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An example of the equation $u_t = u_{xx}+f(x, t, u)$ with distinct maximum and minimum solutions of a mixed problem

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The purpose of the present note is to give an example of an equation

$$u_t = u_{xx}+f(x, t, u) \tag{1}$$

which has two different solutions satisfying the same boundary data. The example is one for which the assumptions which ensure the existence of a maximum solution and a minimum one for (1) are satisfied, i.e. the assumptions of theorems 2 and 3 of [1]. Hence, the maximum solution is different from the minimum one. Let us remark that in th. 2 and th. 3 of [1] it is not mentioned at any place that the function $f(x, t, u)$ is "locally Lipschitz continuous" as was wrongly reported by the reviewer of [1] in Mathematical Reviews, Vol. 22, 8A, August 1961, rev. Nr 6930.

We take the function $f(x, t, u)$ defined as follows: $f(x, t, u)$ is defined for $-\pi/2 \leqslant x \leqslant \pi/2$, $0 \leqslant t \leqslant \pi/4$ and arbitrary u by the formula

$$f(x, t, u) = \begin{cases} -\sqrt{\cos^2 x-u^2}+u & \text{if} \quad |u| \leqslant |\cos x|\,, \\ u & \text{if} \quad |\cos x| \leqslant |u|\,. \end{cases}$$

The function $f(x, t, u)$ is Hölder continuous in x and u, that is for every point (x, u) $(-\pi/2 \leqslant x \leqslant \pi/2$, u arbitrary) there exist a neighbourhood of that point and a constant M and an exponent a, $0 < a < 1$, such that for every couple of points $(\bar{x}, \bar{u}), (\bar{\bar{x}}, \bar{\bar{u}})$ of that neighbourhood the inequality

$$|f(\bar{x}, t, \bar{u})-f(\bar{\bar{x}}, t, \bar{\bar{u}})| \leqslant M[|\bar{x}-\bar{\bar{x}}|^a+|\bar{u}-\bar{\bar{u}}|^a]$$

holds.

The proof of the Hölder continuity of $f(x, t, u)$ will include several cases.

a) Write $d = |f(x, t, \bar{u})-f(x, t, \bar{\bar{u}})|$ and suppose that $|\bar{u}| \leqslant |\bar{\bar{u}}| \leqslant |\cos x|$. We have

$$\begin{aligned} d &= |-\sqrt{\cos^2 x-\bar{u}^2}+\bar{u}+\sqrt{\cos^2 x-\bar{\bar{u}}^2}-\bar{\bar{u}}| \\ &\leqslant |\bar{u}-\bar{\bar{u}}|+|\sqrt{\cos^2 x-\bar{\bar{u}}^2}-\sqrt{\cos^2 x-\bar{u}^2}| \\ &\leqslant |\bar{u}-\bar{\bar{u}}|+\sqrt{\bar{\bar{u}}^2-\bar{u}^2} \leqslant |\bar{u}-\bar{\bar{u}}|+\sqrt{|\bar{u}|+|\bar{\bar{u}}|}\,|\bar{u}-\bar{\bar{u}}|^{1/2}\,. \end{aligned}$$

b) If $|\cos x| \leqslant |\bar{u}|, |\bar{\bar{u}}|$ then $d = |\bar{u} - \bar{\bar{u}}|$.

c) If $|\bar{u}| \leqslant |\cos x| \leqslant |\bar{\bar{u}}|$ then

$$d = |-\sqrt{\cos^2 x - \bar{u}^2} + \bar{u} - \bar{\bar{u}}| \leqslant \sqrt{\cos^2 x - \bar{u}^2} + |\bar{u} - \bar{\bar{u}}|$$
$$\leqslant \sqrt{\bar{\bar{u}}^2 - \bar{u}^2} + |\bar{u} - \bar{\bar{u}}| \leqslant |\bar{u} - \bar{\bar{u}}| + \sqrt{|\bar{u}| + |\bar{\bar{u}}|}\,|\bar{u} - \bar{\bar{u}}|^{1/2}.$$

The above inequalities show that for $(x, t, \bar{u}), (x, t, \bar{\bar{u}})$ belonging to the domain of f

$$(2) \qquad |f(x, t, \bar{u}) - f(x, t, \bar{\bar{u}})| \leqslant K |\bar{u} - \bar{\bar{u}}|^{1/2}$$
$$\text{for} \quad \bar{u}, \bar{\bar{u}} \in [u-\varrho, u+\varrho], \quad 0 < \varrho, \quad \varrho \text{ arbitrary},$$

where $K = \sup\{|\bar{p} - \bar{q}|^{1/2} + \sqrt{|\bar{p}| + |\bar{q}|}\}$ and sup is taken over $\bar{p}, \bar{q} \in [u-\varrho, u+\varrho]$, that is over a certain neighbourhood of u.

a') Write $d' = |f(\bar{x}, t, u) - f(\bar{\bar{x}}, t, u)|$. If $|\cos\bar{x}|, |\cos\bar{\bar{x}}| \leqslant |u|$ then $d' = 0$.

b') Suppose that $|u| \leqslant |\cos\bar{x}|, |\cos\bar{\bar{x}}|$. Then

$$d' = |-\sqrt{\cos^2\bar{x} - u^2} + u + \sqrt{\cos^2\bar{\bar{x}} - u^2} - u|$$
$$\leqslant |\sqrt{\cos^2\bar{\bar{x}} - u^2} - \sqrt{\cos^2\bar{x} - u^2}| \leqslant \sqrt{|\cos^2\bar{x} - \cos^2\bar{\bar{x}}|}$$
$$\leqslant \sqrt{|\cos\bar{x}| + |\cos\bar{\bar{x}}|}\,|\cos\bar{x} - \cos\bar{\bar{x}}|^{1/2} \leqslant \sqrt{2}\,|\bar{x} - \bar{\bar{x}}|^{1/2}.$$

c') Suppose that $|\cos\bar{x}| \leqslant |u| \leqslant |\cos\bar{\bar{x}}|$. Then $d' = |u + \sqrt{\cos^2\bar{\bar{x}} - u^2} - u| = \sqrt{\cos^2\bar{\bar{x}} - u^2}$. But $\cos^2\bar{x} \leqslant u^2$ and consequently $\cos^2\bar{\bar{x}} - u^2 \leqslant \cos^2\bar{\bar{x}} - \cos^2\bar{x}$. Hence

$$d' \leqslant \sqrt{\cos^2\bar{\bar{x}} - \cos^2\bar{x}} \leqslant \sqrt{2}\,|\bar{x} - \bar{\bar{x}}|^{1/2}.$$

We see now that $d' \leqslant \sqrt{2}\,|\bar{x} - \bar{\bar{x}}|^{1/2}$ for $(\bar{x}, t, u)$, $(\bar{\bar{x}}, t, u)$ belonging to the domain of $f(x, t, u)$.

Now take the function $u(x, t) = \cos x \cdot \cos t$. We have

$$u_t = -\cos x \sin t, \qquad u_{xx} = -\cos x \cos t.$$

Hence

$$u_t - u_{xx} = -\cos x \sin t + \cos x \cos t = -\cos x \sqrt{1 - \cos^2 t} + u$$
$$= -\sqrt{\cos^2 x - \cos^2 x \cos^2 t} + u = -\sqrt{\cos^2 x - u^2} + u.$$

Hence u satisfies (1) in $R = \big((x, t)\colon -\pi/2 \leqslant x \leqslant \pi/2,\ 0 < t \leqslant \pi/4\big)$. Take the function $v(x, t) = \cos x$. We have $v_t - v_{xx} = 0 + \cos x = -\sqrt{\cos^2 x - \cos^2 x} + v = -\sqrt{\cos^2 x - v^2} + v$. Hence v satisfies (1) in R. Obviously $u \neq v$ in R and u and v satisfy the same boundary data: $u(x, 0) = v(x, 0) = \cos x$ in $[-\pi/2, \pi/2]$ and $u(-\pi/2, t) = v(-\pi/2, t) = (\pi/2, t) = v(\pi/2, t) = 0$ for $t \in [0, \pi/4]$.

On the other hand, the function $p = 3e^t - 1$ majorizes the boundary data of u and v and $p_t > p_{xx} + f(x, t, p)$ in R because $p > \cos x$ for $x \in [-\pi/2, \pi/2]$. The function $q = -3e^t + 1$ satisfies in R the inequality $q_t < q_{xx} + f(x, t, q)$ and takes on values less than u and v on the parabolic boundary of R. Hence all assumptions of theorem 2 and theorem 3 of [1] are satisfied for (1). Therefore there exist a maximum and a minimum solution of (1), satisfying the same boundary data as the functions u and v do. But $u \neq v$. Hence the maximum solution is different from the minimum one.

Reference

[1] W. Mlak, *Parabolic differential inequalities and the Chaplighin's method*, Annales Polonici Mathematici 8 (1960), p. 139-152.

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