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COUNTEREXAMPLE ON SEMIRADIAL BANACH ALGEBRAS

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Introduction. A Banach algebra $(A, \|\cdot\|)$ is called ***semiradial*** if every non-trivial closed right ideal of A contains a non-zero element a with $0 \neq |a| = \varrho_1(a)$, where $|\cdot|$ is a second norm on A defined by

$$|a| = \sup_{b \neq 0} \frac{\|ab\|}{\|b\|} \quad \text{for all } a \in A,$$

and $\varrho_1(a)$ is the spectral radius of the element a with respect to the norm $\|\cdot\|$. This concept is defined in [4] and the aim is to characterize the H^*-algebras in the sense that every semisimple H^*-algebra is semiradial and, conversely, every semiradial Hilbert algebra is isometrically isomorphic to an H^*-algebra. It is also proved in [4] that the norm $|\cdot|$ is a scalar multiple of the original norm $\|\cdot\|$ on minimal left ideals within closed two-sided ideals.

In this paper we give an example of a semiradial Banach algebra in which the scalar is not unique but depends on the choice of the minimal ideal. For the definitions and the notation used in this paper see [1] and [2].

Example. Let $X = C([0, 1])$ be the space of all continuous real-valued functions on $[0, 1]$ with the sup-norm, and let $Y_0 = l_1([0, 1])$ be the space of functions f on $[0, 1]$ such that $\sum_t |f(t)| < \infty$ with the norm

$$\|f\| = \sum_t |f(t)|, \quad f \in l_1([0, 1]).$$

Let $Y = Y_0 \oplus C$ with the norm

$$\|(f, a)\| = \|f\| + |a| \quad \text{for all } (f, a) \in Y,$$

where $f \in Y_0$ and $a \in C$.

Then it is shown in [2] that $\langle X, Y\rangle$ is a strictly norming dual pair under the duality

$$\langle x, (f, a)\rangle = \sum_t f(t)x(t) + \frac{a}{2}\int_0^1 x(t)\,dt.$$

Now consider the nuclear algebra $N\langle X, Y\rangle$ on $\langle X, Y\rangle$ defined by

$$N\langle X, Y\rangle = \{T \in B\langle X, Y\rangle \colon \nu(T) < \infty\},$$

where $\nu(T)$ is the nuclear norm of T defined as

$$\nu(T) = \inf\Big\{\sum_n \|x_n\|\,\|y_n\|\Big\}$$

and inf is taken over all sequences $\{x_n\} \in X$ and $\{y_n\} \in Y$ such that

$$Tx = \sum_{n=1}^{\infty} \langle x, y_n\rangle x_n \quad \text{for } x \in X.$$

Then it is known that $\big(N\langle X, Y\rangle, \nu(\cdot)\big)$ is a Banach algebra, and also an operator algebra. Hence by [2], Theorem 2.5, $N\langle X, Y\rangle$ is a radial Banach algebra and, by [4], it is a semiradial Banach algebra with the second norm $|\cdot|$ defined by

$$|S| = \sup_{T\neq 0} \frac{\nu(ST)}{\nu(T)}, \qquad S \in N\langle X, Y\rangle.$$

In order to find two different minimal left ideals in $N\langle X, Y\rangle$ with different multiples, we prove a series of lemmas.

LEMMA 1. *For every $x \in X$ and $y \in Y$, $|x\otimes y|$ is equal to the operator norm $\|y\otimes x\|_{\mathrm{op}}$ of the operator $y\otimes x$ of rank one.*

Proof. For $x \in X$ and $u \in Y$ we have, with $v \in X$ and $u \in Y$,

$$(1)\qquad \|y\otimes x\|_{\mathrm{op}} = \sup_{v\neq 0} \frac{\|(y\otimes x)(v)\|}{\|v\|} = \sup_{v\neq 0} \frac{\|\langle v, y\rangle x\|}{\|v\|}$$

$$= \sup_{\substack{v\neq 0\\ u\neq 0}} \frac{|\langle v, y\rangle|\,\|x\|\,\|u\|}{\|v\|\,\|u\|} = \sup_{\substack{v\neq 0\\ u\neq 0}} \frac{\|\langle v, y\rangle u\|\,\|x\|}{\|v\|\,\|u\|} = \sup_{\substack{v\neq 0\\ u\neq 0}} \frac{\|(u\otimes v)^*(y)\|\,\|x\|}{\|v\|\,\|u\|}$$

$$= \sup_{u\otimes v\neq 0} \frac{\nu\big((u\otimes v)^*(y)\otimes x\big)}{\nu(u\otimes v)} = \sup_{u\otimes v\neq 0} \frac{\nu\big((y\otimes x)(u\otimes v)\big)}{\nu(u\otimes v)} \leqslant \sup_{T\neq 0} \frac{\nu\big((y\otimes x)T\big)}{\nu(T)} = |y\otimes x|.$$

Now for every $T \in N\langle X, Y\rangle$ we obtain

$$\nu(T) = \inf \sum_n \|y_n\|\,\|x_n\| = \inf \sum_n \nu(y_n\otimes x_n)$$

with

$$Tx = \sum_{n=1}^{\infty} \langle x, y_n \rangle x_n, \quad \text{i.e.} \quad T = \sum_n y_n \otimes x_n.$$

Hence for a given $\varepsilon > 0$ there are sequences $\{x_m\}$ in X and $\{y_m\}$ in Y such that

$$\sum_m \nu(y_m \otimes x_m) \leqslant \nu(T) + \varepsilon \quad \text{and} \quad T = \sum_m y_m \otimes x_m.$$

For every $S \in N\langle X, Y\rangle$, we have

$$|S| = \sup_{T\neq 0} \frac{\nu(ST)}{\nu(T)} = \sup_{T\neq 0} \frac{\nu\big(S\sum_m y_m \otimes x_m\big)}{\nu(T)}$$

$$= \sup_{T\neq 0} \frac{\nu\big(\sum_m y_m \otimes Sx_m\big)}{\nu(T)} \leqslant \sup_{T\neq 0} \frac{\sum_m \nu(y_m \otimes Sx_m)}{\nu(T)}$$

$$= \sup_{T\neq 0} \frac{\sum_m \|y_m\| \, \|Sx_m\|}{\nu(T)} \leqslant \sup_{T\neq 0} \frac{\|S\|_{\text{op}} \sum_m \|y_m\| \, \|x_m\|}{\nu(T)}$$

$$\leqslant \|S\|_{\text{op}} \sup_{T\neq 0} \frac{\nu(T) + \varepsilon}{\nu(T)}.$$

Since ε is arbitrary, we obtain

$$|S| \leqslant \|S\|_{\text{op}} \quad \text{for all } S \in N\langle X, Y\rangle. \tag{2}$$

From (1) and (2) we get $|y\otimes x| = \|y\otimes x\|_{\text{op}}$.

As an immediate consequence of Lemma 1 we get

$$|y\otimes x| = \sup_{u\otimes v} \frac{|\langle v, y\rangle| \, \|x\|}{\|v\|}, \quad v \in X, u \in Y.$$

Now, let L be a minimal left ideal in $N\langle X, Y\rangle$. Then by [4] we have $|S| = Q_L \nu(S)$, $S \in L$, where Q_L is a constant with $0 \leqslant Q_L \leqslant 1$.

LEMMA 2. *For every $y\otimes x \in L$ with $x \in X$ and $y \in Y$,*

$$Q_L = \sup_{u\otimes v} \frac{|\langle v, y\rangle|}{\|v\| \, \|y\|}, \quad u \in Y, v \in X.$$

Proof. We have

$$Q_L = \frac{|y\otimes x|}{\nu(y\otimes x)} = \sup_{u\otimes v} \frac{|\langle v, y\rangle| \, \|x\|}{\nu(y\otimes x)\|v\|}$$

$$= \sup_{u\otimes v} \frac{|\langle v, y\rangle| \, \|x\|}{\|y\| \, \|x\| \, \|v\|} = \sup_{u\otimes v} \frac{|\langle v, y\rangle|}{\|y\| \, \|v\|}.$$

It is clear that the set $L = \{y\otimes x\}_{x\in X}$ is a minimal left ideal in $N\langle X, Y\rangle$ for every $y \in Y$. Let $y_1 = (0, 1) \in Y$. Then $\|y_1\| = 1$, and so $L_1 = \{y_1\otimes x\}_{x\in X}$ is a minimal left ideal in $N\langle X, Y\rangle$. Also, for any fixed element v_0 in X there exists an element y_2 in Y with $\|y_2\| = 1$ and $|\langle v_0, y_2\rangle| = \|v_0\|$ (since $\langle X, Y\rangle$ is a strictly norming dual pair). Let $L_2 = \{y_2\otimes x\}_{x\in X}$ be another minimal left ideal in $N\langle X, Y\rangle$. Thus $y_1\otimes x \in L_1$ and $y_2\otimes x \in L_2$ for every $x \in X$.

LEMMA 3. *We have* $Q_{L_1} = \frac{1}{2}$ *and* $Q_{L_2} = 1$.

Proof. We obtain

$$Q_{L_1} = \frac{|y_1\otimes x|}{\nu(y_1\otimes x)} = \sup_{u\otimes v} \frac{|\langle v, y_1\rangle|}{\|y_1\|\,\|v\|}$$

$$= \sup_{u\otimes v} \left|\frac{1}{2}\int_0^1 v(t)\,dt\right| \|v\|^{-1} = \frac{1}{2}\sup_{u\otimes v}\left|\int_0^1 v(t)\,dt\right| \|v\|^{-1} = \frac{1}{2}.$$

We have also

$$Q_{L_2} = \frac{|y_2\otimes x|}{\nu(y_2\otimes x)} = \sup_{u\otimes v}\frac{|\langle v, y_2\rangle|}{\|y_2\|\,\|x\|} \geqslant \frac{|\langle v_0, y_2\rangle|}{\|v_0\|},$$

and for the particular element v_0 in X we get $Q_{L_2} \geqslant 1$.

Thus, we have constructed two minimal left ideals L_1 and L_2 (within the same closed two-sided ideal) in $N\langle X, Y\rangle$ in such a way that their multiples Q_{L_1} and Q_{L_2} are not equal and the example is complete.

Conclusion. From the above example we conclude that the definition of the second norm $|\cdot|$ for a semiradial Banach algebra does not give a nice characterization of H^*-algebras and in order to avoid this case we need to define the norm $|\cdot|$ on the Banach algebra $(A, \|\cdot\|)$ as

$$|a| = \max\left\{|a|_l = \sup_{b\neq 0}\frac{\|ab\|}{\|b\|},\ |a|_r = \sup_{b\neq 0}\frac{\|ba\|}{\|b\|}\right\} \quad \text{for every } a \in A$$

in which all the results remain without change. Note that in this case the above example cannot be used as a counterexample as we show in the following.

Applying the same method as in the proof of Lemma 1, we get

$$|y\otimes x|_r = \|(y\otimes x)^*\|_{\mathrm{op}}, \quad x \in X,\ y \in Y.$$

In fact,

$$\|(y\otimes x)^*\|_{\mathrm{op}} \leqslant \nu(y\otimes x) = \|y\|\,\|x\| = \|y\|\,|\langle x, u_0\rangle|$$

for some $u_0 \in Y$ with $\|u_0\| = 1$ and $|\langle x, u_0\rangle| = \|x\|$. By the strictly norming property of $\langle X, Y\rangle$, for $u \in Y$ we obtain

$$\|y\|\,|\langle x, u_0\rangle| = \frac{\|\langle x, u_0\rangle y\|}{\|u_0\|} \leqslant \sup_{u\neq 0}\frac{\|\langle x, u\rangle y\|}{\|u\|} = \sup_{u\neq 0}\frac{\|(y\otimes x)^* u\|}{\|u\|} = \|(y\otimes x)^*\|_{\mathrm{op}}.$$

Thus $|y\otimes x|_r = \|(y\otimes x)^*\|_{\mathrm{op}} = \nu(y\otimes x)$. But $\|y\otimes x\|_{\mathrm{op}} \leqslant \|(y\otimes x)^*\|_{\mathrm{op}}$, since the pair $\langle X, Y\rangle$ is strictly norming (see [1]). Hence $|y\otimes x|_l \leqslant |y\otimes x|_r$, and so

$$|y\otimes x| = |y\otimes x|_r = \nu(y\otimes x).$$

Note that the minimal left ideals constructed above with different multiples have the same multiple in this case, and so far we have no counterexample.

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