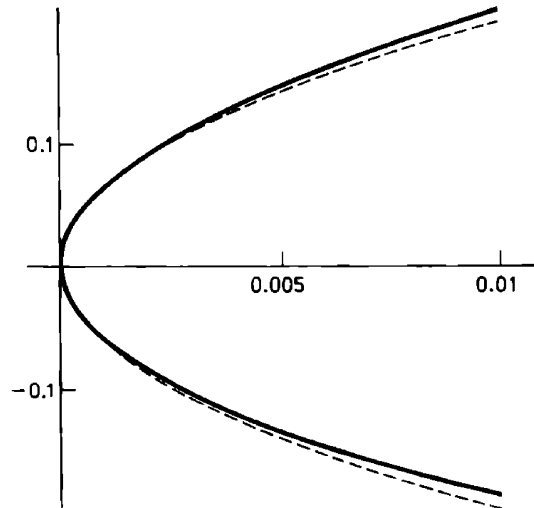


Fig. 1. Comparison of bifurcation diagrams of the systems (2.44) and (2.45). The curve $\Delta = 0$ is plain line and the parabola $\varepsilon_1^2 - 4\varepsilon_0 = 0$ is dotted line

illustrates that the bifurcation diagram is only approximately given by the bifurcation diagram of the truncated system. We see that the approximation becomes quite good when the parameters are small and we are interested in a small region around the singular point.

3. Analysis of the example of Shi Songling. We analyse in the light of Section 2 the system (1.2). For any $\lambda, \varepsilon, \delta$, the system has two singular points $(0, 0)$ and $(0, 1)$. $(0, 0)$ and $(0, 1)$ are foci. System (1.2) has a limit cycle around the point $(0, 1)$.

PROPOSITION. *The limit cycle of system (1.2) around the point $(0, 1)$ comes from a Hopf bifurcation at $\beta = 0$ of the system*

$$(3.1) \quad \dot{x} = -y - 10x^2 + \beta xy + y^2, \quad \dot{y} = x + x^2 - 25xy.$$

Proof. (3.1) has a Hopf bifurcation at $(0, 1)$ for $\beta = 0$. The technique of Section 2 gives us that the limit cycle is attractive and exists for $\beta > 0$ small. For $\beta \in [0, 5]$, system (3.1) has no other singular point than $(0, 0)$ and $(0, 1)$, and a unique singular point at infinity, which is a saddle. The saddle point at infinity has its stable manifold along the equator and its unstable manifold pointing inwards. So the limit cycle cannot disappear for a saddle-connection at infinity. It has no interaction with the origin, since $\dot{x} < 0$ on the line $1 + x - 25y = 0$. The only thing that could happen could be a pitchfork bifurcation (or a similar bifurcation), in which three cycles would appear from the original limit cycle (the more general case would be an odd number of limit cycles). At $\beta = 5$ there is an odd number of cycles around $(0, 1)$, since we have a Poincaré–Bendixson domain limited inside by the repulsive point $(0, 1)$, and outside by the equator at infinity and the line $1 + x - 25y = 0$. At least one of these cycles comes from the Hopf bifurcation.

Now, for $\lambda = \varepsilon = \delta = 0$. The point $(0, 0)$ is a weak focus and we have a degenerate Hopf bifurcation of order 3. In fact, if we compute the Poincaré normal form, we get

$$(3.2) \quad \operatorname{Re}(c_1) = \operatorname{Re}(c_2) = 0, \quad \operatorname{Re}(c_3) = 35625/8.$$

Therefore there exists a perturbation which has three limit cycles. Here it is remarkable that this perturbation can be achieved within the quadratic systems only.

We first consider the following system

$$(3.3) \quad \begin{aligned} \dot{x} &= \lambda x - y - 10x^2 + (5 + \delta)xy + y^2, \\ \dot{y} &= x + \lambda y + x^2 + (8\varepsilon - 25 - 9\delta)xy. \end{aligned}$$

For this system we get an equation (with $\mu = (\varepsilon, \delta) \in \mathbb{R}^2$)

$$(3.4) \quad \dot{r} = \lambda r + \operatorname{Re}(c_{1,\mu})r^3 + \operatorname{Re}(c_{2,\mu})r^5 + \operatorname{Re}(c_{3,\mu})r^7 + O(r^9).$$

We have

$$(3.5) \quad \operatorname{Re}(c_{1,\mu}) = -\varepsilon,$$

$$(3.6) \quad \operatorname{Re}(c_{2,\mu}) = [540\delta^3 + 4125\delta^2 - 1376\varepsilon\delta^2 + 1152\varepsilon^2\delta - 6750\varepsilon\delta + 7125\delta - 320\varepsilon^3 - 2680\varepsilon^2 - 7040\varepsilon]/36,$$

$$(3.7) \quad \operatorname{Re}(c_{3,0}) = 35625/8 \Rightarrow \operatorname{Re}(c_{3,\mu}) \cong 35625/8.$$

We observe that, for small μ , the coefficients of the principal part of (3.4) alternate in sign, starting with $\operatorname{Re}(c_{3,\mu})$ positive. We also have that

$$(3.8) \quad |\lambda| = 10^{-200} \ll \operatorname{Re}(c_{1,\mu}) = -\varepsilon = 10^{-52} \ll |\operatorname{Re}(c_{2,\mu})| \cong 7125/36 \times 10^{-13}.$$

These facts, together with the proof of part (ii) of the theorem give us an understanding of why we do have three limit cycles in this case.

Remark 1. When real-diagonalized the example of Shi Songling in (1.2) is a perturbation of system (3.3) in which $\lambda = -\frac{1}{2} 10^{-200}$. The perturbation is of order about 10^{-200} and does not affect (3.5), (3.6), (3.7). This perturbation is preferred to the one of (3.3), since it keeps $(0, 1)$ as a singular point.

Remark 2. If we neglect the $O(r^9)$ term in (3.4), the number of limit cycles around the origin is equal to the number of positive roots of

$$(3.9) \quad \lambda r + \operatorname{Re}(c_{1,\mu})r^3 + \operatorname{Re}(c_{2,\mu})r^5 + \operatorname{Re}(c_{3,\mu})r^7 = 0$$

which is three. Indeed we have a positive discriminant and Routh–Hurwitz criterion for roots with positive real parts is verified.

Remark 3. The calculation of $\operatorname{Re}(c_1)$, $\operatorname{Re}(c_2)$, $\operatorname{Re}(c_3)$ in (3.2), (3.5) and (3.6) were done on Macsyma. We give a general formula for $\operatorname{Re}(c_2)$ in the next section.

4. Construction of quadratic systems with limit cycles via Hopf bifurcations. We want to exploit here the method described in Section 2 in order to get quadratic systems with as many limit cycles as it is possible, via this method. So we start with an arbitrary quadratic system with two singular points. We can suppose that the origin is a singular point. We start with the origin as a weak focus (see definition below) in order to get a Hopf bifurcation. Modulo a linear change of coordinates, the linear part around the origin is $\dot{x} = -\omega y$, $\dot{y} = \omega x$, $\omega > 0$. A rotation and change of scale brings the other singular points to $(0, 1)$, and a change of scale in time brings ω to 1. We start therefore with the system

$$(4.1) \quad \dot{x} = -y + ax^2 + bxy + y^2, \quad \dot{y} = x + cx^2 + dxy.$$

The goal is to try to degenerate simultaneously or successively the two

singular points, in order to have degenerate Hopf bifurcations of order ≥ 2 . In doing so we must not obtain an integrable system, one for which every trajectory is closed. Fortunately there exists a characterization of the quadratic integrable systems which gives in our case ([8])

(4.1) is integrable iff one of the following conditions is satisfied:

(I) $c = a + 1 = 0$;

(II) $c(d + 2a) = b(a + 1)$ and

$$(4.2) \quad c(d + 2a)^3 + b(d - a)(d + 2a)^2 - b^3(d + 2a) - b^3 = 0;$$

(III) $d - 3a - 5 = -b + 5c = a + 2(c^2 + 1) = 0$.

We call a *weak focus* a singular point of a system, with two purely imaginary eigenvalues of the linearized system. The condition for it to have a degenerate Hopf bifurcation of order at least 2 is that $\text{Re}(c_1) = 0$ in (2.5). For a system

$$(4.3) \quad \dot{x} = -y + f(x, y), \quad \dot{y} = x + g(x, y)$$

with f and g quadratic we get

$$(4.4) \quad \text{Re}(c_1) = [f_{xy}(f_{xx} + f_{yy}) - g_{xy}(g_{xx} + g_{yy}) - f_{xx}g_{xx} + f_{yy}g_{yy}]/16.$$

For system (4.1) this gives

$$(4.5) \quad \text{Re}(c_1) = [b(a + 1) - c(d + 2a)]/8.$$

The calculation (on Macsyma) of $\text{Re}(c_2)$ for system (4.1) gives

$$(4.6) \quad \begin{aligned} \text{Re}(c_2) = & (-5cd^3 + 9acd^2 + 5cd^2 - abd^2 - bd^2 - 20c^3d - 19bc^2d + \\ & + b^2cd + 18a^2cd + 8acd + 10cd - a^2bd + 18abd + 19bd - 40ac^3 - \\ & - 18abc^2 - 10bc^2 - 9ab^2c - 5b^2c - 40a^3c - 4a^2c + 20ac + \\ & + 5ab^3 + 5b^3 + 20a^3b + 40a^2b + 40ab + 20b)/288. \end{aligned}$$

Remark. We can make a heuristic remark on Bautin's theorem, which states that at most three limit cycles can arise from a singular point by variation of the coefficients of a quadratic system. In (4.1) we have four independent parameters that can vary, in order to annihilate the $\text{Re}(c_1)$ of (2.5). Since the manifold of integrable systems is of codimension 2 (cf. (4.2)), it is natural that we can annihilate two of the $\text{Re}(c_i)$ before arriving at an integrable system.

We now show the relationship between the coefficients of the normal form as obtained by the method of Section 2 and the focal quantities obtained in Bautin [3]. These can be found for the normalized system (4.1) in [18]

$$(4.7) \quad V_1 = b(a + 1) - c(d + 2a),$$

$$(4.8) \quad V_2 = (b - 5c) [bc(2a + d + 1) - (d + 2a)(a + 1)(d + 1)],$$

$$(4.9) \quad V_3 = -c(2c^2 + 2 + a) [bc(2a + d + 1) - (d + 2a)(a + 1)(d + 1)].$$

We have that

$$(4.10) \quad \operatorname{Re}(c_1) = V_1/8.$$

Also $\operatorname{Re}(c_2)$ given in (4.6) is an irreducible polynomial over the rationals but it can be factorized under the condition $\operatorname{Re}(c_1) = 0$. Since for $c = 0$ and $\operatorname{Re}(c_1) = 0$ the system is integrable by (4.2), we use the condition $\operatorname{Re}(c_1) = 0$ in the form

$$(4.11) \quad d = -2a + b(a + 1)/c.$$

We get for $\operatorname{Re}(c_2)$

$$(4.12) \quad \begin{aligned} \operatorname{Re}(c_2) = & -[b(5c - b)(c^2 + abc^2 + bc^2 + 2a^3c + 3a^2c - c - a^3b - \\ & - 3a^2b - 3ab - b)]/(48c^2) \\ = & (b - 5c) [bc(2a + d + 1) - (d + 2a)(a + 1)(d + 1)]/48 = V_2/48. \end{aligned}$$

A similar calculation can be made for $\operatorname{Re}(c_3)$ and V_3 .

PROPOSITION. (4.1) *cannot have two weak foci, one of which is degenerate, unless it is integrable.*

Proof. $(0, 0)$ is degenerate to order at least 2 if $\operatorname{Re}(c_1) = 0$. Through (4.5) we get the equation

$$(4.13) \quad \operatorname{Re}(c_1) = [b(a + 1) - c(d + 2a)] = 0.$$

We now localize (4.1) at $(0, 1)$, with $Y = y - 1$. (4.1) becomes

$$(4.14) \quad \dot{x} = bx + Y + ax^2 + bxY + Y^2, \quad \dot{y} = x(1 + d) + cx^2 + dxY,$$

$$(4.15) \quad (0, 1) \text{ is a weak focus iff } b = 0 \text{ and } 1 + d < 0.$$

With $b = 0$, (4.13) becomes $c(d + 2a) = 0$. $b = 0$ and $c = 0$, or $b = 0$ and $d + 2a = 0$ implies that (4.1) is integrable by (II) of (4.2).

Since it is not possible to obtain limit cycles by degenerating simultaneously the two weak foci we try to do as in the example of Shi Songling: degenerate the point $(0, 1)$ as a weak center, then add parameters in order to create limit cycles around $(0, 1)$. Once $(0, 1)$ is only a focus and the limit cycles around it are structurally stable, we try to create limit cycles around $(0, 0)$ by a Hopf bifurcation of higher order.

THEOREM. *By successive Hopf bifurcations around $(0, 1)$ and $(0, 0)$ it is possible to create up to four limit cycles for system (4.1). In the case of four limit cycles, three of them must necessarily be around one singular point and the last one around the other singular point.*

Proof. We consider (4.1) which has a degenerate Hopf bifurcation at $(0, 0)$ if (4.13) is satisfied. We suppose that $b \neq 0$ and that, when $b = 0$ (4.1) has gone through a Hopf bifurcation at the point $(0, 1)$. For $b = 0$, system (4.1), localized at $(0, 1)$, gives

$$(4.16) \quad \dot{x} = Y + ax^2 + Y^2, \quad \dot{y} = x(1+d) + cx^2 + dxY.$$

We change x for $X = -x\sqrt{-d-1}$. (4.10) becomes

$$(4.17) \quad \begin{aligned} \dot{X} &= -Y\sqrt{-d-1} - aX^2/\sqrt{-d-1} - Y^2\sqrt{-d-1}, \\ \dot{Y} &= X\sqrt{-d-1} + cX^2/(-d-1) - dXY/\sqrt{-d-1}. \end{aligned}$$

We annihilate $\text{Re}(c'_1)$ by (4.5) for system (4.17)

$$(4.18) \quad \text{Re}(c'_1) = c(d+2a)/8(d+1)^2 = 0.$$

Since $b \neq 0$, (4.13) and (4.18) together imply $a+1=0$. In the case $a+1=c=0$ we get an integrable system. The case $d+2a=a+1=0$ gives $d=2$ which contradicts (4.15). For $b \neq 0$ we have a limit cycle around $(0, 1)$ if $\text{sgn}(b \text{Re}(c'_1)) < 0$, and b is small enough. If (4.1) also satisfies $\text{Re}(c_1) = \text{Re}(c_2) = 0$, $\text{Re}(c_3) \neq 0$ around $(0, 0)$, then a perturbation of the form

$$(4.19) \quad \begin{aligned} \dot{x} &= \lambda x - y + ax^2 + (b+\delta)xy + y^2, \\ \dot{y} &= x + \lambda y + cx^2 + (d+8\varepsilon/c(a+1)\delta/c)xy \end{aligned}$$

with $|\lambda| \ll |\varepsilon| \ll |\delta|$ will have three limit cycles around the origin. (Since $\text{Re}(c_1) = -\varepsilon$ and for $|\varepsilon| \ll |\delta|$, $\text{Re}(c_2)$ depends essentially on δ .)

Then the system

$$(4.20) \quad \begin{aligned} \dot{x} &= \lambda x - y + ax^2 + (b+\delta)xy + y^2, \\ \dot{y} &= x + cx^2 + (d+8\varepsilon/c+(a+1)\delta/c)xy \end{aligned}$$

will have the required properties. We remark that in this procedure $(0, 1)$ and $(0, 0)$ are singular points throughout the two perturbations involved in the two bifurcations.

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