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ON THE EVALUATION OF FUNCTIONS ON DIGITAL COMPUTERS

1. Introduction. Mathematical tables are of great value in paper-and-pencil calculations, their traditional form is, however, uneconomical for electronic computers. First, because a table of function values needs a great part of the computer store, second, because the speed of the computer is not suitably used. Therefore many of the methods for numerical evaluation of function values do not use tables at all, but heavily depend upon the formulae used for calculation.

The aim of this paper is to give mixed methods, i.e. methods of calculation of function values on the basis of simple formulae with the use of small tables of the given function. These methods are presented here on the examples of the logarithmic and the exponential functions. Some general remarks will be given earlier.

Suppose that we have, after the use of a reduction formula, to calculate $f(x)$ for $a \leqslant x < b$. Let h be any number such that $0 < h < b-a$. Denote by

$$s_k = a+k\cdot h \qquad (k = 0, 1, 2, \ldots). \tag{1.1}$$

If for a given x we choose i such that $s_i \leqslant x < s_{i+1}$ then we obtain

$$f(x) = f\big(s_i+(x-s_i)\big) = f(s_i+y),$$

where

$$y = x-s_i, \qquad 0 \leqslant y < h.$$

After the above transformations the interval of length $b-a$ is diminished to an interval of length h and the function $f(s_k+y)$ may be expanded into an exponential series for any $a \leqslant s_k < b$.

The value of the index i is given by

$$i = \left[\frac{x-a}{h}\right], \tag{1.2}$$

where $[z]$ is the integral part of the number z. From (1.1) and (1.2) we obtain

$$y = x - s_i = x - a - h\left[\frac{x-a}{h}\right] = h\left\{\frac{x-a}{h}\right\},$$

where $\{z\}$ denotes the fractional part of the number z.

It is worth pointing out that if h is an integral power of 2 then the multiplication and division by h may be replaced by an appropriate shifting operation, usually being a fast one.

In the further part of the paper we assume $h = 2^n$, where n is integral, and the function $f(s_k + y)$ is expanded actually into a Taylor series in the neighbourhood of the point $s_k + h/2$.

2. Calculation of the values of the natural logarithm. We shall use the following series

$$\text{(2.1)} \qquad \log z = -2\left[\frac{1-z}{1+z} + \frac{1}{3}\left(\frac{1-z}{1+z}\right)^3 + \frac{1}{5}\left(\frac{1-z}{1+z}\right)^5 + \dots\right].$$

Let $x = m\cdot 2^c$, where $1/2 \leqslant m < 1$, and let $h = 2^{-4}$, $i = [2^4 m]$ and $a_i = hi + 2^{-5}$. Then

$$\log x = c\log 2 + \log m = c\log 2 + \log a_i + \log\left(1 + \frac{m - a_i}{a_i}\right).$$

An expansion of $\log(1 + m - a_i/a_i)$ into the series (2.1) yields

$$\text{(2.2)} \qquad \log\left(1 + \frac{m-a_i}{a_i}\right) = 2y + \frac{2}{3}y^3 + \frac{2}{5}y^5 + R_1,$$

where

$$\text{(2.3)} \qquad y = \frac{-1 + \left(1 + \dfrac{m-a_i}{a_i}\right)}{1 + \left(1 + \dfrac{m-a_i}{a_i}\right)} = \frac{m-a_i}{m+a_i} = \frac{\{2^4 m\} - 2^{-1}}{2^4 m + [2^4 m] + 2^{-1}}$$

and

$$|y| < 2^{-5} \quad \text{and} \quad |R_1| < \frac{2}{7}2^{-35}.$$

It is known that for the Chebyshev polynom of 5th order holds

$$|T_5(2^5 y)| \leqslant 1 \quad \text{for} \quad |y| < 2^{-5}$$

and that

$$T_5(2^5 y) = 2^{29}y^5 - 5\cdot 2^{17}y^3 + 5\cdot 2^5 y,$$

thus

$$\frac{2}{5}y^5 = 2^{-11}y^3 - 2^{-23}y + R_2,$$

where $|R_2| = \frac{2}{5}2^{-29}|T_5(2^5 y)| \leqslant \frac{2}{5}2^{-29}$.

Using the last inequality, calculating from it y^5, and substituting it into (2.2) we get

$$\log x = c\log 2 + \log a_i + (2 - 2^{-23})y + \left(\frac{2}{3} + 2^{-11}\right)y^3 + R, \tag{2.4}$$

where y is given by (2.3) and $|R| = |R_1 + R_2| < 2^{-30}$.

The method of calculating $\log x$ is therefore as follows

a) calculate $i = [2^4 m]$,

b) calculate $\log x$ from (2.4).

The numbers $\log a_i = \log \frac{2i+1}{32}$ $(i = 8, 9, \ldots, 15)$ are constants of the programme.

The programme for the Elliott-803 computer using the described method is 2,7 times faster (it calculates the logarithm in 47,5 msec) than the programme B 101 given in the Elliott library of programmes and uses 3 memory locations less.

3. Calculation of the values of the exponential function. Since $e^x = 2^{x\log_2 e}$, it is sufficient to deal with the exponential function having the base 2. Make the following notations

$$t = x\log_2 e, \qquad v = [t], \qquad u = \{t\}. \tag{3.1}$$

We have

$$e^x = 2^t = 2^v \cdot 2^u.$$

One needs only calculating 2^u, for $0 \leqslant u < 1$, because v is integer and because the multiplication of any number of the form $m \cdot 2^c$ by an integer power of 2 may be substituted by the addition of the exponent to the number c.

Let us introduce the following notations

$$i = [2^2 u], \qquad a_i = 2^{-2} i + 2^{-3},$$

$$y = u - a_i = 2^{-2}\{2^2 u\} - 2^{-3} \qquad (|y| \leqslant 2^{-3}). \tag{3.2}$$

If we expand 2^u into a Taylor series in the neighbourhood of the point a_i, and if we eliminate the term of 5th order using a Chebyshev polynom $T_5(2^3 y)$, we obtain

$$e^x = 2^v \cdot 2^{a_i}\left[1 + \left(\log 2 - \frac{2^{-19}}{3}\log^5 2\right) y + \frac{\log^2 2}{2} y^2 + \right.$$
$$\left. + \left(\frac{\log^3 2}{6} + \frac{2^{-11}}{3}\log^5 2\right) y^3 + \frac{\log^4 2}{24} y^4 + R\right], \tag{3.3}$$

where $|R| < 0{,}75 \cdot 2^{-28}$.

The method of calculating e^x is as follows:

a) calculate u, v from (3.1),

b) calculate $i = [2^2 u]$,

c) use formula (3.3), where y is given by (3.2).

The numbers $2^{a_i} = 2^{\frac{1}{8}(2i+1)}$ $(i = 0, 1, 2, 3)$ are constants of the programme.

It is easy to see that the relative error of the obtained result is independent of x if the number u is calculated with the same number of significant digits as are given in x.

The programme for the Elliott-803 computer which calculates the exponential function by this method is 1,8-2,5 times faster (the maximum time is 60,5 msec) than the library programme B 102 and uses 3 memory locations more.

4. Elliott-803 computer programmes. Here are given programmes written in the machine code (TI-code) of Elliott [1], however, for the convenience of the reader some minor changes have been made, as follows: a) capital letters (e.g. A, V) denote addresses of working locations, b) a notation like $\langle 5\rangle$ denotes the address of the constant 5.

The following subroutine calculates the natural logarithm of the floating-point number being in the accumulator. The numbers $L+i$ $(i = 8, 9, \ldots, 15)$ are addresses of the numbers $\log\frac{2i+1}{32}$.

0,	+0				
1,	42	18,	:	41	18,
2,	20	X	:	03	$\langle 511\rangle$
3,	27	X	:	05	$\langle 256\rangle$
4,	65	4096	:	63	$\langle \log 2\rangle$
5,	20	R	:	30	X
6,	03	$\langle 15\cdot 2^{34}\rangle$	:	04	$\langle 2^{33}\rangle$
7,	20	A	:	04	X
8,	51	1	:	65	4096
9,	10	A	:	27	X
10,	52	$\langle 16\rangle$	:	10	X
11,	46	15,	:	65	4096
12,	64	A	:	20	A
13,	63	A	:	40	14,
14,	63	$\langle \frac{1}{12}+2^{-14}\rangle$	:	60	$\langle 1-2^{-24}\rangle$
15,	63	A	:	60	R
16,	00	X	/	60	L
17,	00	0,	/	40	1
18,	74	2	:	40	18,

The subroutine calculating the value of the exponential function e^x, where x is a floating-point number in the accumulator, is given now.

It is advisable to give the constant $-255\log 2$ in the programme in the form $-254.9999\log 2$.

The number	denotes the adress of
$P-1$	2^0
$P+i$ $(i=0,1,2,3)$	$2^{\frac{1}{8}(2i+1)}$
A	$\frac{1}{24}\log^4 2$
B	$\frac{1}{6}\log^3 2+\frac{1}{3}2^{-11}\log^5 2$
C	$\frac{1}{2}\log^2 2$
D	$\log 2-\frac{1}{3}2^{-19}\log^5 2$

0,	+0			
1,	20	V	: 20	Y
2,	03	⟨511⟩	: 27	Y
3,	07	⟨242⟩	: 45	5,
4,	30	V	: 60	P−1
5,	40	28,	: 04	⟨10⟩
6,	41	9,	: 02	0
7,	21	I	: 36	V
8,	63	⟨$\log_2 e$⟩	: 40	22,
9,	07	⟨−4⟩	: 10	V
10,	41	29,	: 60	⟨$-255\log 2$⟩
11,	45	12,	: 74	19
12,	44	11,	: 30	Y
13,	52	⟨$\frac{1}{2}\log_2 e$⟩	: 52	⟨1⟩
14,	10	V	: 45	16,
15,	10	V	/ 54	1
16,	40	18,	: 01	0
17,	10	V	/ 50	−1
18,	16	V	: 54	2
19,	20	I	: 57	0
20,	05	⟨2^{37}⟩	: 42	26,
21,	65	4096	: 05	⟨40⟩
22,	20	Y	: 63	A
23,	60	B	: 63	Y
24,	60	C	: 63	Y
25,	60	D	: 63	Y
26,	60	P−1	: 04	V
27,	00	I	/ 63	P
28,	00	0,	/ 40	1
29,	62	⟨$-255\log 2$⟩	: 45	12,
30,	06	0,	/ 40	1

Reference

[1] J. Łukaszewicz, *Programowanie dla maszyny Elliott-803 ze szczególnym uwzględnieniem autokodu Mark 3*, PWN, Warszawa 1966.

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OBLICZANIE WARTOŚCI FUNKCJI NA MASZYNACH CYFROWYCH

STRESZCZENIE

Omawiane są metody numerycznego wyznaczania wartości funkcji uwzględniające kilka wyrazów jej rozwinięcia w szereg potęgowy oraz korzystające z niewielkich tablic tej funkcji. Szczegółowo omówiony jest logarytm i funkcja wykładnicza. Dla nich też znajdują się programy w języku wewnętrznym maszyny cyfrowej Elliott-803.

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НАХОЖДЕНИЕ ЗНАЧЕНИИ ФУНКЦИИ НА ЭЛЕКТРОННЫХ ВЫЧИСЛИТЕЛЬНЫХ МАШИНАХ

РЕЗЮМЕ

Рассматриваются методы нахождения численного значения функции учитывающие несколько членов её разложения в степенной ряд и пользующиеся небольшими таблицами этой функции. Подробно рассмотрены логарифмическая и показательная функции. Находятся тоже служащие им программы в коде электронической машины Elliott-803.
