A CHARACTERIZATION OF \( \alpha \)-CONVOLUTIONS

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For the terminology and notation used here, see [2]. In particular, \( \mathfrak{B} \) denotes the class of all probability measures defined on Borel subsets of the positive half-line. By \( E_a (a \geq 0) \) we denote the probability measure concentrated at the point \( a \). For any \( a \) \((a > 0)\), the transformation \( T_a \) of \( \mathfrak{B} \) onto itself is defined by means of the formula \( (T_a P)(\mathcal{A}) = P(a^{-1}\mathcal{A}) \), where \( P \in \mathfrak{B} \), \( \mathcal{A} \) is a Borel set, and \( a^{-1}\mathcal{A} = \{ a^{-1}x : x \in \mathcal{A} \} \). The transformation \( T_a \) is defined by assuming \( T_a P = E_0 \) for all \( P \in \mathfrak{B} \).

A commutative and associative \( \mathfrak{B} \)-valued binary operation \( \circ \) defined on \( \mathfrak{B} \) is called a generalized convolution if it satisfies the following conditions:

(i) \( E_0 \circ P = P \) for all \( P \in \mathfrak{B} \);

(ii) \( (aP + bQ) \circ R = a(P \circ R) + b(Q \circ R) \), whenever \( P, Q, R \in \mathfrak{B} \) and \( a \geq 0, b \geq 0, a + b = 1 \);

(iii) \( (T_a P) \circ (T_a Q) = T_a (P \circ Q) \) for any \( P, Q \in \mathfrak{B} \) and \( a \geq 0 \);

(iv) if \( P_n \to P \), then \( P_n \circ Q \to P \circ Q \) for all \( Q \in \mathfrak{B} \), where the convergence is the weak convergence of probability measures;

(v) there exists a sequence \( c_1, c_2, \ldots \) of positive numbers such that the sequence \( T_{c_n} E_1^n \) weakly converges to a measure \( Q \) different from \( E_0 \) (the power \( E_1^n \) is taken here in the sense of the operation \( \circ \)).

The class \( \mathfrak{B} \) with a generalized convolution \( \circ \) is called a generalized convolution algebra and denoted by \( (\mathfrak{B}, \circ) \). Algebras admitting a non-trivial homomorphism into the real field are called regular.

An algebra \( (\mathfrak{B}, \circ) \) is called quasi-regular if it satisfies the following condition:

(vi) there exists a sequence \( c_1, c_2, \ldots \) of positive numbers such that

\[
\lim_{n \to \infty} c_n = 0 \quad \text{and} \quad \lim_{n \to \infty} T_{c_n} E_1^n = Q \quad \text{and} \quad Q \neq E_0.
\]

It is known that every regular algebra \( (\mathfrak{B}, \circ) \) is quasi-regular (see [2], Theorem 4). The following problem appears still to be open:
**PROBLEM.** Is every quasi-regular convolution algebra regular? (P 826)

The $\alpha$-convolutions, being a modification of the ordinary convolution, are simple examples of regular generalized convolutions. For every $\alpha > 0$, an $\alpha$-convolution is defined by the formula

$$
\int_0^\infty f(x)(P \circ R)(dx) = \int_0^\infty \int_0^\infty f((x^a + y^a)^{1/\alpha})P(dx)R(dy)
$$

for all bounded continuous functions $f$ on the positive half-line.

The aim of the present paper is to give a characterization of the $\alpha$-convolutions. We say that a probability measure $P$ is infinitely divisible in the algebra $(\mathfrak{B}, \circ)$ if for every integer $n$ there exists a probability measure $P_n$ such that $P_n^n = P$. Further, we say that a probability measure $P$ is decomposable if it can be written in the form $P = R_1 \circ R_2$, where $R_1 \neq E_0$ and $R_2 \neq E_0$. It is clear that each infinitely divisible measure is decomposable. Therefore, our result can be regarded as a partial solution of the following problem raised by K. Urbanik:

**PROBLEM.** Suppose that $(\mathfrak{B}, \circ)$ is a quasi-regular convolution algebra and the measure $E_1$ is decomposable. Is then $(\mathfrak{B}, \circ)$ an $\alpha$-convolution algebra? (P 827)

**THEOREM.** Let $(\mathfrak{B}, \circ)$ be a quasi-regular convolution algebra in which the measure $E_1$ is infinitely divisible. Then $(\mathfrak{B}, \circ)$ is an $\alpha$-convolution algebra.

Before proving the Theorem we shall prove some lemmas.

**LEMMA 1.** If an algebra $(\mathfrak{B}, \circ)$ is quasi-regular and there exists a sequence $a_1, a_2, \ldots$ such that $T_{a_n} E_1^n \to P$, where $P \in \mathfrak{B}$ and $P \neq E_0$, then

$$
\lim_{n \to \infty} a_n = 0.
$$

**Proof.** If the algebra $(\mathfrak{B}, \circ)$ is quasi-regular, then there exists a sequence $c_1, c_2, \ldots$ of positive numbers for which

$$
\lim_{n \to \infty} c_n = 0 \quad \text{and} \quad \lim_{n \to \infty} T_{c_n} E_1^n = Q, \quad \text{where } Q \in \mathfrak{B} \text{ and } Q \neq E_0.
$$

Now, let us suppose that there exists a subsequence $a_{k_1}, a_{k_2}, \ldots$ of the sequence $a_1, a_2, \ldots$ such that

$$
\lim_{n \to \infty} a_{k_n}^{-1} = a < \infty.
$$

We have

$$
T_{a_{k_n}^{-1}} T_{a_{k_n}} T_{c_{k_n}} E_1^{c_{k_n}} = T_{c_{k_n}} E_1^{c_{k_n}} \to Q.
$$

On the other hand,

$$
T_{a_{k_n}^{-1}} T_{a_{k_n}} T_{c_{k_n}} E_1^{c_{k_n}} = T_{a_{k_n}^{-1}} T_{c_{k_n}} T_{a_{k_n}} E_1^{c_{k_n}} \to T_0 P = E_0.
$$
Hence, \( Q = E_0 \), which contradicts the hypothesis.

**Lemma 2.** If for some integer \( k \) there exists a probability measure \( P \) such that \( P^{ck} = E_1 \), then there exists a point \( a \) for which \( E_a^{ck} = E_1 \).

**Proof.** It is easy to verify that for some point \( a \)

\[
P\left(\left\{ x: a - \frac{1}{n} \leq x \leq a + \frac{1}{n}\right\}\right) > 0 \quad \text{for any integer } n > 0.
\]

Let us introduce the notation

\[
\mathcal{A}_n = \left\{ x: a - \frac{1}{n} \leq x \leq a + \frac{1}{n}\right\}
\]

and

\[
P_n(\mathcal{A}) = \frac{P(\mathcal{A}_n \cap \mathcal{A})}{P(\mathcal{A}_n)} \quad \text{for all Borel sets } \mathcal{A}.
\]

Then we have \( P = a_nP_n + \beta_nR_n \), where \( R_n \) is a probability measure concentrated on the set \([0, \infty) - \mathcal{A}_n\) and \( a_n > 0 \). Taking into account the formula

\[
E_1 = a_n^kP_n^{ck} + \sum_{r=1}^{n} \binom{n}{r} a_n^r \beta_n^{n-r} P_n^{cr} \circ Q^{(n-r)}
\]

and the inequality \( a_n^k > 0 \), we infer that the measure \( P_n^{ck} \) is concentrated at the point 1. Consequently, \( P_n^{ck} = E_1 \). Further, it is easy to verify that

\[
\lim_{n \to \infty} P_n^{ck} = E_1^{ck}.
\]

Thus \( E_a^{ck} = E_1 \), which completes the proof.

**Proof of the Theorem.** From lemmas 1 and 2 it follows that for the measure \( E_1 \) there exists a sequence \( a_1, a_2, \ldots \) such that

1. \( E_1 = T_{a_n} E_1^n \)
2. \( \lim_{n \to \infty} a_n = 0 \).

First of all, we prove that

\[
\lim_{n \to \infty} \frac{a_n}{a_{n+1}} = 1.
\]

Of course, it suffices to prove that for every convergent subsequence \( a_{n_k}/a_{n_k+1} \) of sequence \( a_n/a_{n+1} \)

\[
\lim_{k \to \infty} \frac{a_{n_k}}{a_{n_k+1}} = 1.
\]
We prove that

\[
\lim_{k \to \infty} \frac{a_{n_k}}{a_{n_k + 1}} < \infty.
\]

Contrary to this let us suppose that

\[
\lim_{k \to \infty} \frac{a_{n_k + 1}}{a_{n_k}} = 0.
\]

Of course, we have the formula

\[
T_{a_{n_k + 1}} E_{1}^{(n_k + 1)} = E_{1}.
\]

On the other hand, putting \( d_k = a_{n_k + 1}/a_{n_k} \),

\[
T_{a_{n_k + 1}} E_{1}^{(n_k + 1)} = T_{d_k a_{n_k}} E_{1}^{(n_k + 1)} = T_{d_k a_{n_k}} E_{1}^{(n_k + 1)} \circ T_{d_k a_{n_k}} E_{1} \to E_{0}.
\]

Hence \( E_0 = E_1 \) which gives a contradiction. Formula (4) is thus proved.

Let \( r_k = a_{n_k}/a_{n_k + 1} \) and \( r_k \to r \). Then we have the relation

\[
T_{a_{n_k}} E_{1}^{(n_k + 1)} = T_{a_{n_k}} E_{1}^{(n_k + 1)} \circ T_{a_{n_k}} E_{1} \to E_{1} \circ E_{0} = E_{1}.
\]

On the other hand,

\[
T_{a_{n_k}} E_{1}^{(n_k + 1)} = T_{r_k a_{n_k + 1}} E_{1}^{(n_k + 1)} = T_{r_k} E_{1} \to T_{r} E_{1}.
\]

Hence \( E_1 = E_r \), and \( r = 1 \). Formula (3) is thus proved.

From (2) and (3) it follows that for any pair \( x, y \) of positive numbers there exist subsequences \( a_{n_1}, a_{n_1}, \ldots \) and \( a_{m_1}, a_{m_2}, \ldots \) of the sequence \( a_1, a_2, \ldots \) such that

\[
\lim_{k \to \infty} \frac{a_{n_k}}{a_{m_k}} = \frac{y}{x}.
\]

Moreover, we can assume without loss of generality that the limit

\[
s = \lim_{k \to \infty} \frac{a_{n_k}}{a_{m_k + n_k}},
\]

perhaps infinite, does exist. First of all, we prove that the limit \( s \) is finite. Let us suppose to the contrary that

\[
\lim_{k \to \infty} v_k = 0, \quad \text{where} \quad v_k = \frac{a_{n_k + m_k}}{a_{n_k}}.
\]

Setting \( w_k = a_{n_k}/a_{m_k} \), we have

\[
T_{a_{n_k + m_k}} E_{1}^{(n_k + m_k)} = E_{1}.
\]
On the other hand,
\[ T_{{x^k}a_{{n^k}+{m^k}}} E_{1}^{c_{n^k}+{m^k}} = T_{{v^k}a_{{n^k}}} E_{1}^{c_{n^k}} \circ T_{{v^k}a_{{m^k}}} E_{1}^{c_{m^k}} = T_{{v^k}a_{{n^k}}} E_{1} \circ T_{{v^k}a_{{m^k}}} E_{1} \rightarrow E_{0} \circ E_{0} = E_{0}. \]

Hence \( E_{0} = E_{1} \), which is impossible. The finiteness of the limit \( s \) is thus proved.

Using the notations \( s_{{n^k}+{m^k}} = a_{{n^k}}/a_{{m^k}} \) and \( w_{{m^k}} = a_{{n^k}}/a_{{m^k}} \), we obtain the following equations:
\[ T_{{x^k}a_{{n^k}+{m^k}}} E_{1}^{c_{n^k}+{m^k}} = T_{{x^k}a_{{n^k}}} E_{1}^{c_{n^k}} \circ T_{{x^k}a_{{m^k}}} E_{1}^{c_{m^k}} = T_{{x^k}a_{{n^k}}} E_{1} \circ T_{{x^k}a_{{m^k}}} E_{1} \rightarrow T_{{x^k}a_{{n^k}}} E_{1} \circ T_{{x^k}a_{{m^k}}} E_{1} = E_{x} \circ E_{y}, \]
\[ T_{{x^k}a_{{n^k}+{m^k}}} E_{1}^{c_{n^k}+{m^k}} = T_{{x^k}a_{{n^k}+{m^k}}} E_{1}^{c_{n^k}+{m^k}} \rightarrow T_{{x^k}a_{{n^k}+{m^k}}} E_{1} = E_{x}. \]

Hence we have
\[ E_{x} \circ E_{y} = E_{x}. \] (5)

We define an auxiliary function \( g(x, y) \) by means of the formulas
\[ g(x, 0) = x, g(0, y) = y \quad \text{and} \quad g(x, y) = sx \quad \text{for} \quad x > 0, y > 0. \] The function \( g \) satisfies the equation
\[ E_{x} \circ E_{y} = E_{g(x,y)}. \] (6)

It is easy to see that \( g \) is the only function satisfying (6).

As a direct consequence of equation (6) and of the uniqueness of its solution, we obtain
\[ g(x, y) = g(y, x), \] (7)
\[ g(g(x, y), z) = g(x, g(y, z)), \] (8)
\[ g(xz, yz) = zg(x, y) \] (9)
for all non-negative numbers \( x, y \) and \( z \).

Now, we prove that the function \( g \) is continuous in the quadrant \( x, y \geq 0 \). Let \( x_{n} \rightarrow x \) and \( y_{n} \rightarrow y \). Moreover, suppose that the sequence \( g(x_{n}, y_{n}) \rightarrow z \), where \( 0 \leq z \leq \infty \). The equation \( z = \infty \) is impossible. Indeed, setting \( p_{n} = x_{n}/g(x_{n}, y_{n}) \) and \( q_{n} = y_{n}/g(x_{n}, y_{n}) \) by (6), we have
\[ E_{p_{n}} \circ E_{q_{n}} = E_{1}. \] If \( p_{n} \rightarrow 0 \) and \( q_{n} \rightarrow 0 \), then \( E_{p_{n}} \circ E_{q_{n}} \rightarrow E_{0} \) and \( E_{1} = E_{0} \). It is impossible. Hence
\[ z < \infty \quad \text{and} \quad E_{x} \circ E_{y} = \lim_{n \to \infty} (E_{x_{n}} \circ E_{y_{n}}) = \lim_{n \to \infty} E_{z_{n}} = E_{z}. \]

From this equation it follows that \( g(x, y) = z \). Thus the function \( g \) is continuous.

From (6) we obtain
\[ E_{1}^{z} = E_{g(1,1)}. \] (10)
If \( g(1, 1) < 1 \), then, by induction from (10), we get \( E_{1}^{n} = E_{0}^{n(1,1)} \rightarrow E_{0} \). Hence \( T_{a_{n}}E_{1}^{n} \rightarrow E_{0} \). On the other hand, \( T_{a_{n}}E_{1}^{n} = E_{1} \). Since the equation \( E_{0} = E_{1} \) cannot hold true, we have the inequality \( g(1, 1) \geq 1 \). If \( g(1, 1) = 1 \), then from (10) we get \( E_{1}^{2} = E_{1} \) and, consequently, \( E_{1}^{n} = E_{1} \). Hence, it follows that

\[
\lim_{n \to \infty} T_{a_{n}}E_{1}^{n} = \lim_{n \to \infty} T_{a_{n}}E_{1} = E_{1}.
\]

Of course, it is impossible. Therefore,

\[(11) \quad g(1, 1) > 1.\]

By (9), to prove the inequality

\[(12) \quad g(x, y) > x \quad (x \geq 0, y > 0)\]

it suffices to prove it for \( y = 1 \). Let us suppose that there exists a number \( x_{1} \) such that \( g(x_{1}, 1) < x_{1} \). Since \( g(0, 1) = 1 \) and the function \( g \) is continuous, we infer that there exists a number \( x_{0} \) lying between 0 and \( x_{1} \), for which the equation \( g(x_{0}, 1) = x_{0} \) holds. From this equation, (8) and (9) we obtain, by induction, \( g(x_{0}, g^{n}(1, 1)) = x_{0} \). Setting \( z_{n} = x_{0}/g^{n}(1, 1) \) we get, by (9), \( g(z_{n}, 1) = z_{n} \). From inequality (11) it follows that

\[
\lim_{n \to \infty} z_{n} = 0.
\]

Thus, by the continuity of \( g \), the last equation implies \( g(0, 1) = 0 \) which contradicts the definition of \( g(0, 1) = 1 \). This completes the proof of (12).

Now, we prove that for all \( x \geq 0 \)

\[(13) \quad g(x, y_{1}) > g(x, y_{2}), \quad \text{whenever } y_{1} > y_{2}.\]

If \( y_{2} = 0 \), then (13) is a consequence of (12) and the definition of \( g \).

Suppose that \( y_{2} > 0 \). Since \( g(0, y_{2}) = y_{2} \) and, by (12), \( g(y_{1}, y_{2}) > y_{1} \), we infer, by virtue of continuity of \( g \), that there exists a number \( y \) satisfying the inequality \( 0 < y < y_{1} \) for which the equation \( g(y, y_{2}) = y_{1} \) holds. Hence, taking into account (7), (8) and (12), we obtain

\[
g(x, y_{1}) = g(x, g(y, y_{2})) = g(g(x, y), y_{2})
= g(g(y, x), y_{2}) = g(y, g(x, y_{2})) = g(g(x, y_{2}), y) > g(x, y_{2})
\]

which completes the proof of (13).

Bohnenblust proved (see [1], p. 630-632) that the continuous functions \( g \) satisfying conditions (7)-(9), (13) and condition \( g(0, x) = x \) are of the form

\[
g(x, y) = (x^{a} + y^{a})^{1/a}, \quad \text{where } 0 < a < \infty.
\]
From this and from (6) it follows that for every \( x \geq 0, y \geq 0 \) the equation

\[
E_x \circ E_y = E_{(x^\alpha + y^\alpha)^{1/\alpha}},
\]

where \( \alpha \) is a positive constant, holds. Now, it is easy to verify that for convex linear combinations \( P \) and \( R \) of the measures \( E_a \) (\( a \geq 0 \)) the formula

\[
\int_0^\infty f(x)(P \circ R)(dx) = \int_0^\infty \int_0^\infty f((x^\alpha + y^\alpha)^{1/\alpha})P(dx) R(dy),
\]

where \( f \) is a bounded, continuous function on \([0, \infty)\), holds. Since the convex linear combinations of the measures \( E_a \) form a dense subset of \( \mathcal{B} \) in the sense of weak convergence, formula (14) holds for all measures \( P \) and \( R \) from \( \mathcal{B} \). In other words, the algebra in question is an \( \alpha \)-convolution algebra.

REFERENCES


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