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Timelike spaces

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INTRODUCTION

The interest in indefinite metrics derives largely from the theory of relativity. On the other hand (quoting Eddington [5, p. 22]): "Assuming that a material particle cannot travel faster than light the intervals along its track must be timelike. We ourselves are limited by material bodies and can only have direct experience of timelike intervals."

This suggests a study of purely timelike metrics, independently of the question whether they are restrictions of indefinite metrics to timelike intervals. The present paper lays the foundations for such a theory.

The principal property of a timelike space $R$ is this: $R$ is partially ordered ($x \leq y$) and a function $\varrho(x, y)$ is defined for $x \leq y$ satisfying $\varrho(x, x) = 0$, $\varrho(x, y) > 0$ for $x < y$ and the "time inequality" $\varrho(x, y) + \varrho(y, z) \leq \varrho(x, z)$ for $x < y < z$. In contrast to the metric case this must be supplemented by other requirements. The function $\varrho$ does not define a topology, which is introduced separately, there must be enough pairs $x, y$ with $x < y$, etc.

In addition, there are interesting spaces where a timelike distance can be defined only locally (they occur also in general relativity.) The basic axioms for timelike and locally timelike spaces and their consequences are given in Section 1.

Our aim is a geometric theory analogous to that of metric $G$-spaces (see [2]), which proved an adequate basis for many different types of geometric developments. The additional axioms leading to timelike and locally timelike $G$-spaces are found in Section 2. Although variations of these axioms are possible they will appear quite natural when their meaning in special cases is examined.

However, some remarks on completeness are necessary. There are two types of completeness, one concerns small and the other large distances. The former states that $\varrho(x_*, y_*) \to 0$ if $x_* \leq y_*$, $x_* \to x$, $y_* \to y$, and not $x \leq y$. The latter type proves very elusive. In the metric case the Hopf-
Rinow Theorem in the general version of Cohn-Vossen (see [4]) states that under certain conditions finite compactness, completeness and geodesic completeness are equivalent. This theorem has no analogue in the timelike case and geodesic completeness (i.e. the indefinite prolongability of a geodesic curve) emerges as the most relevant concept.

Section 2 contains, besides the axioms, the existence of segments and geodesics. Section 3 deals with basic topological properties. Unfortunately we could only in relatively few instances refer to the theory of metric G-spaces because just replacing \( \leq \) by \( \geq \) in the triangle inequality rarely produces a sufficient argument.

The absence of concrete examples beyond Lorentz spaces is a considerable handicap in the study of timelike spaces. The remainder of the paper is therefore devoted to special cases. In Section 4 we discuss products of timelike and metric spaces which include the Lorentz spaces. We then turn to the most important special class of timelike spaces, namely timelike Minkowski spaces, which furnish the local geometries in any differentiable locally timelike G-space.

More specifically, the general theory is discussed in Section 5 and mobility properties in Section 7. There are some unexpected phenomena. For example, the group of stability of a point may (in all dimensions) contain a subgroup which is transitive and abelian on the spheres about the point. There is an interesting open problem, namely the determination of all Minkowskian geometries with pairwise transitive groups of motions. This leads to very simple sounding, but unsolved, problems on projectivities of convex hypersurfaces in affine space, which are of general interest. Section 6 contains all that is known in this respect. The difficulties derive from our lack of information regarding locally compact transformation groups.

We conclude with timelike and locally timelike Hilbert geometries. The above mentioned completeness for small distances is particularly interesting in this case, because it eliminates the timelike form in favor of the locally timelike form, which contains the so-called exterior hyperbolic geometry as a special case.

The present paper is intended as a basis for further investigations. Some subjects in metric G-spaces clearly do not have timelike counterparts, for example the theory of perpendicularity, [2, Chapter II]. But others like mobility, remain and become much more challenging. Of course, there are also entirely novel problems.

Finally we point out that the study of indefinite metrics started in [4] although independent of, and quite different from, the present theory, is closely related to it in purpose.
1. TIMELIKE AND LOCALLY TIMELIKE SPACES

Preorderings, which will turn out to be partial orderings, of the space $R$, or a neighborhood $U(p)$ of a point $p$ in $R$, are essential for timelike spaces and will be denoted by $x \leq y$ or $x \leq_p y$, reserving $x < y$ and $x <_p y$ for the case $x \neq y$. We mention that by definition $x \leq x$ or $x \leq_p x$ for all $x \in R$ or $x \in U(p)$. The sets of pairs of points $x, y$ in $R \times R$ or $U(p) \times U(p)$ for which $x \leq y$, $x < y$, $x \leq_p y$, $x <_p y$ will be denoted respectively by $(\leq)$, $(<)$, $(\leq_p)$, $(<_p)$. Similarly, $(>)$ etc. consists of the pairs $x, y$ for which $y < x$, i.e. in the notation used by some authors $(>)^{-1} = (<)$.

The axioms for a timelike space are

$T_1$. $R$ is a (non-empty) Hausdorff space.

$T'_2$. A preordering $x \leq y$ is defined in $R$. The set $(<)$ is open in $R \times R$ and each neighborhood $W(q)$ of a given point $q$ contains points $x, y$ with $x < q$ and $q < y$.

$T'_3$. A continuous real valued function $\varrho(x, y)$ is defined on $(\leq)$ and satisfies

$$\varrho(x, x) = 0, \quad \varrho(x, y) > 0 \quad \text{for} \quad x < y,$$

and the "time inequality"

$$\varrho(x, y) + \varrho(y, z) \leq \varrho(x, z) \quad \text{for} \quad x < y < z.$$

The axioms for a locally timelike space arise from these by localizing $T'_2$ and $T'_3$ and adding a consistency condition.

$R$ is locally timelike if it satisfies $T_1$ and

$T_2$. Each point has a neighborhood $U(p)$ in which a preordering $x \leq_p y$ is defined. The set $(<_p)$ is open in $U(p) \times U(p)$ (or $R \times R$) and each neighborhood $W(q)$ of a given point $q \in U(p)$ contains points $x, y$ with $x <_p q$ and $q <_p y$.

$T_3$. A continuous real valued function $\varrho_p(x, y)$ is defined on $(\leq_p)$ and satisfies

$$\varrho_p(x, x) = 0, \quad \varrho_p(x, y) > 0 \quad \text{if} \quad x <_p y$$

and

$$\varrho_p(x, y) + \varrho_p(y, z) \leq \varrho_p(x, z) \quad \text{if} \quad x <_p y <_p z.$$

$T_4$. If $(<_p) \cap (<_q) \neq \emptyset$ then $(<_p) \cap (>_q) = \emptyset$ and $\varrho_p(x, y) = \varrho_q(x, y)$ for $(x, y) \in (<_p) \cap (<_q)$. If $(<_p) \cap (>_q) \neq \emptyset$ then $(<_p) \cap (<_q) = \emptyset$ and $\varrho_p(x, y) = \varrho_q(y, x)$ for $(x, y) \in (<_p) \cap (>_q)$.

A timelike space may be considered as the special case of a locally timelike space where $U(p) = U(q) = R$ and $(<_p) = (<_q)$ for any two
points $p$, $q$. This remark frequently obviates separate definitions or proofs for the timelike case which are obtained by simply omitting the subscript $p$. Where this is clear we will not discuss the timelike space explicitly. The following are two simple but useful examples

(1) \[ x \leq_p y \text{ (and } x \leq y \) are partial orderings. \]

For $x <_p y$ and $y \leq_p x$ give $x <_p x$ and $q_p(x, y) + q_p(y, x) \leq q_p(x, x) = 0$, hence $q_p(x, y) = 0$ and $x = y$.

We say that $y$ lies between $x$ and $z$ and write $(x y z)_p$ (or $(x y z)$) if $x <_p y <_p z$ and \[ q_p(x, y) + q_p(y, z) = q_p(x, z). \]

(2) \[(u x y)_p \text{ and } (u y z)_p \text{ if and only if } (u x z)_p \text{ and } (x y z)_p. \]

Let $(u x y)_p$ and $(u y z)_p$; then
\[
q'_p(u, z) = q_p(u, y) + q_p(y, z) = q'_p(u, x) + q'_p(x, y) + q_p(y, z)
\leq q'_p(u, x) + q'_p(x, z) \leq q'_p(u, z),
\]
hence $(u x z)_p$ and $(x y z)_p$. The converse is proved in the same way.

If in a locally timelike space we replace, for an arbitrary set of points $p$, the set $(\leq_p)$ by $(\geq_p)$ and define a new distance by $q'_p(x, y) = q(p, y, x)$, all axioms will be satisfied.

The space is consistently ordered if $(<_p) \cap (>_q) = \emptyset$ for any $p, q$. The space possesses a consistent ordering if by replacing, for a suitable set of points $p$, the set $(<_p)$ by $(\geq_p)$ and $q_p$ by $q'_p$, the space can be made consistently ordered. If such a change is possible, we may and will assume that the original space is consistently ordered. There are very interesting spaces which do not possess consistent orderings (see Section 8).

In a consistently ordered space we call chain $C_{xy}$ from $x$ to $y$ a set of points $u_0 = x, u_1, \ldots, u_k = y$ with $u_{i-1} \leq_{p_i} u_i$ for some $p_i$. If $C_{xy}$ exists, we write $x \leq y$. Then $x < y$ and $y \leq z$ imply $x \leq z$. We put
\[
\sigma(x, y, C_{xy}) \equiv \sum q_{p_i}(u_{i-1}, u_i).
\]

Because of $T_4$ this is independent of the choice of the $p_i$ for which $u_{i-1} \leq_{p_i} u_i$.

(3) \[ \text{In a consistently ordered locally timelike space, if } xy = \sup_{C_{xy}}(\sigma(x, y, C_{xy})) \]
is finite for all $x, y$ with $x \leq y$, then $xy$ defines a timelike space.

The proof is obvious. (3) singles out those locally timelike spaces which can be identified with timelike spaces.

A local $T$-curve in a locally timelike space is a continuous map $x(t)$ of an interval $[a, \beta]$ in $\mathcal{H}$ into a neighborhood $U(p)$ satisfying $T_2$ such that either (a) $x(t_1) < p x(t_2)$ for all $t_1 < t_2$ in $[a, \beta]$ or (b) $x(t_2) < p x(t_1)$
for all \( t_1 < t_2 \) in \([a, \beta]\). We refer to the two possibilities as cases (a) and (b) respectively. To treat them simultaneously we put

\[
x(t_1)x(t_2) = \begin{cases} 
g_p(x(t_1), x(t_2)) & \text{in case (a),} 
g_q(x(t_2), x(t_1)) & \text{in case (b).}
\end{cases}
\]

For a partition \( \Delta: a = t_0 < t_1 < \ldots < t_k = \beta \) of \([a, \beta]\) set

\[
L(x, \Delta) = \sum_{i=1}^{k} x(t_{i-1})x(t_i)
\]

and define the length \( L(x) \) of \( x(t) \) by

\[
L(x) = \inf_{\Delta} L(x, \Delta).
\]

Because of the time inequality

(4) \[ L(x) \leq L(x, \Delta) \leq x(a)x(\beta). \]

For any \( \Delta \) put \( ||\Delta|| = \max_{i} (t_{i+1} - t_i) \). Then we have as for metric spaces:

(5) If \( \Delta_n \) is a sequence of partitions of \([a, \beta]\) with \( ||\Delta_n|| \to 0 \) then \( L(x, \Delta_n) \to L(x) \).

Choose \( \Delta_u: u_0 = a < u_1 < \ldots < u_n = \beta \) such that

\[
L(x, \Delta_u) < L(x) + \varepsilon.
\]

In \( \Delta_u \) choose \( t_i^* (i = 1, \ldots, n-1) \) as the last element smaller than \( u_i \) and put \( t_0^* = a, t_n^* = \beta \). Then for large \( n \)

\[
t_0^* = a < t_1^* < \ldots < t_n^* = \beta,
\]

also \( \lim_{i \to \infty} t_i^* = u_i \).

If \( \Delta'_u \) denotes this partition of \([a, \beta]\), then

\[
L(x, \Delta'_u) \leq L(x, \Delta_u) \quad \text{and} \quad \lim_{i \to \infty} x(t_{i-1}^*)x(t_i^*) = x(u_{i-1})x(u_i).
\]

Therefore

\[
\limsup L(x, \Delta_u) \leq \lim L(x, \Delta'_u) = L(x, \Delta_u) < L(x) + \varepsilon
\]

and (5) follows from \( \liminf L(x, \Delta'_u) \geq L(x) \).

If \( a \leq t' < t'' \leq \beta \) we denote by \( L^{t,t'}_q(x) \) the length of \( x(t)|[t', t''] \) (i.e. the restriction of \( x(t) \) to \([t', t'']\)).

We conclude from (5) that length is additive:

(6) For any partition \( \Delta: a = t_0 < t_1 < \ldots < t_k = \beta \)

\[
L(x) = \sum_{i=1}^{k} L^{t_i}_{t_{i-1}}(x).
\]

(7) If \( t_* \to t_0 \) then \( L^{t_*}_{t_0}(x) \to L^{t_0}_{t_0}(x) \).
For either \( t_\nu = t_0 \) or \( t_\nu < t_0 \) and \( L^{\nu}_0(x) \leq x(t_\nu) x(t_0) \to 0 \) or \( t_\nu > t_0 \) and \( L^{\nu}_0(x) \leq x(t_\nu) x(t_0) \to 0 \). Now the additivity (6) of length yields the assertion.

A continuous map \( x(t) \) of \([\alpha, \beta]\) into a locally timelike space is a \( T\)-curve if a partition \( \alpha = t_0 < t_1 < \ldots < t_k = \beta \) exists for which each \( x(t) \mid [t_{i-1}, t_i] \) is a local \( T\)-curve. We define the length of \( x(t) \) by

\[
L(x) = \sum_{i=1}^{k} L^{\nu}_{t_{i-1}}(x)
\]

and conclude from \( T_4 \) and (6) that this definition is independent of the partition chosen.

The assertions (5), (6), (7) remain true for \( T\)-curves, but \( L(x, A) \) in (5) will, in general, be defined for large \( \nu \) only.

In timelike spaces there is, of course, no difference between local \( T\)-curves and \( T\)-curves, moreover always \( L(x) \leq \rho(x(\alpha), x(\beta)) \).

In consistently ordered spaces it suffices to consider the case (a) only. But there are interesting spaces which cannot be consistently ordered with \( T\)-curves traversing the same \( U(p) \) twice satisfying (a) the first time and (b) the second.

Almost the same proof as for lower semi-continuity of length in the metric case yields upper semi-continuity in the present case (G, p. 20)(1). (8) If \( x_\nu(t) \) \((\nu \in [\alpha, \beta], \nu = 0, 1, 2, \ldots)\) are \( T\)-curves in a locally timelike space and \( x_\nu(t) \) tends uniformly to \( x_0(t) \) then

\[
\limsup L(x_\nu) \leq L(x).
\]

In the timelike case \( x_\nu(t) \to x(t) \) for each \( t \) suffices.

The length of a \( T\)-curve with \( \alpha < \beta \) may vanish: If \( R = \mathcal{R} \) with the usual order and \( \rho(t_1, t_2) < |t_1 - t_2|^{\rho}, \sigma > 1 \) then \( R \) becomes a timelike space in which all local \( T\)-curves have length 0. Such curves correspond to non-rectifiable curves in the metric case. Therefore we define:

A \( T\)-curve \( x(t) \) is rectifiable if \( L^2_{t_1}(x) > 0 \) for \( t_1 < t_2 \). On a rectifiable curve \( x(t), t \in [\alpha, \beta] \), we can introduce arclength as parameter:

\[
s(t) = L^t_0(x), \quad s(\alpha) = 0.
\]

The function \( s(t) \) is continuous by (7) and strictly increasing by rectifiability. So the inverse \( t(s) \) is defined. Then

\[
y(s) = x(t(s)), \quad 0 \leq s \leq L(x)
\]

is the representation in terms of arclength:

\[
L^s_{t_1}(y) = s_2 - s_1, \quad \text{for} \quad s_1 < s_2.
\]

(1) Paper [2] is quoted as G.
Because of additivity it suffices to prove \( L^b_0(y) = s \) and this follows from the uniform continuity of \( t(s) \). Also, \( L(x) = L(y) \).

A local \( T \)-curve, \( x(t) \in U(p) \) for \( t \in [a, b] \) is a segment if
\[
L(x) = x(a) \times x(b) = e_p(x(a), x(b)) \quad \text{or} \quad e_p(x(b), x(a)).
\]
We have (in case (a)) for \( a < t_1 < t_2 < b \) from (4) and (6)
\[
e_p(x(a), x(b)) = L^b_{t_1}(x) + L^b_{t_2}(x) + L^b_{t_2}(x) \\
\leq e_p(x(a), x(t_1)) + e_p(x(t_1), x(t_2)) + e_p(x(t_2), x(b)) \\
\leq e_p(x(a), x(b)).
\]
Therefore
\[
L^b_{t_1} = e_p(x(t_1), x(t_2)) \quad \text{for} \quad a \leq t_1 < t_2 \leq b.
\]
This shows

(9) A segment is rectifiable and its restriction to any sub interval is a segment.

(10) A rectifiable local \( T \)-curve \( x(t) \) is a segment if and only if its arclength representation \( y(s) \) satisfies
\[
e_p(y(s_1), y(s_2)) = s_2 - s_1 \quad \text{for} \quad s_1 < s_2
\]
or
\[
e_p(y(s_1), y(s_1)) = s_2 - s_1 \quad \text{for} \quad s_1 < s_2.
\]
In the first case, if the condition is satisfied, then for any \( \Delta \)
\[
L(\Delta, y) = e_p(y(0), y(L(x))) = e_p(x(a), x(b)).
\]
The converse is contained in the preceding argument.

\( x(t) \) defined on a connected set \( \tau \) of the \( t \)-axis (which contains more than one point) is a partial geodesic, if for each \( t_0 \in \tau \) an \( \epsilon(t_0) > 0 \) exists such that \( x(t)|[t_0 - \epsilon(t_0), t_0 + \epsilon(t_0)] \cap \tau \) is a segment. Partial geodesics can be partially ordered: \( \{\tau, x(t)\} \leq \{\tau', x'(t)\} \) if \( \tau \subset \tau' \) and \( x(t) = x'(t) \) for \( t \in \tau \). Then each well ordered set \( \{\tau_i, x_i(t)\}, i \in I \) of partial geodesics has an upper bound, namely, the obvious partial geodesic defined on \( \bigcup \tau_i \). By Zorn’s Lemma each partial geodesic lies on a maximal partial geodesic. Maximal partial geodesics are called geodesics. Thus we have

(11) A partial geodesic, in particular a (proper) segment, can be extended to a geodesic.

On a partial geodesic \( \{\tau, x(t)\} \) we can introduce “arc length” as parameter: select \( t_0 \in \tau \) and any real \( a \), put \( s(t_0) = a \) and
\[
s(t) - a = L^b_0(x) \quad \text{for} \quad t > t_0, \quad a - s(t) = L^b_0(x) \quad \text{for} \quad t < t_0.
\]
Then \( t(s) \) is defined, monotone and continuous and with \( y(s) = x(t(s)), \)
\[
y(a) = x(t_0)
\]
\[
L^b_{s_2}(y) = s_2 - s_1 \quad \text{for} \quad s_1 < s_2.
\]
We will always assume that for a partial geodesic \( \{ \tau, x(t) \} \) the parameter is arc length, i.e. \( L^2_{\text{arc}}(x) = t_2 - t_1 \) for \( t_1 < t_2 \) and \( t_1, t_2 \in \tau \).

This implies for segments that arc length is parameter with a possible shift of the origin

\[
t_2 - t_1 = \varrho_p(x(t_1), x(t_2)) \quad \text{or} \quad \varrho_p(x(t_2), x(t_1))
\]

if

\[
a \leq t_1 < t_2 \leq \beta = a + x(a)x(\beta).
\]

According to our agreement for consistently ordered spaces we have:

(12) If \( x(t) \) is a geodesic in a timelike space then \( x(t') < x(t'') \) for \( t' < t'' \).

A geodesic \( \{ \tau, x(t) \} \) in a timelike space is a line if \( x(t)\| [t', t''] \) is a segment for any \( t' < t'' \) in \( \tau \).

The axioms T1-4 do not contain the existence of segments, which must be derived from additional axioms. The first step of the existence proof works, however, in any locally timelike space.

A linear set \( \lambda(q, r) = \{ \tau, x(t) \} \) is a map \( t \rightarrow x(t) \) of a subset \( \tau \) of an interval \([0, \beta]\) in some \( U(p) \) with the following properties

\[
0 \in \tau \quad \text{and} \quad x(0) = q, \quad \beta \in \tau \quad \text{and} \quad x(\beta) = r,
\]

either \( x(t_1) \leq_p x(t_2) \) for all \( t_1, t_2 \) in \( \tau \), or \( x(t_2)_p \leq x(t_1) \) for all \( t_1, t_2 \) in \( \tau \), and, with the previous notation,

\[
x(t_1)x(t_2) = t_2 - t_1 \quad \text{for} \quad 0 \leq t_1 \leq t_2 \leq \beta;
\]

which implies \( \beta = qr \).

Since \( \tau = \{ 0 \} \) implies \( q = r \) we assume \( q \neq r \).

The \( \lambda(q, r) \) can be partially ordered like partial geodesics and Zorn's Lemma yields the existence of a maximal linear set containing a given \( \lambda(q, r) \). Notice that \( \{ q, r \} \) is a linear set. Thus we have

(13) A given linear set \( \lambda(q, r) \), in particular, \( \{ q, r \} \) with \( q < r \) in \( U(p) \), lies in a maximal linear set \( \mu(q, r) \).

So far we only used the existence of the partial ordering \( (\leq_p) \) and not the additional properties postulated in T2' or T2. They serve the purpose of making the theory non-trivial. To see this we consider two examples where \( R \) is the \((x_1, x_2)\)-plane. First we define \( x < y \) by \( x_1 < y_1 \) and \( x_2 = y_2 \); we put \( q(x, y) = y_1 - x_1 \). Then all axioms are satisfied except that \( (\prec) \) is not open in \( R \times R \).

Next define \( x < y \) by \( x_1 + 1 < y_1 \) and \( |x_2 - y_2| \leq |x_1 - y_1| \) and define

\[
q_1(x, y) = y_1 - x_1 - |y_2 - x_2|;
\]

then not each \( W(q) \) contains points \( x, y \) with \( x < q < y \).
However, defining $x \prec y$ by $x_1 \prec y_1$ and $|x_2 - y_2| \leq |x_1 - y_1|$ and using $g_1(x, y)$, the axioms $T_1$, $T_2$, $T_3'$ are satisfied. We deduce from $T_2$ or $T_2'$ (with $U(p) = R$ in the latter case)

(14) Given $W(p) \subset U(p)$ then $W'(p) \subset W(p)$ and points $u, v$ in $W(p)$ exist such that $u \prec u, x \prec v$ for every $x \in W'(p)$.

For $u, v$ with $u \prec u, p \prec v$ exist and the openness of $(\prec, p)$ guarantees that $u \prec x \prec v$ for $x$ in a suitable $W'(p)$.

The most important implication is that any two points $x, y$ in $W'(p)$ have a common predecessor $u$, and a common successor $v$.

We observe that in the $(x_1, x_2)$-plane with the metric $g_1$ and the second definition of $x \prec y$ the set $[\tau, x(t)]$ with $\tau = [0, 2]$ and

$$x(t) = (t, 0) \text{ for } 0 \leq t \leq 1, \quad x(t) = (t + 1, 1) \text{ for } 1 \leq t \leq 2$$

is a maximal linear set $\mu[0, 0, (3, 1)]$ for which $x(t)$ is not continuous. although $x(t) \to t$ is an isometry, i.e. $d(x(t_1), x(t_2)) = |t_1 - t_2|$ for $t_1 < t_2$.

2. SEGMENTS AND GEODESICS IN TIMELIKE G-SPACES

We now list our additional axioms. Those for a timelike $G$-space are, besides $T_1$, $T_2'$, $T_3'$,

$G_1$. The space, $R$, is locally compact, connected and has a countable base.

$G_2'$. If $x \prec y$, $x \to x$, $y \to y$, and not $x \preceq y$ then $g(x_r, y_r) \to 0$.

$G_3'$. If $x \to q$, $z \to q$ and $(x_r, y_r, z_r)$ then $y_r \to q$.

$G_4'$. If $q < r$ then points with $(q, x, r)$ exist and the closure of all $x$ with this property is compact.

$G_5'$. Each point has a neighborhood $V(p)$ such that for $x, y$ in $V(p)$ with $x \prec y$ points $u$ and $z$ with $(u, x, y, z)$ exist.

$G_6'$. If $(u_1, x, y)$, $(u_2, x, y)$ and $g(u_1, x) = g(u_2, x)$ then $u_1 = u_2$.

If $(x, y, z_1)$, $(x, y, z_2)$ and $g(y, z_1) = g(y, z_2)$ then $z_1 = z_2$.

The axioms for a locally timelike $G$-space $R$ are $T_{1-4}$, $G_1$ and

$G_2$. If $x \prec y$, $x \to x \in U(p)$, $y \to y \in U(p)$ and not $x \preceq y$ then $g(x, y) \to 0$.

$G_3$. If $x \to q \in U(p)$, $z \to q \in U(p)$ and $(x, y, z)_p$ then $y \to q$.

$G_4$. There is a neighborhood $U'(p) \subset U(p)$ such that for $q, r$ in $U'(p)$ and $q \prec r$ points $x$ with $(q, x, r)_p$ exist.

$G_5$. There is a neighborhood $V(p) \subset U(p)$ such that for $x, y$ in $V(p)$ and $x \prec y$ a point $u$ in a given neighborhood of $x$ with $(u, x, y)_p$ and a point $z$ in a given neighborhood of $y$ with $(x, y, z)_p$ exist.
If \((u_1, x, y)_p\), \((u_2, x, y)_p\) and \(q_p(u_1, x) = q_p(u_2, x)\) then \(u_1 = u_2\).
If \((x, y, z_1)_p\), \((x, y, z_2)_p\) and \(q_p(y, z_1) = q_p(y, z_2)\) then \(z_1 = z_2\).

The two sets of axioms are strict analogues except for \(G_4\), \(G'_4\) and \(G_5, G'_5\). In the latter case we could omit the condition “in a given neighborhood of \(x\) (or \(y\))” because the considerations of this section will show that after shrinking \(U'(p)\) and \(U(p)\) the axiom \(G_5\) will be satisfied with the existence of neighborhoods of \(x\) and \(y\) when postulated without this existence. We chose the given formulation for emphasis.

As to \(G_4\) and \(G'_4\), a local form of the second part of \(G_4\) or \(G'_4\) follows from \(G_2\) or \(G'_2\) and this suffices for locally timelike \(G\)-spaces.

1. Given \(W(p)\) with compact \(\overline{W(p)} \subset U(p)\), there is a \(W_1(p) \subset W(p)\) such that \(q, r\) in \(W_1(p)\) and \((q, x, r)_p\) imply \(x \in \overline{W(p)}\).

\((U(p) = R\) in the timelike case. We will not mention this modification every time.)

If (1) were false we could find a sequence of point triples \(q_v, r_v, x_v\) in \(W(p)\) with \((q_v, x_v, r_v)_p\), \(q_v \rightarrow p\), \(r_v \rightarrow p\) and \(x_v \notin W(p)\). This contradicts \(G_3\) or \(G'_3\). The compactness of \(W(p)\) is not used, but this is the most useful case and exhibits the analogy to \(G_4\). The axioms \(G_2, G'_2\) are important completeness requirements which have no counterparts in the metric case.

In the following considerations, in as far as they are local, we will always assume that \(qr = q_p(q, r)\) and not \(qr = q_p(r, q)\).

\(G_6\) and \(G'_6\) play a greater role in timelike spaces than the corresponding axioms for metric \(G\)-spaces. To understand this fully we prove

2. If \(T_{1-4}\) and \(G_{1-4}\) hold and the \(\overline{W(p)}\) of (1) lies in the \(U'(p)\) of \(G_4\) then for a maximal linear set \(\mu(q, r) = \{\tau, x(t)\}\) with \(q, r\) in \(W_1(p)\) the set \(\tau\) is the interval \([0, qr]\).

In the timelike case \(T_1, T_2, T_3\) and \(G_1, G_{2-4}\) imply the same for any maximal linear set.

However, \(x(t)\) need not be continuous.

First we deduce from \(T_{1-4}\) and \(G_{1-3}\) that \(\tau\) is closed for \(q, r\) in \(W_1(p)\). It follows from (1) that \(x(t) \in \overline{W(p)}\). Let \(t_0 \epsilon \tau\) and \(t_0 \rightarrow t_0\). We want to show \(t_0 \epsilon \tau\) and may assume \(0 \leq t_0 < qr\). Because of (2) a subsequence \(\{x(t_k)\}\) of \(\{x(t)\}\) tends to a point \(x_0\).

Let \(0 \leq t' < t_0 < t' < qr\) where \(t', t''\) lies in \(\tau\). Then

\[ q_p(x(t'), x(t_k)) \rightarrow t_0 - t' > 0, \quad q_p(x(t_k), x(t'')) \rightarrow t'' - t_0 > 0 \]

hence by \(G_2\)

\[ x(t') < x_0 \quad \text{and} \quad q_p(x(t'), x_0) = t_0 - t', \]

\[ x_0 < x(t'') \quad \text{and} \quad q_p(x_0, x(t'')) = t'' - t_0. \]
Therefore, if \( \mu(q, r) \) did not contain a point corresponding to \( t_0 \) the set 
\( \mu(q, r) \cap x_0 \) would still be linear contradicting maximality. The proof
for the timelike case uses the second part of \( G_4^\prime \) instead of (1).

Assume now that \( \tau \neq [0, qr] \). Then \( a, \beta \) in \( \tau \) with \((a, \beta) \cap \tau = \emptyset \)
would exist since \( \tau \) is closed. By \( G_4 \) there is an \( x_0 \) with \((x(a), x_0, x(\beta))_p \)
since \( x(a), x(\beta) \) lie in \( W(p) \subset U'(p) \). We deduce from (1.2) that
\[ (x(t'), x_0, x(t''))_p \quad \text{for} \quad 0 \leq t' \leq a \quad \text{and} \quad \beta \leq t'' \leq qr. \]

In particular \((q, x_0, r)_p \) and \( x_0 \in W(p) \). Therefore \( \mu(q, r) \cap x_0 \) would
still be linear.

The distance \( q_1(x, y) \) of Section 1 satisfies \( T_1, T_2', T_3' \) and \( G_1, G_2', G_4' \)
(the latter is proved under more general hypotheses in Section 4) and we noticed that \( \mu(q, r) \) with noncontinuous \( x(t) \) exist.

We now show

(3) If \( G_3 \) or \( G_5 \) hold in addition to the hypothesis of (2) then \( x(t) \) is con-
tinuous, hence \( \mu(q, r) \) is a segment.

We must show \( x(t_*) \to x(t_0) \) for \( t_* \to t_0 \) and \( t_* < \tau \). We assume \( t_0 > 0 \),
the case \( t_0 = 0 \) is treated like \( t_0 = qr \). Since each subsequence of \( \{x(t_*)\} \)
has an accumulation point, it suffices to show that \( x(t_*) \to x_0 \) implies \( x_0 = x(t_0) \). As in the proof (2) we find for \( 0 < t' < t_0 \) that
\[ x(t') < x(t_0) \quad \text{and} \quad e_p(x(t'), x_0) = t_0 - t' = e_p(x(t'), x(t_0)). \]

Then \((q, x(t'), x(t_0))_p \) and \((q, x(t'), x_0)_p \) and \( G_4 \) gives \( x(t_0) = x_0 \).

We will denote a segment from \( q \) to \( r \) by \( \sigma(q, r) \) in the timelike case
and \( \sigma_p(q, r) \) in the locally timelike case.

\( T_1' \) and \( G_4 \) imply

(4) There is at most one segment \( \sigma_p(q, r) \) if a point \( u \) with \((u, q, r)_p \) or a point \( r \n
with \((q, r, r)_p \) exists.

For let \((u, q, r)_p \) and assume two distinct segments \( x(t), x'(t) \) \( (0 \leq t \leq e_p(q, r)) \) from \( q \) to \( r \) exist. Then \( x(t_0) \neq x'(t_0) \) for some \( t_0 \) with \( 0 < t_0 < e_p(q, r) \) and \( e_p(q, x(t_0)) = e_p(q, x'(t_0)) = t_0 \). On the other hand (1.2)
yields \((u, q, x(t_0))_p \) and \((u, q, x'(t_0))_p \) contradicting \( G_5 \).

We combine our results to obtain the following important facts:

(5) Theorem. In a timelike \( G \)-space a segment \( \sigma(q, r) \) exists whenever \( q < r \).
It is unique when either \( u \) with \((u, q, r)_p \) or \( v \) with \((q, r, v)_p \) exists. In par-
ticular, \( \sigma(q, r) \) is unique when \( q, r \) lie in a \( V(p) \) of \( G_5 \).

(6) Theorem. In a locally timelike space there is at most one segment \( \sigma_p(q, r) \) when \( u \) with \((u, q, r)_p \) or \( v \) with \((q, r, v)_p \) exists, in particular when \( q, r \) lie in the \( V(p) \) of \( G_5 \).

Each point has a neighborhood \( W_1(p) \) such that \( \sigma_p(q, r) \) exists for \( q, r \) with \( q < r \) in \( W_1(p) \) and all these \( \sigma_p(q, r) \) lie in a compact set \( \bar{W} \).
It should be mentioned that the use of Zorn's Lemma could have been avoided in the existence proof of segments and geodesics, but the arguments would have been longer.

(7) Theorem. In a timelike or locally timelike $G$-space the extension of a partial geodesic (in particular of a proper segment) to a geodesic $\{\tau, x(t)\}$ is unique and $\tau$ is open.

The uniqueness follows from $G_6$, $G'_6$ and the openness of $\tau$ from $G_5$, $G'_5$.

This implies that a geodesic $\{\tau, x(t)\}$ cannot traverse the same segment in opposite direction (i.e. that $[\alpha, \beta]$ and $[\beta', \alpha']$ in $\tau$ exist with $\beta - \alpha = \alpha' - \beta' > 0$ and $x(a + t) = x(a' - t)$ for $0 \leq t \leq \beta - \alpha$), although allowing reversal of the local partial ordering might seem to make this possible. A geodesic may, of course, be closed and traverse one segment infinitely often in the same direction.

We cannot conclude that for a geodesic $\tau$ is the entire real axis, because our axioms do not contain a completeness requirement corresponding to finite compactness postulated for metric $G$-spaces. The reason for omitting such a postulate will be discussed in the next section.

We now prove various facts concerning the extent of segments and convergence of geodesics.

(8) If $x(t)$ is a geodesic and $x(t) | [\alpha, \beta] \subset V(p) \cap W_1(p)$ then it is a segment ($W_1(p) = R$ in the timelike case).

Assume this is not true. Then there is a $t' \epsilon (a, b)$ such that $x(t) | [a, t']$ is a segment and $x(t) | [a, t]$ is not for $t > t'$. For if $g_\alpha(x(a), x(t_1)) > t_1 - a$ and $t_2 > t_1$ then

$$g_\alpha(x(a), x(t_2)) > g_\alpha(x(a), x(t_1)) + g_\alpha(x(t_1), x(t_2)) > t_1 - a + t_2 - t_1.$$ 

Since $x(a), x(t') \epsilon V(p)$, there is a $v$ with $(x(a), x(t'), v) \epsilon W_1(p)$ and hence a segment $\sigma_\alpha(x(t'), v)$. By (1.2) the segment $x(t) | [a, t']$ continued by $\sigma_\alpha(a(t'), v)$ would be a segment and lead to a geodesic providing a second continuation of $x(t) | [a, t']$ to a geodesic.

(9) Corollary. In a timelike space each geodesic is a line if and only if for given $x < y$ points $u, v$ with $(u \ x \ y)$ and $(x \ y \ v)$ exist.

The sufficiency follows from (8) and the necessity is obvious.

(10) Let $x, y \epsilon W''(p)$ and $x < y$ where $W''(p) \subset \overline{W''(p)} \subset V(p) \cap W_1(p)$. Then $\sigma_\alpha(x, y) \subset \sigma_\alpha(u, v)$ with $u < p x < p y < p v$, $\sigma_\alpha(u, v) \subset W''(p) \cap u \smallsetminus v$ and $u, v \epsilon \overline{W''(p)} - W''(p)$.

We consider the geodesic $x(t)$ for which $x(t) | [0, \ g_\alpha(x, y)]$ is $\sigma_\alpha(x, y)$.

If $(x \ y \ z)_p$ and $z \epsilon W_1(p)$ then $\sigma_\alpha(y, z)$ is $x(t) | [t_0, \ g_\alpha(x(t), x(t_0)) \ g_\alpha(x(t), z)]$. Therefore, if $t_0 = \sup t$ where $x(t) | [a, t] \subset W''(p)$, it follows that $x(t_0) \epsilon \overline{W''(p)} - W''(p)$. 

Similarly for \( t_1 = \inf t \) with \( x(t) : [t, t_0] \to W''(p) \) then \( x(t_1) \in \overline{W''(p)} - W''(p) \) and \( x(t) : [t_1, t_0] \) is a segment by (8).

(11) For a given geodesic \( \{\tau_0, x_0(t)\} \) and \( t_0 \in \tau_0 \) there is an \( \eta > 0 \) with the following property:

if \( \{\tau_v, x_v(t)\} (v = 1, 2, \ldots) \) are geodesics, \([a, t_0 - \varepsilon] \subset \bigcap_{r=0}^{\infty} \tau_r (a < t_0 - \varepsilon, 0 < \varepsilon \leq \eta)\), and \( x_v(t) \to x_0(t) \) for \( t \in [a, t_0 - \varepsilon] \), then \([a, t_0 + \varepsilon] \subset \tau_0 \bigcap_{r=0}^{\infty} \tau_r\) for a suitable \( N \) and \( x_v(t) \to x_0(t) \) for \( t \in [a, t_0 + \varepsilon] \).

Put \( p = x(t_0) \) and choose \( \eta > 0 \) such that \( x_0(t) : [t_0 - 2\eta, t_0 + 2\eta] \subset W''(p) \). Let \( \alpha' = \min(a, t_0 - 2\varepsilon) \), \( q_v = x_v(\alpha') \). For large \( v \) the points \( q_v \), \( x_v(t_0 - \varepsilon) \) lie in \( W''(p) \). By (10) there is a segment \( \sigma_\rho(q_v, r_v) \) containing \( x_v(t) : [\alpha', t_0 - \varepsilon] \) which lies in \( W''(p) \) except for \( r_v \in \overline{W''(p)} - W''(p) \), moreover \( \sigma_\rho(q_v, r_v) \) is \( x_v(t) : [\alpha', \alpha' + q_\rho(q_v, r_v)] \). Any accumulation point of \( \{r_v\} \) lies in \( \overline{W''(p)} - W''(p) \).

An accumulation point of \( y_v \in \sigma_r \) must lie on \( x_0(t) \) because of the uniqueness of prolongation. The continuity of the distance \( \rho(p, y) \) now shows that \( x_v(t) \) is defined for \( t \in [a, t_0 + \varepsilon] \) and large \( v \) and also that \( x_v(t) \to x_0(t) \) in this interval.

(12) Theorem. Let \( \{\tau_v, x_v(t)\} (v = 0, 1, 2 \ldots) \) be geodesics and \([a, \beta] \subset \bigcap_{r=0}^{\infty} \tau_r (a < \beta)\), moreover \( x_v(t) \to x_0(t) \) for \( t \in [a, \beta] \). Let \( \tau^* \) be the set of those \( t \) which lie in all but a finite number of \( \tau_r \). Then \( \tau^* \supset \tau_0 \) and \( x_v(t) \to x_0(t) \) for \( t \in \tau_0 \). The convergence is uniform on any \([t_1, t_2] \subset \tau_0 \).

Put \( \tau^\prime = \tau^* \cap [a, \infty) \) and assume \( \tau^\prime \not\subset \tau_0 \). Then \( t_0 \in \tau_0 \) exists such that \( \tau^\prime \cap [t > t_0] = \emptyset \) but \( \tau^* \supset [a, t_0] \). For each \( \varepsilon > 0 \) there is an \( N(\varepsilon) \) with \( \tau_r \supset [a, t_0 - \varepsilon] \) for \( r \geq N(\varepsilon) \). Choose \( \varepsilon \) such that (11) is applicable. Then (11) would give \( \tau_v \supset [a, t_0 + \varepsilon] \) for large \( v \), so \( \tau^* \supset [a, t_0 + \varepsilon] \) which is impossible. With a similar argument for \( \tau^\prime = \tau^* \cap (-\infty, \beta] \) this shows \( \tau^* \supset \tau_0 \).

Convergence on \( \tau_0 \) also follows from (11) and uniform convergence in closed intervals is equivalent to the easily proved statement \( x_v(t_r) \to x_0(t_0) \) for \( t_r \to t_0 \in \tau^* \).

A partial geodesic \( \{\tau, x(t)\} \) in a timelike space is a progressing (receding) ray if \( \tau = [a, \infty) \) (\( \tau = (\infty, a] \)), \( x(t_1) < x(t_2) \) for \( t_1 < t_2 \) and

\[ \rho(x(t_1), x(t_2)) = t_2 - t_1 \]  for  \( t_1 < t_2 \).

Assume \([a, \infty), x(t)\) and \([\beta, \infty), y(t)\) are two progressing rays for which \( y(\beta) = x(\alpha') \) with \( \alpha' > a \).

If \( y(t) \) is a subray of \( x(t) \), i.e., \( y(t + \beta) = x(t + \alpha') \), we can evidently find sequences \( t_r \to \infty, t'_r \to \infty \) such that \( x(t_r) > y(t'_r) \) and, writing \( \alpha^b \)
instead of \( g(a, b) = x(t, y(t')) \to 0 \). In metric \( G \)-spaces the existence of such sequences is also sufficient for \( y(t + \beta) = x(t + a') \) (see G, p. 137; \( a' > a \) is essential). One verifies easily in the Lorentz plane (see Section 4) that in timelike spaces this is not correct even if we require additionally that a sequence \( t''_r \to \infty \) with \( y(t'_r) < x(t'_r) \) and \( y(t'_r) x(t'_r) \to \infty \) exists.

So it looks as though the metric theorem has no timelike analogue; however, it does, and the analogue leads to a generalization of the result in the metric case.

(13) **Theorem.** Let \( \{[a, \infty), x(t)\} \) and \( \{[\beta, \infty), y(t)\} \) be progressing rays with \( y(\beta) = x(a'), a' > a \). If there are sequences \( t_r, t'_r, t''_r \) tending to \( \infty \) such that with \( x(t_r) = x_r, y(t'_r) = y'_r \) etc. either

\[
\begin{align*}
x_r < y'_r &< x'_r \quad \text{and} \quad x_r x'_r - x'_r y'_r - y'_r x''_r \to 0 \\
or
\end{align*}
\]

then \( y(t + \beta) = x(t + a') \) for \( t \geq 0 \).

There is, of course, an analogous theorem for receding rays.

For an indirect proof assume \( z = y(\gamma + \beta) \neq x(\gamma + a') \) for some \( \gamma > 0 \) and put \( x(a) = x, x(a') = y(\beta) = y \). Then \( xz - xy - yz = \delta > 0 \).

In the first case we have for large \( r \)

\[
\begin{align*}
xx''_r &= xy + yx_r + x_r x''_r < xy + y' + x'_r x''_r \\
&= xy + yz + y'_r + x''_r - x'_r y'_r,
\end{align*}
\]

hence

\[
xx''_r = xz + yz'_r + y'_r x''_r - \delta + (x'_r x''_r - x'_r y'_r - y'_r x''_r).
\]

This would yield \( xx''_r < xz + yz'_r + y'_r y''_r \) for large \( r \).

In the second case, for large \( r \),

\[
xy'_r < xx'_r - y'_r x'_r = xy + yx'_r - y'_r x'_r < xy + yz + yz'_r - y'_r x'_r - y''_r x''_r,
\]

hence

\[
xy'_r < xz + yz'_r - \delta + (y''_r x''_r - y'_r x'_r - y''_r x''_r)
\]

and so \( xy'_r < xz + yz'_r \) for large \( r \).

The same argument with signs reversed yields for metric \( G \)-spaces that \( y(t + \beta) = x(t + a') \) if either \( x'_r y'_r + y'_r x''_r - x'_r x''_r \to 0 \) or \( y'_r x'_r + x'_r y''_r - y'_r x''_r \to 0 \). This contains the statement that \( y(t + \beta) = x(t + a') \) if \( x'_r y'_r \to 0 \) as the special case \( x'_r = x''_r \) or \( y'_r = y''_r \) which does not make sense in the timelike case.
3. TOPOLOGICAL PROPERTIES. COMPLETENESS

A line element $L_p$ at a point $p$ of a locally timelike space is a maximal set of segments $\sigma_p(q, r)$ such that: each of these $\sigma_p(q, r)$ contains $p$ as relative interior point and any two have a further common point. It follows from (2.4) that the intersection of any two segments in $L_p$ is an element of $L_p$. Each $\sigma_p(q, r)$ with $q <_p p <_p r$ lies in exactly one $L_p$.

A geodesic $H$ which contains (in an obvious sense) one segment in $L_p$ contains all. Therefore we call $L_p$ a line element of $H$ at $p$. The cardinal number of distinct line elements of $H$ at $p$ is the multiplicity of $H$ at $p$. The point $p$ is a simple point of $H$ if its multiplicity is 1, otherwise it is a multiple point. $H$ is simple if it has no multiple points. As for $G$-spaces (G, pp. 44-45) one proves

1. The multiplicity of a geodesic at a point is finite or countable. A geodesic has an at most countable number of multiple points.

2. The geodesic $\{\tau, x(t)\}$ is simple if and only if either $x(t_1) \neq x(t_2)$ for $t_1 \neq t_2$ or if $x(t_1) = x(t_2)$ for $t_1 = t_2$ implies $\tau = \mathcal{R}$ and $x(t_1 + t) = x(t_2 + t)$ for all $t$.

It follows from (2.5.6) and $T_2', T_2$ that a (locally) timelike $G$-space $R$ contains with each point $p$ segments containing this point, therefore, $\dim_p R \geq 1$.

Whether a locally timelike $G$-space is always a topological manifold is not known. This is not even known for metric $G$-spaces. In all interesting special cases it will be a manifold. Nevertheless there is a certain interest in the topological properties which can be inferred from the axioms. The presently best way for metric $G$-spaces is not that of $G$, but proving that the space is locally homogeneous in the sense of Montgomery, see [4, Theorem (3.2)]. This proof cannot be carried over to timelike $G$-spaces since it uses in an essential way, that there are unique segments connecting any two points in a suitable neighborhood of a given point. However, the approach taken in $G$ can be carried over with some modifications. We will briefly indicate, mostly without complete proofs, how this is done.

In the first place it follows from the remark after (1.14), (2.5.6) and $G_1$ that

3. A (locally) timelike $G$-space is arcwise and locally arcwise connected.

Next one observes the following application of (2.6.12).

4. LEMMA. Every point $p$ of a (locally) timelike $G$-space has a neighborhood $W_2(p)$ such that $\sigma_p(q, r)$ exists and is unique for $q < r$ in $W_2(p)$. If $q_r, r_r$ ($r = 0, 1, 2, \ldots$) lie in $W_2(p), q_r < r_r, q_r \to q_0, r_r \to r_0$ and $x_r(t)$ represents $\sigma_p(q_r, r_r)$ with $x_r(0) = q_r$ then $x_r(t_0) \to x_0(t_0)$ for $t_r \to t_0$. 
This is used to prove

\((1')\) If two distinct line elements \(L^1_p, L^2_p\) exist at \(p\), then \(\dim_p R \geq 2\).

(The proof produces a set homeomorphic to a 2-simplex with \(p\) corresponding to a vertex.)

If \(\sigma^2_p \in L^2_p\). Let \(\sigma_p(p, q) \subset \sigma^1_p\) and \(\sigma_p(p, r) \subset \sigma^2_p\) consist of those points on \(\sigma^k_p\) for which \(p \leq x\). Using \(T_2\) and the results of the preceding section we can find \(q'\) with \((p q')_p\) and \(r'\) with \((p r')_p\) such that \(q' < r'\) and \(\sigma_p(q', r')\) as well as all segments \(\sigma_p(p, y)\) with \(y \in \sigma_p(q', r')\) lie in the \(W_2(p)\) of (4). It then follows from \((1')\) that \(\bigcup_{p} \sigma_p(p, y)\) is homeomorphic to a 2-simplex.

\((5)\) A locally timelike G-space of dimension one consists of one simple geodesic. A one-dimensional timelike G-space consists of one line.

Because of (3) no geodesic can have multiple points, and there si at least one geodesic \(\{r, x(t)\}\). We want to show that any point \(v\) lies on \(x(t)\). Let \(t_0 \in r\) and \(q = x(t_0)\). Using the remark after (1.14) we find points \(u_0 = q, u_1, \ldots, u_k = v\) such that for each \(i = 0, \ldots, k-1\) either a segment \(\sigma_{u_i}(u_i, u_{i+1})\) or segments \(\sigma_{u_i}(u_i, v)\) and \(\sigma_{u_i}(u_{i+1}, v)\) exist. Applying \((1')\) repeatedly, we see that all these segments must lie on \(x(t)\).

The same argument slightly refined yields:

\((6)\) If \(\dim_p R = 1\) for some \(p\), then \(\dim_p R = 1\) for all \(p\).

The second part of (5) is seen as follows: if \(R\) is timelike, then \(x(t_1) < x(t_2)\) for \(t_1 < t_2\) by (1.12) and a segment \(\sigma(x(t_1), x(t_2))\) exists. \(x(t)|_{[t_1, t_2]}\) is the only arc in \(R\) from \(x(t_1)\) to \(x(t_2)\) and hence is a segment.

\((7)\) Theorem. A two dimensional locally timelike G-space is a manifold.

For simplicity we take a timelike G-space in which all geodesics are lines. The argument is the same in the general case using the uniformity expressed in (2.10) for localization.

Because \(\dim_p R = 2\) for all \(p\) by (6) there are two distinct line elements at each point \(p\). There are points \(u, v, r, s\) with \((u v v), (r p s), u < r < s\) but not \((u r s)\). We prolong each \(\sigma(u, x), x \in \sigma(r, s)\) to a \(\sigma(u, x^*\) so that the \(x^*\) form an arc \(R\). It suffices to show that any sufficiently small neighborhood \(U\) of \(p\) is contained in \(\bigcup_{x^*} \sigma(u, x^*) = V^*\).

If this were not correct we could, possibly by shrinking \(\sigma(r, s)\), produce the following situation. \(u^*\) is close to \(u\) not in \(V^*\), \((p u^* e)\) and \(y < e\) for \(y \in \bigcup_{x \in \sigma(r, s)} \sigma(p, x) = V\).

Put \(2\beta = \min_{x \in r} c > 0\) and let \(x\) be the map of \(V\) in \(\bigcup_{x \in \sigma(x, c)} \) which maps \(x\) on \(x^*\) with \((x x' c)\) and \(x' c = \beta\). Put \(V^* = \pi V\).
We show \( \dim V' \geq 2 \) following the method of [G, p. 53] where references to the facts used are found. It suffices to prove that any set \( F' \subset V' \) which separates \( \sigma' = \pi \sigma(r, s) \) from \( u' = \pi u \) has at least dimension 1. The set \( \pi^{-1} F' = F \) is closed and separates \( \sigma(r, s) \) from \( p \), so does a continuum \( F_0 \) in \( F \), hence \( F_0 \) contains a point \( r_0^* \) of \( \sigma(p, r) \) and a point \( s_0^* \) of \( \sigma(p, s) \). Then \( F'_0 = \pi F_0 \subset F' \), moreover \( \pi r_0^* , \pi s_0^* \) lie in \( F'_0 \) and are distinct. Since \( F'_0 \) is a continuum, \( \dim F'_0 \geq 1 \). So \( \dim V' \geq 2 \).

Now \( \bigcup_{x' \in V'} \sigma(x', e) \) contains the product of \( V' \) and a segment, and has at least dimensions 3.

For our next topic, completeness, as well as for later purposes, we need some auxiliary facts.

In a timelike space \( R \) the set of those points whose distance from a given point exceeds a certain number, plays in many respects the rôle of open ball in metric spaces. We, therefore, introduce the notation \( (\sigma \geq 0) \)

\[
F(x, \sigma) = \{ y: x < y \text{ and } xy > \sigma \},
\]

\[
P(x, \sigma) = \{ u: u < x \text{ and } ux > \sigma \}.
\]

We put \( F(x, 0) = F(x), P(x, 0) = P(x) \). These sets consist respectively of all points which follow or precede \( x \) and are called the future and the past of \( x \). The closures of \( F(x, \sigma), P(x, \sigma), F(x), P(x) \) are denoted by \( \overline{F}(x, \sigma), \overline{P}(x, \sigma), \overline{F}(x), \overline{P}(x) \). We prove:

(8) In a timelike \( G \)-space (for \( \sigma \geq 0 \)) if \( x < y \) then \( F(x, xy + \sigma) = \overline{F}(y, \sigma) - y \); if \( u < x \) then \( P(x, ux + \sigma) = \overline{P}(u, \sigma) - u \).

For let \( z \in \overline{F}(y, \sigma) - y \). For \( z \in \overline{F}(y, \sigma) \) the assertion follows from \( xz \geq xy + yz > xy + \sigma \).

Let \( z \in \overline{F}(y, \sigma) \) and \( z \neq z \). From \( xz \geq xy + yz > xy + \sigma \) follow \( x < z \) and \( xz \geq xy + \sigma \).

Choose \( w \) with \( (x \ w \ y) \) so close to \( x \) that \( w < z \) and \( w < z \) for large \( v \). This is possible because \( (x) \) is open. We have

\[ wz \geq wy + yz \geq xy + yz \geq xy + \sigma, \quad \text{hence} \quad wz \geq wy + \sigma. \]

If \( wz > wy + \sigma \) then \( xz \geq xw + wz > xw + wy + \sigma = xy + \sigma \). Let \( wz = wy + \sigma \). Then \( (x \ w \ z) \) is impossible. This is clear for \( \sigma = 0 \) because of \( G'_6 \). If \( \sigma > 0 \) choose \( v \) with \( (w \ v \ z) \) and \( wy = wy \). Then \( (x \ w \ v) \) and we have again a contradiction to \( G'_6 \).

Therefore in either case

\[ xz > xw + wz \geq xw + wy + \sigma = xy + \sigma. \]
The following slightly stronger statement for $\sigma = 0$ is also needed. 

(9) If $x < y < z$, $y \rightarrow y \neq x$, $z \rightarrow z \neq y$ and $x < y$ then $xz > xy$. If $v < x$, $u < v$, $v \rightarrow v \neq x$, $u \rightarrow u \neq v$ and $v < x$ then $ux > vx$.

For we have $xy \rightarrow xy > 0$ and $xz \geq xy + yz \geq xy \rightarrow xy$.

Choosing $w$ as before we have $w < y$, $w < z$ for large $v$, hence

$$wz \geq wy + yz \geq wy \rightarrow wy$$

therefore $wz = wy$ and the proof proceeds as before.

In locally compact metric spaces satisfying postulates analogous to $G_{i,s}$ completeness, geodesic completeness ($r = R$ for a geodesic $\{r, x(t)\}$) and finite compactness (bounded infinite sequences have accumulation points) are equivalent. In Riemannian geometry this is essentially the content of the Hopf-Rinow Theorem.

This theorem does not seem to have an analogue in timelike spaces. Apparently there is no concept of completeness or finite compactness which is shared by all spaces which we want to admit. Examples will be found in the next two sections.

No really interesting timelike space has the property that a sequence of points $x_v$ with $p < x_v$ and $0 < \gamma \leq px_v \leq \beta$ has an accumulation point. However Lorentz spaces are finitely compact with the following definition (see Section 5):

A timelike space is finitely compact if a sequence of points $x_v$ with $p < q \leq x_v$ and $px_v \leq \beta$, or $x_v \leq q < p$ and $x_v \leq \beta$, has an accumulation point.

A timelike space is complete if $x_v < x_{v+\mu}$ ($v, \mu = 1, 2, \ldots$) and $\rho(x_v, x_{v+\mu}) \leq \gamma_v$ (or $x_{v+\mu} < x_v$) and $\rho(x_{v+\mu}, x_v) \leq \gamma_v$ and $\gamma_v \rightarrow 0$ imply that $\{x_v\}$ converges.

This could be formulated locally with $x_v \in U(p)$, $x_v < p, x_{v+\mu}$, etc. However, we are interested in $G$-spaces where $U(p)$ may be taken as compact; then $\{x_v\}$ has at least one accumulation point and it can be proved that it has only one.

We prove

(10) A finitely compact timelike $G$-space is complete.

For let $x_v < x_{v+\mu}$ and $x_v x_{v+\mu} \leq \gamma_v$, $\gamma_v \rightarrow 0$. Then $x_1 x_v \geq x_1 x_2 > 0$ for $v \geq 2$ and $x_1 x_v \leq \gamma_1$. Therefore finite compactness yields the existence of an accumulation point for every subsequence. We must show that only one accumulation point exists. If there were two, $q$ and $q'$, then we would have subsequences $\{i_v\}$ and $\{j_v\}$ of $\{v\}$ with $i_v < j_v$, $x_{i_v} \rightarrow q$, $x_{j_v} \rightarrow q'$. Also let $\{k_v\} \in \{i_v\}$ with $k_v > j_v$. Then

$$x_1 x_{i_v} \geq x_1 x_{i_v} + x_{i_v} x_{j_v}, \quad x_1 x_{k_v} \geq x_1 x_{k_v} + x_{i_v} x_{k_v}.$$
Now $G_2$ implies $x_1 < q$, $x_1 < q'$ and the inequalities yield $x_1 q = \lim x_1 x_i = \lim x_1 x_{i^v} = x q'$ contradicting (9).

For comparison with the metric case note that we did not use all axioms for a G-space, only the validity of $G'_2$ and (9), (the proof of (9) uses $G'_2$).

The converse which is valid under certain conditions in the metric case, makes no sense here, because completeness deals only with sequences satisfying $x_r < x_{r+\mu}$.

A (locally) timelike G-space is geodesically complete if $\tau = \mathcal{R}$ for every geodesic $\{\tau, x(t)\}$.

This definition could be applied to any locally timelike space, but would often lead to absurdities, for example, if no proper segments exist.

Assume that for a geodesic $\{\tau, x(t)\}$ in a timelike G-space the set $\tau$ has a finite upper bound $\beta$. If $t_1 < t_2 < \ldots$ and $t_r \to \beta$ then $x(t_r) < x(t_{r+\mu})$ but—in contrast to the metric case—$x(t_r) x(t_{r+\mu}) > t_{r+\mu} - t_r$. Therefore we cannot conclude that $\{x(t_r)\}$ converges if the space is complete. The only result in this direction is:

(11) For a timelike G-space in which all geodesics are lines completeness implies geodesic completeness. The converse does not hold.

That the converse is false follows from (4) in Section 5. The following sections will show that geodesic completeness is the most relevant concept. It should be remembered that $G_2$ has the nature of a completeness postulate but without an analogue in metric spaces.

We conclude this section by defining motion. This is simple for timelike spaces:

A motion of a timelike space $R$ is a topological map $\Phi$ of $R$ on itself which maps $<$ on itself and preserves distances, i.e. if $x < y$ if and only if $\Phi x < \Phi y$ and $d(\Phi x, \Phi y) = d(x, y)$.

The definition is necessarily more involved for locally timelike spaces owing to the largely arbitrary choice of the $U(p)$ and the possibility that the space cannot be consistently ordered.

A motion of a locally timelike space $R$ is a topological map $\Phi$ of $R$ on itself with the following property: Each point has a neighborhood $N(p) \subset U(p)$ such that $\Phi N(p) \subset U(\Phi p)$ and either $\Phi x <_{\Phi p} \Phi y$ and $d_{\Phi p}(\Phi x, \Phi y) = d_p(x, y)$ for all pairs $x, y$ with $x < y$ in $N(p)$ or $\Phi y <_{\Phi p} \Phi x$ and $d_{\Phi p}(\Phi y, \Phi x) = d_p(x, y)$ for all pairs $x, y$ with $x < y$ in $N(p)$.

Since this definition is cumbersome we observe:

(12) A topological map of a (locally) timelike G-space on itself is a motion, if and only if it maps T-curves on T-curves and preserves the lengths of all T-curves.
4. PRODUCTS OF TIMELIKE AND METRIC SPACES

Our next aim is significant examples to elucidate the theory. In this section $S_1$ is a timelike space (the results all carry over to the case where $S_1$ is locally timelike), $S_2$ is a metric space and $R = S_1 \times S_2$. The distance in $S_i$ is denoted by $\sigma_i$, points in $S_1$ by $a$, $b$, $c$ (with sub- and superscripts), points in $S_2$ by $x$, $y$, $z$ and points in $R$ by $p = (a, x)$, $q = (b, y)$, $r = (c, z)$, etc. The set $(<)$ is defined by $p = (a, x) < q = (b, y)$ meaning $a < b$ and $\sigma_1(a, b) > \sigma_2(x, y)$. Transitivity follows for $r = (c, z)$ from

\begin{equation}
\sigma_1(a, c) \geq \sigma_1(a, b) + \sigma_1(b, c) > \sigma_2(x, y) + \sigma_2(y, z) \geq \sigma_2(x, z).
\end{equation}

Moreover, $T'_2$ for $S_1$ implies $T'_2$ for $R$.

We impose different distances $\varrho_{a}$ $(a \geq 1)$ on $R$

\[ \varrho_{a}(p, q) = [\sigma_1^a(a, b) - \sigma_2^a(x, y)]^{1/a}. \]

For $\varrho_1(p, q) = \sigma_1(a, b) - \sigma_2(x, y)$ we find from (1) that $\varrho_1(p, q) + \varrho_1(q, r) \geq \varrho_1(q, r)$ with equality only if $\sigma_1(a, b) + \sigma_1(b, c) = \sigma_1(a, c)$ and $\sigma_2(x, y) + \sigma_2(y, z) = \sigma_2(x, z)$.

To prove that the other $\varrho_a$ satisfy the time equality we observe:

(2) If $h_1 > k_1 \geq 0$, $h_2 > k_2 \geq 0$ and $a > 1$ then

\[ (h_1^a - k_1^a)^{1/a} + (h_2^a - k_2^a)^{1/a} \leq [(h_1 + h_2)^a - (k_1 + k_2)^a]^{1/a} \]

with equality only when either $k_1 = k_2 = 0$ or $h_1 : k_1 = h_2 : k_2$.

Put $\varepsilon_i = (h_i^a - k_i^a)\frac{1}{1/a}$ or $h_i^a = \varepsilon_i^a + k_i^a$. By Minkowski's Inequality [8, p. 31],

\[ [(\varepsilon_1 + \varepsilon_2) + (k_1 + k_2)^a]^{1/a} \leq [(\varepsilon_1 + \varepsilon_2)^a + (k_1 + k_2)^a]^{1/a} \]

with equality only if $k_1 = k_2 = 0$ or $\varepsilon_1 : k_1 = \varepsilon_2 : k_2$.

This is equivalent to the assertion, from which we conclude:

(3) If $a > 1$ then

\[ \varrho_{a}(p, q) + \varrho_{a}(q, r) \leq \varrho_{a}(p, r) \]

with equality for $p < q < r$ only if $(a \ b \ c)$, $\sigma_2(x, y) + \sigma_2(y, z) = \sigma_2(x, z)$ and either $x = y = z$ or $\sigma_1(a, b) : \sigma_2(x, y) = \sigma_1(b, c) : \sigma_2(y, z)$.

We denote the space $R$ with the distance $\varrho_{a}$ by $R_{a}$ and have proved that $R_{a}$ is timelike for all $a$. We turn to the axioms $G_1$, $G'_2$. Obviously

(a) $G_1$ holds in $R_{a}$ $(a \geq 1)$ if it holds for $S_1$ and $S_2$.

(b) $G'_2$ holds in $R_{a}$ $(a \geq 1)$ if it holds in $S_1$.

For if $p_{r} = (a_{r}, x_{r}) \rightarrow (a, x) = p$, $q_{r} = (b_{r}, y_{r}) \rightarrow (b, y) = q$, $p_{r} \leq q_{r}$ and $p \leq q$ does not hold, then $a \leq b$ does not hold, hence $\sigma_1(a_{r}, b_{r}) \rightarrow 0$
and 
\[ \sigma_1(a_r, b_r) \geq \sigma_2(x_r, y_r) \quad \text{gives} \quad \varrho_a(p_r, q_r) \to 0. \]

(c) \( G'_3 \) holds in \( R_a \) (\( a \geq 1 \)) if it holds in \( S_1 \).

This follows from the condition for equality in (3). To discuss \( G'_4 \) we define \((x y z)\) in \( S_2 \) as for \( S_1 \), i.e. \( x, y, z \) are distinct and \( \sigma_2(x, y) + \sigma_2(y, z) = \sigma_2(x, z) \). If \( S_2 \) is finitely compact then the points \( y \) satisfying \((x y z)\) lie (for fixed \( x, z \)) in a compact set and if for given \( x, z \) a point \( y \) with \((x y z)\) exists then any two points of \( S_2 \) can be joined by a segment, [G, p. 29]. Therefore the conditions for equality on (3) yield

(d) \( G'_4 \) holds in \( R_a \) (\( a \geq 1 \)) if it holds in \( S_1 \) and \( S_2 \) is finitely compact and contains with \( x \neq z \) a point \( y \) with \((x y z)\).

Similarly one deduces from (3)

(e) \( G'_5 \) holds in \( R_a \) (\( a \geq 1 \)) if it holds in \( S_1 \) and \( S_2 \) and either any two points \( a, b \) in \( S_1 \) with \( a < b \) or any two points in \( S_2 \) can be joined by a segment.

So far no difference between the cases \( a = 1 \) and \( a > 1 \) has appeared. However

(f) If \( S_1 \) and \( S_2 \) satisfy \( G'_6 \) then \( R_a \) does for \( a > 1 \) but in general not for \( a = 1 \).

For, assume \((p q r_i)\) with \( \varrho_a(q, r_1) = \varrho_a(q, r_2) \) (\( a > 1 \)). Then, if \( r_i = (c_i, z_i) \) the conditions for equality in (3) give

\[ \sigma_1(a, b) + \sigma_1(b, c_i) = \sigma_1(a, c_i), \quad \sigma_1(a, b) > 0, \quad \sigma_1(b, c_i) > 0, \]

\[ \sigma_2(x, y) + \sigma_2(y, z_i) = \sigma_2(x, z_i) \]

and either \( x = y = z_i \) or \( x, y, z_i \) are distinct and

\[ \sigma_1(a, b) : \sigma_2(x, y) = \sigma_1(b, c_i) : \sigma_2(y, z_i). \]

If \( x = y \) then \( x = y = z_1 = z_2 \) and \( \sigma_1(b, c_1) = \sigma_1(b, c_2) \) so \( c_1 = c_2 \) and \( r_1 = r_2 \). If \( x, y, z_i \) are distinct then the above relations imply \( \sigma_1(b, c_1) = \sigma_1(b, c_2) \) and \( \sigma_2(y, z_1) = \sigma_2(y, z_2) \) and \( G'_6 \) in \( S_1 \) gives \( c_1 = c_2 \) and \( z_1 = z_2 \) hence \( r_1 = r_2 \).

The negative assertion for \( a = 1 \) is one of several enunciated in:

(4) If \( S_1 \) is a timelike \( G \)-space and \( S_2 \) is a metric \( G \)-space, then \( R_1 \) satisfies neither \( G'_6 \) nor (3.8) or (3.9). Also, \( R_1 \) contains maximal linear sets \( \mu(q, r) = \{(0, \varrho_1(q, r)), \; p(t)\} \) for which \( p(t) \) is not continuous.
There are points \( a, b, c_1, c_2 \) in \( S_1 \) and \( x_1, x_2 \) in \( S_2 \) such that \((a \ b \ c_1), (a \ c_1 \ c_2) \) and \( \sigma_2(x_1, x_2) = \sigma_1(c_1, c_2) \). Putting \( p = (a, x_1), \ q = (b, x_1), \ r_i = (c_i, x_i) \) we find

\[
e_1(p, q) = \sigma_1(a, b), \quad e_1(q, r_1) = \sigma_1(b, c_1),
\]

\[
e_1(q, r_2) = \sigma_1(b, c_2) - \sigma_2(x_1, x_2) = \sigma_1(b, c_1),
\]

\[
e_1(p, r_2) = \sigma_1(a, c_2) - \sigma_2(x_1, x_2) = \sigma_1(a, c_1) = e_1(p, q) - e_1(q, r_2).
\]

So \((p, q, r_1) \) and \( e_1(q, r_1) = e_1(q, r_2) \) but \( r_1 \neq r_2 \).

That (3.8) and (3.9) do not hold is seen in a similar way: There are points \( a, b, c \) in \( S_1 \) and \( y, z, z \) in \( S_2 \) with \((a \ b \ c) \) and \( \sigma_1(b, c) = \sigma_2(y, z), \ z \rightarrow z \) and \( \sigma_2(y, z) < \sigma_2(y, z) \). Then with \( p = (a, y), \ q = (b, y), \ r = (c, z), \ r \rightarrow r \neq q \) of (3.8) are satisfied but

\[
e_1(p, r) = \sigma_1(a, b) + \sigma_1(b, c) - \sigma_2(y, z) = \sigma_1(a, b) = e_1(p, q)
\]

hence \( F(p, e_1(p, q)) \) does not contain \( F(q) - q \). Putting \( q_r = q \) yields a negative answer to (3.9).

To construct a discontinuous \( p(t) \) consider a segment \([0, \sigma_1(a, b)] \), \( a(t) \) from \( a \) to \( b \ (a < b) \) in \( S_1 \). Choose values \( 0 < t' < t' + \epsilon < \sigma_1(a, b) \) and points \( x, z \) in \( S_2 \) with \( \sigma_2(x, z) = \epsilon \). Define

\[
p(t) = (a(t), x) \quad \text{in} \quad [0, t'], \quad p(t) = (a(t + \epsilon), z) \quad \text{in} \quad [t', \sigma_1(a, b) - \epsilon].
\]

One readily verifies

\[
e_1(p(t_1), p(t_2)) = t_2 - t_1 \quad \text{for} \quad 0 \leq t_1 \leq t_2 \leq \sigma_1(a, b) - \epsilon,
\]

so \( p(t) \) defines a maximal linear set but is not continuous.

Combining our results we have

(5) Theorem. If \( S_1 \) is a timelike \( G \)-space and \( S_2 \) is a metric \( G \)-space, then \( R_n \) is a timelike \( G \)-space if \( \alpha > 1 \). The space \( R_1 \) satisfies all axioms but \( G' \) and has the other properties listed in (4).

Note. Minor modifications of our arguments show that \( R_n \) is for \( \alpha > 1 \) a locally timelike \( G \)-space if \( S_1 \) is a locally timelike \( G \)-space and \( S_2 \) is a metric \( G \)-space.

Now we turn to completeness:

(6) Theorem. If \( S_1 \) is a complete timelike space and \( S_2 \) is a complete metric space then \( R_n \) is a complete timelike space for \( \alpha > 1 \), but in general not for \( \alpha = 1 \).

Let \( \gamma_r < \gamma_{r+1} \) and \( g_a(p_r, p_{r+1}) < \gamma_r \) and \( \gamma_r \rightarrow 0, \alpha > 1 \). Then

\[
\epsilon_r = \sigma_1(a_{r-1}, a_r) - \sigma_2(x_{r-1}, x_r) > 0.
\]
Using the Mean Value Theorem and (1) we find

\[ q^\alpha_s(p, p_{r+\mu}) = q^\alpha_s(a_r, a_{r+\mu}) - \sigma^\alpha_2(x_r, x_{r+\mu}) \]

\[ \geq \left( \sum_{r+1}^{r+\mu} \sigma_1(a_{i-1}, a_i) \right)^\alpha - \left( \sum_{r+1}^{r+\mu} \sigma_2(x_{i-1}, x_i) \right)^\alpha = \left( \sum_{r+1}^{r+\mu} \epsilon_i \right)^\alpha \alpha c_{r+\mu}^{-1} \]

where

\[ \sum_{r+1}^{r+\mu} \sigma_2(x_{i-1}, x_i) \leq c_{r+\mu} \leq \sum_{r+1}^{r+\mu} \sigma_1(a_{i-1}, a_i). \]

Therefore

\[ \sum_{r+1}^{r+\mu} \epsilon_i \left( \sum_{r+1}^{r+\mu} \sigma_2(x_{i-1}, x_i) \right)^\alpha^{-1} \leq a^{-1} \gamma^\alpha \rightarrow 0. \]

There are two cases:

1) All \( \sigma_2(x_{i-1}, x_i) = 0 \) then \( q^\alpha_s(p, p_{r+\mu}) = \sigma_1(a_r, a_{r+\mu}) \) and the assertion follows from the completeness of \( S_1 \).

2) Not all \( \sigma_2(x_{i-1}, x_i) \) vanish. The last inequality with \( \nu = 1 \) shows that both series

\[ \sum_{i} \epsilon_i \quad \text{and} \quad \sum_{i} \sigma_2(x_{i-1}, x_i) \]

converge. The latter and

\[ \sigma_2(x_r, x_{r+\mu}) \leq \sum_{r+1}^{r+\mu} \sigma_2(x_{i-1}, x_i) \]

shows that \( \{x_r\} \) is a fundamental sequence in \( S_2 \), so that \( x_r \) tends to a point \( x \) in \( S_2 \). Moreover,

\[ \sigma^\alpha_1(a_r, a_{r+\mu}) = q^\alpha_s(p, p_{r+\mu}) + \gamma^\alpha \sigma^\alpha_2(x_r, x_{r+\mu}) \leq \gamma^\alpha \]

with \( \gamma^\alpha \rightarrow 0 \), hence \( x_r \) converges to a point \( a \) in \( S_1 \) and \( (a_r, x_r) \rightarrow (a, x) \).

The space \( L_1^{n+1} \) which will be defined presently is an example for the negative part of (6) as well as of (7) and (8).

7) If \( S_1 \) and \( S_2 \) are finitely compact then so is \( R_\alpha \) for \( a > 1 \), but not necessarily for \( a = 1 \).

For, let \( p < q < r \) and \( q^\alpha_s(p, r) \leq \beta \) \((a > 1)\). Then

\[ \sigma_1(a, b) - \sigma_2(x, y) = \gamma > 0, \quad \sigma_1(b, c_r) \geq \sigma_2(y, z_r) \]

and

\[ \beta^\alpha \geq \sigma^\alpha_1(a, c_r) - [\sigma_1(a, c_r) - \gamma]^\alpha + [\sigma_1(a, c_r) - \gamma]^\alpha - \sigma^\alpha_2(x, z_r), \]

\[ \sigma_2(x, z_r) - \sigma_2(x, y) \leq \sigma_2(y, z_r) \leq \sigma_1(b, c_r) \leq \sigma_1(a, c_r) - \sigma_1(a, b) \]
or\n\[\sigma_2(x, z_r) \leq \sigma_1(a, c_r) - \gamma.\]
Therefore
\[\beta^a \geq \sigma_1^a(a, c_r) - (\sigma_1(a, c_r) - \gamma)^a,\]
which implies \(\sigma_1(a, c_r) \leq \beta^a\) with a suitable \(\beta^a < \infty\) and hence \(\sigma_2(x, z_r) \leq \beta^a\).
The finite compactness of \(S_1\) and \(S_2\) now yields a subsequence \(\{k\}\) of \(\{r\}\) for which \(c_k\) and \(z_k\) converge to points \(c, z\); so that \(r_k \to (c, z)\).

An immediate consequence of (3) is

(8) If \(S_1\) is a geodesically complete timelike \(G\)-space and \(S_2\) is a metric \(G\)-space, then \(R_a\) is geodesically complete for \(a > 1\) but not necessarily for \(a = 1\).

We now consider the most important special case where \(S_1\) is the real axis with the usual order and distance \(\sigma_1(u_1, u_2) = u_2 - u_1\) and \(S_2\) is the \(n\)-dimensional \((n \geq 1)\) euclidean space. We introduce coordinates \(x^1, \ldots, x^n\) so that the distance takes the standard form
\[\sigma_2(x, y) = e(x, y) = [\sum (x^i - y^i)^2]^{1/2}.\]
In this case we denote \(\sigma_a\) by \(\lambda_a\) and the space by \(L_a^{n+1}\). Then for \(p = (u, x)\) and \(q = (v, y)\) the relation \(p < q\) means \(v - u > e(x, y)\) and
\[\lambda_a(p, q) = [(v - u)^a - e^a(x, y)]^{1/a}, \quad a \geq 1.\]
In particular, \(\lambda_2(p, q)\) defines the \((n + 1)\)-dimensional Lorentz space \(L_2^{n+1}\).

(9) In \(L_a^{n+1}\) \((a \geq 1)\) the affine lines
\[r(t) = [1 - \lambda_a^{-1}(p, q)t]p + t\lambda_a^{-1}(p, q)q, \quad p < q, \quad -\infty < t < \infty\]
are geodesics and they are lines, i.e.
\[\lambda_2(r(t_1), r(t_2)) = t_2 - t_1 \quad \text{for} \quad t_1 < t_2.\]
They are the only geodesics when \(a > 1\), there are others if \(a = 1\), in particular such for which the corresponding set \(\tau\) is bounded.

This implies that \(L_a^{n+1}\) is geodesically complete for \(a > 1\) but not for \(a = 1\).

All statements in (9) except the very last follow from (3). Cosider \(x(s) = (s, 0, \ldots, 0)\) and \(u(s) = s + \arctan s\). Then \(r(s) = (u(s), x(s))\) is a line (but \(s\) is not arc length) because
\[r(s_1) < r(s_2) < r(s_3) \quad \text{for} \quad s_1 < s_2 < s_3\]

(2) Strictly speaking \(L_2^{n+1}\) is the restriction to the pairs \((u, x) < (v, y)\) of the indefinite metric \([(v - u)^2 - e^2(x, y)]^{1/2}\) defined for all pairs which is usually denoted as Lorentz space.
and

\[ \lambda_{1}(r(s_1), r(s_2)) = \arctan s_2 - \arctan s_1 \quad \text{for} \quad s_1 < s_2. \]

However, the range of the length \( t \) if \( s = 0 \) corresponds to \( t = 0 \), is \( \tau = (\pi/2, \pi/2) \).

The sequence \( r(v) \) \((v = 1, 2, \ldots)\) also shows that \( L_{1}^{n+1} \) is neither complete nor finitely compact, confirming the last statement in (6) and (7). Using the preceding results we have

(10) The spaces \( L_{a}^{n+1} \) \((a > 1)\) are finitely compact and geodesically complete \( G \)-spaces in which all geodesics are lines.

The spaces \( L_{1}^{n+1} \) satisfy all axioms but \( G_{i} \). They are neither complete, nor finitely compact nor geodesically complete.

However, \( L_{1}^{n+1} \) has the property that any segment can be extended to a geodesic \( \{ \tau, x(t) \} \) for which \( \tau = \mathcal{R} \). This is easily verified.

We mentioned that \( L_{2}^{n+1} \) does not have the property that every sequence of points \( q_{r} \) with \( p < q_{r} \) and \( 0 < a \leq \lambda_{2}(p, q_{r}) \leq \beta \) has an accumulation point. This is true for all \( a \): If \( e(x_{r}, r_{r+1}) = r^{1/\alpha}, q = (0, x_{1}), p_{r} = ((v+1)^{1/\alpha}, x_{r+1}) \) then \( \lambda_{a}(q, p_{r}) = 1 \) but no subsequence of \( \{ p_{r} \} \) converges.

Finally we mention the obvious fact

(11) If \( \Phi_{1} \) is a motion of the timelike space \( S_{1} \) and \( \Phi_{2} \) is a motion of the metric space \( S_{2} \), then \( (a, x) \rightarrow (\Phi_{1} a, \Phi_{2} x) \) defines a motion of \( R_{a} \) \((a \geq 1)\).

5. TIMELIKE MINKOWSKI SPACES

We now discuss the type of timelike \( G \)-space which is basic in the sense that under differentiability hypotheses any timelike \( G \)-space behaves locally like a space of this type.

We call these spaces Minkowski spaces with a twofold justification: they are the obvious analogue to the metric Minkowski spaces and they comprise the Lorentz space \( L_{2}^{4} \) which in relativity is often called the Minkowski space.

A \emph{timelike Minkowski space} is defined by the following properties: it is a timelike \( G \)-space for which the underlying space \( R \) is the \( n \)-dimensional affine space \( A^{n} \) \((n \geq 2)\) with the usual topology and the translations of \( A^{n} \) are motions.

In order to discribe these spaces we need certain functions: In terms of affine coordinates the function \( f(x) = f(x', \ldots, x^{n}) \) is a \emph{gauge function} if...
(a) \( f(x) \) is defined on an open convex set \( D \neq \emptyset \) which is a cone with the origin as apex, i.e. \( \lambda D = D \) for \( \lambda > 0 \),

(b) \( f(\lambda x) = \lambda f(x) \) for \( \lambda > 0 \),

(c) \( f(x) > 0 \),

(d) \( f(x_*) \to 0 \) if \( x_* \to x \not\in D \),

(e) \( f((1 - \theta)x + \theta y) > (1 - \theta)f(x) + \theta f(y) \) for \( 0 < \theta < 1 \) unless \( x = \lambda y \) with \( \lambda > 0 \).

It follows from (b) and (c) that the origin 0 does not belong to \( D \). The property expressed by (e) may be called strong concavity of \( f(x) \) in analogy to strong convexity, see [G, p. 99]. It has two important equivalent forms:

(e') \( f(x + y) > f(x) + f(y) \) unless \( x = \lambda y \) with \( \lambda > 0 \).

(e'') The set \( f(x) \geq \varrho > 0 \) is strictly convex, i.e. \( f(x) \geq \varrho \), \( f(y) \geq \varrho \) and \( x \neq y \) imply \( f((1 - \theta)x + \theta y) > \varrho \).

This equivalence is established as for convex \( f(x) \) [G, pp. 99, 100] and can, in fact, be reduced to the convex case by observing that \(-f(x)\) is convex. Therefore \( f(x) \) is continuous, and if we put \( f(0) = 0 \) then (d) implies that \( f(x) \) is continuous at 0.

Denote the boundary of \( D \) by \( C \).

(1) The set \( D \cap C \) possesses at 0 a supporting hyperplane which intersects \( D \cap C \) only at 0.

For, denote by \( E \) the intersection of all closed half spaces bounded by hyperplanes through 0 and containing the set \( \{ x, f(x) \geq 1 \} \). Since this set is strictly convex, there is a hyperplane through 0 intersecting \( E \) only at 0. But any half space whose boundary contains 0 and which contains \( f(x) \geq 1 \) also contains \( f(x) \geq \varrho \) by (b) for any \( \varrho \), so \( E \supset D \) and \( E \supset D \cap C \). It is easily seen that actually \( E = D \cap C \).

We now establish the relation between Minkowski spaces and gauge functions.

(2) Theorem. Let \( R \) be a timelike Minkowski space with affine coordinates \( x = (x^1, \ldots, x^n) \) and distance \( \varrho(x, y) \). Then \( \varrho(x, y) = f(y - x) \), where \( f(x) \) is a gauge function and \( x < y \) is equivalent to \( f(y - x) > 0 \).

Conversely, if a gauge function \( f(x) \) in \( \mathbb{A}^n \) is given and \( x < y \) is defined by \( f(y - x) > 0 \), then \( \varrho(x, y) = f(y - x) \) defines a timelike Minkowski space.

We prove the second part first. The topological properties \( T_1, G_1 \) are trivial. The continuity of \( \varrho(x, y) \) follows from that of \( f(x) \), and so does the openness of \((<)\). The existence of \( x, y \) with \( x < q < y \) in a given
$W(q)$ is clear from (b), the time inequality is contained in (e'). $G'_2$ follows from (d).

Let $(x \ y \ z)$ or $f(y-x)+f(z-y) = f(z-x)$; then (e') gives $(y-x) = \lambda(z-y)$ with $\lambda > 0$ or

$$y = \frac{1}{1+\lambda} x + \frac{\lambda}{1+\lambda} z, \quad z = -\frac{x}{\lambda} + \frac{1+\lambda}{\lambda} y,$$

which proves $G'_2$ and $G'_6$ and reversing the argument gives $G'_3, G'_4$.

Now let $R = \mathcal{A}^n$ be a timelike Minkowski space with $x' = x + a$ as motions. Then $x < y$ implies $x+a < y+a$, in particular $0 < y-x$ and $\varrho(x, y) = \varrho(0, y-x)$. Put $f(x) = \varrho(0, x)$. Then $\varrho(x, y) = f(y-x)$ for $x < y$. Let $D$ be the set $f(x) > 0$. If $0 < x, 0 < y$ then $x + x+y$ and $\varrho(0, x+y) \geq \varrho(0, x) + \varrho(x, x+y)$.

(a) If $x \in D, y \in D$ then $x + y \in D$ and $f(x+y) \geq f(x) + f(y)$.

We call $z$ a midpoint of $x$ and $y$ if $x < z < x$ and $\varrho(x, z) = \varrho(z, y) = \varrho(x, y)/2$.

(β) If $z$ is a midpoint of $x$ and $y$ then so is $u = x+y-z$.

For $u-x = y-z, y-u = z-x$ give $f(u-x) = f(y-z) > 0$ hence $x < u$, similarly $u < y$ and $f(u-x) = f(y-z) = f(x-y)/2 = f(z-x) = f(y-u)$. This implies

(γ) If a midpoint $z$ of $x$ and $y$ exists and is unique then $z = (x+y)/2$.

If the segment $\sigma(x, y)$ is unique ($x < y$) then any two points $v, w$ with $v < w$ on $\sigma(x, y)$ have a unique midpoint because of (1.2) and (2.5). We conclude from (γ) that $\frac{3}{4}x + \frac{1}{4}y$ and $\frac{1}{2}x + \frac{3}{4}y$ are the midpoints of $x, z$ and $z, x$ respectively. Using the continuity of $f(x)$ we find that $\sigma(x, y)$, if unique, is the affine segment consisting of the points $x_\theta = (1-\theta)x + \theta y$ ($0 \leq \theta \leq 1$) and that

$$\varrho(x, x_\theta) = \theta \varrho(x, y) \quad \text{for} \quad 0 \leq \theta \leq 1.$$  

Now segments are locally unique, see (2.5) and hence locally affine segments. This shows that any geodesic curve $x(t), a \leq t \leq b$, is an affine segment, hence the only geodesic curve from $x(a)$ to $x(b)$. On the other hand, there is a $\sigma(x(a), x(b))$ and this is a geodesic curve. It follows that the geodesics are affine lines and that for $x < y$ the geodesic through $x$ and $y$ has the form

(δ) $z(t) = (1-t\varrho^{-1}(x, y))x + t\varrho^{-1}(x, y)y$, $-\infty < \epsilon < \infty$, $t$ arclength.

Applying (δ) with $x = 0$ we find $f(\lambda y) = \lambda f(y)$ for $\lambda > 0$.

This and (a) prove (e) via (e'), and (d) follows from $G'_2$. Because $0 < a, 0 < b$ imply $0 < a+b$, hence $0 < (a+b)/2$, the set $D$ is convex.

A consequence of this discussion is:
A timelike Minkowski space is geodesically complete and all its geodesics are lines.

In general the space will not be complete, but there is a simple criterion for completeness. To formulate it we remember that $f(x-a) > 0$ is the future $F(a)$ of $a$.

$F(a, \sigma)$ denotes the set of $x$ satisfying $\varrho(a, x) > \sigma$, in our case $f(x-a) > \sigma$. The closure $\overline{F}(a, \sigma)$ is $\{x, f(x-a) \geq \sigma\}$. The boundary of $F(a)$ is, according to the language of relativity, the light cone $C(a)$. With the previous notation $F(0) = D$, $C(0) = C$.

A timelike Minkowski space is complete or finitely compact if and only if no hyperplane exists which separates a generator of $C(0)$ from $\overline{F}(0, 1)$.

We could, of course, have used any point $a$ and any $\sigma > 0$ instead of 0 and 1. A generator $G$ of $C(0)$ is a ray with origin 0 lying on $C(0)$. It is clear that a hyperplane separating $C = C(0)$ from $\overline{F} = \overline{F}(0, 1)$ must be parallel to $G$. Because of (4.10) we must prove:

1) If the condition in (4) is satisfied then the space is finitely compact.

2) If it is not satisfied then the space is not complete.

To show 1) let $0 < y$ or $f(y) > 0$ and $y \leq z$, with $f(z) = \varrho(0, z) \leq \beta$. No generator $G'$ of $C(y)$ lies outside $\overline{F}(0, \beta)$ because then a suitable hyperplane through $G'$ would separate $F(0, \beta)$ from the generator of $C$ parallel to $G'$. Therefore the intersection of $\{x, f(x) \leq \beta\}$ and $\overline{F}(y)$ is compact and the sequence $\{z_n\}$ has an accumulation point.

For 2) let $H$ be a hyperplane separating the generator $G$ of $C$ from $\overline{F}$. Let $G_1$ be any other generator of $C$ and in the plane determined by $G$ and $G_1$. Let $G, G_1$ be the non-negative $x$- and $y$-axes of an affine coordinate system such that $G_1 \cap H$ is the point $(0, 1)$. Put $p_i = (x_i, y_i)$ ($i = 0, 1, 2, \ldots$) with $x_i = i$ and $y_i = 1 - 2^{-i}$. Then $\varrho(p_0, p_i) < 1/2$ and more generally $\varrho(p_k, p_{k+j}) < 2^{-k}$, but the sequence $\{p_i\}$ does not converge.

An explicit example of a non-complete timelike Minkowski space is given by $D = F(0) = \{x: x^1 > 0, x^2 > 0\}$ and $f(x) = x^1 x^2 (x^1 + x^2)^{-1}$, where $f(x) = 1$ is the branch $x_1 > 0$ of the hyperbola $(x^1 - 1)(x^2 - 1) = 1$. Such examples show that a geodesically complete timelike space with a transitive group of motions need not be complete, whereas any locally compact metric space with a transitive group of motions is complete.

The significance of $G_2$ as a completeness condition may be seen from the following observation: In a timelike Minkowski space with gauge function $f(x)$ restrict $f(x)$ to an open convex subcone of $F(0)$ with apex 0. Then all axioms except $G_2$ are satisfied.
Let $A^n$ be the underlying space of a locally timelike space $G$-space for which the translations are motions. The crucial arguments in the preceding proof were ($\beta$) and ($\gamma$) and both are local. We conclude therefore that the affine lines are geodesics and that ($\delta$) holds. The space satisfies the hypotheses of (1.3) and nothing new is gained.

However, consider a locally timelike space $R$ which is a manifold of dimension $n \geq 2$ and possesses a transitive abelian group of motions. Then no motion except the identity has fixed points. (For if $qa = a$ and $b = qa$ then $qb = qqa = qqa = qa = b$.) The group is therefore simply transitive and the space can be identified with the group space. According to a theorem of Pontrjagin [14, p. 170] the group is the product of $n$ groups isomorphic either to the real numbers or to the circle group.

The universal covering space $R$ is the affine space $A^n$ and the lifted group is the group of translations. So we have the previous case. $R$ possesses a consistent ordering, but the hypothesis of (1.3) will not be satisfied unless $R$ is simply connected. We may express this as follows:

(5) A locally timelike $G$-space which is a topological manifold and possesses a transitive abelian group of motions is locally Minkowskian.

The Möbius strip and the one-sided torus can also be provided with locally timelike Minkowski metrics, but not with arbitrary ones, because the reflection in some line must exist (as for example in $I^2_2$). This leads to spaces which cannot be consistently ordered.

6. PROJECTIVITIES OF CONVEX HYPERSURFACES

Sections 7 and 8 lead to problems concerning convex hypersurfaces which are of considerable independent interest, but have only been partly solved, although they sound very simple. What is known concerning these problems is the content of the present section. In order not to interrupt the discussion later we first establish a lemma of a general nature which is due to Montgomery:

(1) If a Lie group $\Gamma$ acts transitively on the manifold $M$ then so does its identity component $\Gamma_0$.

The number of components of $\Gamma$ is finite or countable and they have the form $\varphi_1 \Gamma_0, \varphi_1 \in \Gamma$. For any point $p \in M$, let $\Gamma_0(p)$ be the orbit of $p$ under $\Gamma_0$. Because $\Gamma$ is transitive on $M$, $\bigcup \varphi_i \Gamma_0(p) = M$ and each $\varphi_i \Gamma_0(p)$ is homeomorphic to $\Gamma_0(p)$, which is the union of a countable number of compact sets. Therefore

$$\dim \varphi_i \Gamma_0(p) = \dim M \quad \text{for all } i.$$
It follows that $\varphi_i \Gamma_0(p)$ contains an open subset of $M$, see [9, p. 46], and hence is open. If $\Gamma_0(p) \neq M$ then $M - \Gamma_0(p)$ would be the union of some of the $\varphi_i \Gamma_0(p)$ and $M$ would not be connected.

Consider now a closed convex hypersurface $K$ in $A^n (n \geq 2)$ and let $I$ be its interior, put $K^o = K \circlearrowright I$. We complete $A^n$ to the $n$-dimensional projective space $P^n$ by adding a hyperplane $x^{n+1} = 0$ and will be concerned with the group $\Gamma_K$ of projectivities which map $K$ on itself. These take also $I$ and $E = P^n - K^o$ into themselves. Conversely, a projectivity which maps $I$ or $E$ on itself also maps $K$ on itself. We show first:

(2) If $\Gamma_K$ is transitive on $K$ then $K$ is an ellipsoid.

The theorem is elementary for $n = 2$, so we may assume $n > 2$. Then $K$ is simply connected and by (1) the identity component $\Gamma_0$ of $\Gamma_K$ acts transitively on $K$. According to a theorem of Montgomery [11, p. 226] $\Gamma_0$ contains a compact subgroup $\Gamma_c$ acting transitively on $K$. Now a result due to the collective effort of several mathematicians, see Nagano [13], implies that $\Gamma_c$ is isomorphic to a subgroup of the orthogonal group $O(n)$ and this gives readily the assertion.

A much more elementary method of deducing the assertion from the existence of $\Gamma_c$ is found at the end of this section.

For $n = 3$ we could have referred to Lie [10] who determined all surfaces possessing transitive groups of projectivities onto themselves. However, he always assumes that the surface is analytic, and in this area innocent looking smoothness hypotheses may change completely the character of a problem, see Theorem 7.

The principal unsolved problem is finding all $K$ for which $\Gamma_K$ is transitive on $I$. The problem would be quite accessible if it were known that under this hypothesis different orbits of $\Gamma_K$ have different closures or that the number of distinct orbits is finite. We solve the problem only in special cases.

For this purpose we consider the metrization of $I$ as a Hilbert geometry (for details see [G, Section 18]).

We put $h(a, a) = 0$ and if $a$, $b$ are distinct points of $I$ let the projective line $ab$ through $a$ and $b$ intersect $K$ in $x$ and $y$. Remembering that for any permutations $(i_1, i_2)$ and $(j_1, j_2)$ of $(1, 2)$

$$R(a_{i_1}, a_{i_2}, x_{j_1}, x_{j_2}) = [R(a_1, a_2, x_1, x_2)]^{\pm 1},$$

where $R(\ )$ denotes the crossratio, we put

$$h(a, b) = |\log R(a, b, x, y)|.

(3)

Then $h(a, b)$ satisfies the axioms for a metric space. With this metric $I$ is finitely compact, the intersections of the projective lines with $I$ are isometric to $R$ and hence are geodesics. They are the only geodesics,
unless there are two proper segments on \( K \) whose convex hull contains points of \( I \). (The proof of [G, (18.5)]) leads to this result, but the formulation of (18.5) does not quite express it.)

We prove first

(4) \( K^0 = K \cup I \) is a simplex if and only if \( \Gamma_K \) has a subgroup \( \Gamma_a \) which is abelian and transitive on \( I \).

The set \( x^i > 0 \) (\( i = 1, \ldots, n \)) in \( A^n \) may be considered as the interior \( I \) of a simplex and the maps \( \tilde{x}^i = \beta^i x^i, \beta^i > 0 \) (\( i = 1, \ldots, n \)) provide a transitive abelian group of projectivities of \( I \).

For a proof of the necessity we observe that for \( \varphi \in \Gamma_a \) the distance \( h(x, \varphi x) \) is independent of \( x \). For if \( y \in I \) then \( \varphi \in \Gamma_a \) with \( \varphi x = y \) exists and \( h(y, \varphi y) = h(\varphi x, \varphi \varphi x) = h(\varphi x, \varphi x) = h(x, \varphi x) \). Let \( u \in K, \varphi \in \Gamma_a \) and \( u \neq \varphi u = v \). We will show that \( u \) is not an extreme point of \( K^0 \). (3) Let \( a_v \in I \) with \( a_v \to u \). Then \( b_v = \varphi a_v \to \varphi u = v \). If \( a_v, b_v \) intersects \( K \) in \( x_v \) and \( y_v \) and the names are such that the order is \( y_v, a_v, b_v, x_v \), then

\[
h(a_v, b_v) = \log R(a_v, b_v, x_v, y_v).
\]

For a subsequence \( \{i\} \) of \( \{v\} \) we have \( x_i \to x^* \), \( y_i \to x^* \). Since \( h(a_v, b_v) \) is independent of \( v \), it follows that \( x^* \neq v \) and \( y^* \neq u \). Therefore \( u \) lies in the interior of a segment on \( K \) and hence is not an extreme point of \( K^0 \).

Thus every element of \( \Gamma_a \) leaves each extreme point of \( K \) fixed. Since \( K^0 \) is the closure of the convex hull of the set of extreme points and \( \dim K^0 = n \), there are at least \( n + 1 \) extreme points which do not lie in a hyperplane. There cannot be more since the elements of \( \Gamma_a \) are projectivities.

A hyperplane contains at most \( n \) extreme points no \( n - 1 \) of which lie in an \( (n - 2) \) flat; otherwise each element of \( \Gamma_a \) would leave the hyperplane pointwise fixed and \( \Gamma_a \) would not be transitive on \( I \). Proceeding in this way we see that \( K^0 \) has precisely \( n + 1 \) extreme points and hence is a simplex.

A point \( p \) of \( K \) is an Euler point if the following holds: \( K \) is differentiable at \( p \), all sections of \( K \) by two-flats through \( p \) which do not lie in the tangent hyperplane have curvatures at \( p \), and these obey the classical relations of Meusnier and Euler. Almost all points of \( K \) are Euler points (see [3, p. 23]). The Gauss curvature at an Euler point can be defined in the usual way as product of the principal curvatures.

Our aim is to show that \( K \) is an ellipsoid if it possesses an Euler point with non-vanishing Gauss curvature and \( \Gamma_K \) is transitive on \( I \).

(3) The definition and the properties of extreme points used here can be found in [1, Section 10].
The proof is based on a lemma whose explicit formulation would be very long.

In the $s$-dimensional euclidean space $E^s$ with Cartesian coordinates $x^1, \ldots, x^s$ let two closed continuous hypersurfaces which are starshaped with respect to the origin be given by

$$|x| = r(u) \quad \text{and} \quad |x| = q(u), \quad u \in S$$

where $S$ is the unit sphere. The numbers

$$\min r(u) = r_m, \quad \max r(u) = r_M, \quad \min q(u) = q_m, \quad \max q(u) = q_M$$

are finite and positive.

Let $\varphi$ be a projectivity of the $E^s$ completed to $P^s$ which leaves $O$ fixed and takes the first hypersurface into the second:

$$\varphi (r(u) u) = q(v) v, \quad v = v(u), \quad |v| = 1.$$  

Then $\varphi$ possesses a unique representation of the form

$$y^j = - \frac{\sum_{j=1}^{s} a_j x^j}{\sum_{j=1}^{s} b_j x^j + 1} \quad \text{or} \quad y = \frac{Ax}{b \cdot x + 1} \quad \text{with} \quad \det A \neq 0.$$  

The lemma states that

$$|a_j| \leq 2 q_M r_m^{-1}, \quad |\det A| \geq r_M^s q_m^s, \quad |b_j| < r_m^{-1}.$$  

The last relation is true if $b = (b_1, \ldots, b_s) = 0$. Assume $b \neq 0$, and put $u_b = b |b|^{-1}$. We conclude from

$$q(v) v = Ar(u) u (b \cdot r(u) u + 1)^{-1} = Au(b \cdot u + r^{-1}(u))^{-1}$$

that $b \cdot r(u) u + 1 \neq 0$ for all $u$. Since $b \cdot u_b > 0$, we see that

$$br(-u_b)(-u_b) + 1 = -|b| r(-u_b) + 1 > 0,$$

hence

$$|b_j| \leq |b| < r^{-1}(-u_b) \leq r_m^{-1}.$$  

Next put $u_k = \varepsilon (\delta_k^1, \ldots, \delta_k^s)$, $v(u_k) = v_k$ where $\varepsilon = 1$ if $b_k \geq 0$, $\varepsilon = -1$ if $b_k < 0$. Then

$$\varphi^2(v_k) = \sum_i ((v_m) v_k^i)^2 = \sum_i (a_k^i)^2 |b_k| + r^{-1}(u_k))^{-2}$$

and from $|b_k| \leq r_m^{-1}$

$$q_M^2 \geq (a_k^i)^2 4^{-1} r_m^2$$

which gives the first inequality in (5).
Finally let $A_i^k$ be the cofactor of $a_i^k$ in the matrix $A$ so that $(A_i^k \det^{-1} A) = A^{-1}$. Put $A_i = (\sum_k (A_i^k)^2)^{1/2} > 0$. Choose $\varepsilon = \pm 1$ such that $\varepsilon \sum_k b_k A_i^k \geq 0$ and put

$$w_i = \varepsilon A_i^{-1} (A_i^1, \ldots, A_i^s).$$

Then

$$\rho^2(w(w_i)) = \sum_i \left( \sum_k a_i^k A_i^k A_i^{-1} \right)^2 \left( \varepsilon \sum_k b_k A_i^k A_i^{-1} + r^{-1}(w) \right)^2 \leq A_i^{-2} \det^2 A r_M^2$$

and

$$\rho^2_m r_M^{-2s} \leq \prod_i A_i^{-2} \det^2 A \leq \det^2 (A_i^k) \det^2 A = \det^2 A,$$

where the second estimate follows from Hadamard's Inequality, [8, p. 34].

As an application we have

(6) Let $\{K_1^i\}$ and $\{K_2^i\}$ be sequences of closed convex hypersurfaces in $E^n$ tending to closed convex hypersurfaces $K^1, K^2$ containing the origin in their interiors. If a projectivity of $E^n$ (completed to $P^n$) exists which leaves 0 fixed and takes $K_1^i$ into $K_2^i$, then there is a projectivity $\varphi$ taking $K^1$ into $K^2$ (leaving 0 fixed).

It is easy to verify with examples that assuming 0 to remain fixed is essential.

Let $K_1^i$ and $K_2^i$ be given by

$$|x| = r_*(u) \quad \text{and} \quad |x| = \varrho_*(u), \quad u \in S.$$

Then for large $\nu$ and suitable $r_1, r_2, \varrho_1, \varrho_2$

$$0 < r_1 \leq r_*(u) \leq r_2, \quad 0 < \varrho_1 \leq \varrho_*(u) \leq \varrho_2.$$

The projectivity $\varrho_*$ has the form

$$y = A_* x (b_* x + 1)^{-1}, \quad \det A_* = \det (a_{ik}^*) \neq 0$$

and we conclude from (4) that

$$|a_{ik}^*| \leq 2 \varrho_2 r_1^{-1}, \quad |\det A_*| \geq r_2^{-s} \varrho_1^s, \quad |b_{ik}| < r_1^{-1}.$$

For a suitable subsequence $\{\mu\}$ of $\{\nu\}$ we have $a_{ik}^* \to a_{ik}, \ b_{ik} \to b_i$ with $\det A \neq 0$ and

$$q: y = A x (b x + 1)^{-1}$$

defines a projectivity satisfying the assertion.

Consider now a closed convex hypersurface $K$ in $E^n$ which possesses an Euler point $p$ with non-vanishing Gauss curvature. Denote by $H$ the tangent hyperplane of $K$ at $p$ and by $H_\sigma$ the hyperplane parallel to $H$ at distance $\sigma > 0$ intersecting $K$, let $p_\sigma$ be the intersection of $H_\sigma$ with the normal to $K$ at $p$. Project $C_\sigma = K \cap H_\sigma$ parallel to this normal on $H$
and dilate it in the ratio $1 : \sqrt{\sigma}$ obtaining $\bar{C}_0$. Then $\bar{C}_0$ tends for $\sigma \to 0+$ to an ellipsoid, namely to the Dupin Indicatrix of $K$ at $p$, see [3, Section 3].

Assume that $\Gamma_K$ is transitive on $I$ and give $q \in I$. A projectivity which maps $H_\sigma$ on a hyperplane $H'_\sigma$ through $q$ sends $C_\sigma$ into $C'_\sigma = H'_\sigma \cap K$. Also, $\varphi_0H_\sigma$ is a projectivity of $H_\sigma$ on $H'_\sigma$.

For a suitable sequence $\sigma_n \to 0+$ the $H'_\sigma$ converge to a hyperplane $H'$ and $C'_\sigma \to C' = H' \cap K$. A motion of $E^n$ takes $C'_\sigma$ into $C^2_\sigma$ in $H$ such that $q$ goes into $p$ and $C^2_\sigma$ converges to an image $C^1_\psi$ of $C'$ under a motion. With $C'_\sigma = C^1_\psi$ we have $\varphi_\rho C^1_\psi = C^2_\psi$ where $\varphi_\rho$ is a projectivity of $H$ which leaves $p$ fixed. It follows from (5) that a projectivity $\psi$ of $H$ will take $C^1_\rho$ into $C^2_\rho$. Therefore $C^1_\rho$ and hence $C'$ is an ellipsoid.

This discussion also shows that all sections of $K$ by hyperplanes parallel to $H^1$ on at least one side of $H'$ must be ellipsoids homothetic to $C'$. Since $q$ was arbitrary in $I$, this yields readily that $K$ is an ellipsoid. Thus we proved:

(7) **Theorem.** Let $K$ be a closed convex hypersurface in $E^n$ which possesses an Euler point with non-vanishing Gauss curvature. If the interior $I$ of $K$ possesses a transitive group of projectivities then $K$ is an ellipsoid.

The simplex and other examples show that the hypothesis concerning the Euler point is essential.

(7) not only substantiates our earlier statement that very harmless looking conditions may prove highly restrictive, but (7) will most probably also be essential in determining all $K$ for which $I_\Gamma$ is transitive on $I$ by allowing inductive reduction of the dimension.

In addition the arguments leading to (7) provide an elementary proof for the fact (deduced under (2) from deep results on transformation groups) that $K$ is an ellipsoid if $\Gamma_K$ possesses a compact subgroup $\Gamma_c$ which is transitive on $K$. Every point is an Euler point hence the Gauss curvature cannot vanish. Let $p \in I$. The orbit $\Gamma_c(p)$ of $p$ under $\Gamma_c$ is compact, hence stays away from $K$. The previous arguments show that through some point $q$ of $\Gamma_c(p)$ there passes a hyperplane $H$ such that the intersections of $K$ with the hyperplanes parallel to $H$ on at least one side of $H$ are homothetic ellipsoids. The same must hold for every point of $\Gamma_c(p)$ and hence for every point of $I$.

Concerning the $K$ with $\Gamma_K$ transitive on $I$ the following conjecture seems reasonable:

The convex bodies $K^0 = K \cap I$ for which $\Gamma_K$ is transitive on $I$ are convex hulls of a finite number of points and solid ellipsoids of dimension $\geq 2$ with the property that the dimension of the hull decreases if one of the points is omitted or one of the ellipsoids is replaced by a lower dimensional convex subset.
7. MINKOWSKI SPACES WITH PAIRWISE TRANSITIVE GROUPS OF MOTIONS

Consider again an \( n \)-dimensional timelike Minkowski space with affine coordinate \( x^1, \ldots, x^n \) and the translations \( x' = x + a \) as motions. In this section we are interested in additional motions and show first:

1. A motion \( \varphi \) of a timelike Minkowski space is an affinity.

The proof uses

2. Given a finite number of points \( a_1, \ldots, a_k \) then \( p \) with \( p < a_i \) \((i = 1, \ldots, k)\) exists.

This is obvious because \( F'(0) \) and hence \( F(p) = F'(0) + p \) is an open cone.

Putting generally \( \varphi x = x' \) we must show that \( a_2 = (1 - t)a_1 + ta_3 \) \((0 < t < 1)\) implies \( a_2' = (1 - t)a_1' + ta_3' \). Let \( p < a_i \) \((i = 1, 2, 3)\); then \( p' < a_i' \). Since translations are motions, we may assume \( p = p' = 0 \). Because the set \( (\leq) \) is open in \( A^n \times A^n \) we have \( b_1 = \mu_1 a_1 < a_3 \) for small \( \mu_1 > 0 \). The ray \( \lambda a_2 \) \((\lambda \geq 0)\) intersects the affine segment from \( b_1 \) to \( a_3 \) in a point \( b_2 \), so with suitable \( \lambda, \mu_2 \) between 0 and 1

\[
b_2 = \mu_2 a_2 = (1 - \lambda)b_1 + \lambda a_3.
\]

It follows from (3) in Section 5 that \( b_2' = (\mu_1 a_1)' = \mu_1 a_1' \) and similarly \( b_2' = \mu_2 a_2' = (1 - \lambda)b_1' + \lambda a_3' \).

The following lemma is needed:

3. In the \((x, y)\)-plane let \( y = g(x) \) be a strictly convex decreasing and differentiable curve in \((\gamma, \infty)\), where \( \gamma \geq 0 \), \( g(x) > 0 \) and \( g(x) \to \infty \) for \( x \to \gamma \). Let the tangent at \( q = (x, g(x)) \) intersect the \( x \)-axis at \( q_x \) and the \( y \)-axis at \( q_y \). If

\[
\beta \geq |q - q_y| : \beta = \lim_{x \to \alpha} g(x)
\]

then \( \gamma = 0 \) and \( \lim_{x \to \infty} g(x) = 0 \).

If \( |q - q_y| : |q - q_x| = \alpha > 0 \) then \( g(x) = k x^{-\alpha} \) \((0 < x < \infty)\).

For \( q_y = (0, g(x) - x g'(x)) \) hence

\[
\beta \geq |q - q_y| : |q - q_x| = -x g'(x)/g(x) \geq \alpha,
\]

whence

\[
\frac{g(x)}{g(y + 1)} \leq \left( \frac{\gamma + 1}{x} \right)^{\beta} \quad \text{for} \quad x \leq \gamma + 1,
\]

\[
\frac{g(x)}{g(y + 1)} \leq \left( \frac{\gamma + 1}{x} \right)^{\alpha} \quad \text{for} \quad x \geq \gamma + 1.
\]
A group $\Gamma$ of motions of a timelike space is called \textit{pairwise transitive} if given $p < q$ and $p' < q'$ with $\varrho(p, q) = \varrho(p', q')$ a motion in $\Gamma$ exists which takes $p$ into $p'$ and $q$ into $q'$. A \textit{triplewise transitive} group is defined analogously for $p < q < r$, $p' < q' < r'$ with $\varrho(p, q) = \varrho(p', r')$, $\varrho(p, r) = \varrho(p', r')$, $\varrho(q, r) = \varrho(q', r')$.

A metric Minkowski space with a pairwise transitive group of motions is euclidean, [G, p. 101]. A timelike Minkowski space with a pairwise transitive group of motions need not be a Lorentz space. It is when the group of motions is triplewise transitive. The problem of finding all timelike Minkowski spaces with pairwise transitive groups of motions leads to difficult unsolved problems on convex bodies and hypersurfaces, including the one concerning the $K$ with $\Gamma_K$ transitive on $I$. The remainder of this section discusses the facts which are known in this direction.

(4) \textit{A timelike Minkowski space with a pairwise transitive group of motions is finitely compact.}

If $p < q$, $p < q'$ and $\varrho(p, q) = \varrho(p, q') = \Delta > 0$ then a motion exists which leaves $p$ fixed and takes $q$ into $q'$. Applying this to $q = 0$ and using (1) we see that the convex hypersurface $K_A, f(x) = \Delta$ possesses a transitive group of central affinities. Since any convex hypersurface is almost everywhere differentiable (even twice differentiable), see [3, p. 23], $K_A$ is everywhere differentiable.

Let $q \in K_A$. The tangent plane $T_q$ of $K_A$ at $q$ intersects the light cone $C(0)$ in a set $S_q$ homeomorphic to $S^{n-2}$. A line $L$ in $T_q$ through $q$ intersects $S_q$ in two points $s, s'$. The ratio $|q - s| : |q - s'|$ of the cartesian distances is an affine invariant. As $S$ traverses $S_q$ this ratio attains a maximum $\beta$ and minimum $\alpha = \beta^{-1}$, which are independent of $q$.

According to (5.4) it suffices to show that no hyperplane separates a generator of $C(0)$ from $K_A$ and this follows at once from (3).

The two dimensional case is easily handled.

(5) \textit{A timelike Minkowski plane with a pairwise transitive group of motions can in suitable affine coordinates $x^1, x^2$ be represented as follows: $x < y$ means $x^1 < y^1$ and $x^2 < y^2$ and $\varrho(x, y) = (y^1 - x^1)^{1-\mu} (y^2 - x^2)^\mu$ with $0 < \mu < 1$.}

\textbf{Note.} $L_2^2$ corresponds to $\mu = 1/2$ and is characterized geometrically by the fact that it can be reflected in each geodesic.

For a proof we choose $x^1, x^2$ such that $E(0)$ is the first quadrant $x^1 > 0, x^2 > 0$. A centro-affinity mapping $q \in K_A$ on $q' \in K_A$ leaves the ratio $|q - q_x| : |q - q_x|$ of (3) fixed hence $x^2 = k(x^1)^{-\alpha}$ with $\alpha > 0$.

Because $f(\lambda x) = \lambda f(x)$ for $\lambda > 0$ the corresponding function is
text{$c(x^1)^{1-\mu}(x^2)\mu$, $0 < \mu < 1$, $c > 0$}, where $c$ may be omitted because different $c$ give isometric spaces.
To discuss the cases $n > 2$ we imbed $A^n$ in the $n$-dimensional projective space $P^n$ by adding a hyperplane $H_\infty$ and a coordinate $x^{n+1}$ so that $H_\infty$ is $x^{n+1} = 0$. The cone $C(0)$ intersects $H_\infty$ in $C_\infty$ which, because of (5.1), is a closed convex hypersurface in $H_\infty$ relative to a suitable $(n-2)$-flat as ideal locus. $C_\infty$ bounds a set $F_\infty$, the intersection of $F(0)$ with $H_\infty$.

A motion of the Minkowski space induces a projectivity of $H_\infty$ on itself which takes $C_\infty$ and $F_\infty$ into themselves.

Denote as stability group of a point $p$ in a timelike space the group of all motions which leave $p$ fixed. If the space has a pairwise transitive group of motions then this stability group is transitive on each sphere $S(p, \Delta) = \{x; p < x \text{ and } \varrho(p, x) = \Delta\}$. Using (6.3) we prove

(6) Theorem. A timelike Minkowski space for which the stability group of a point $p$ possesses a subgroup which is transitive and abelian on the spheres $S(p, \Delta)$ is in suitable affine coordinates given by: $x < y$ meaning $x^i < y^i$ ($i = 1, \ldots, n$) and

$$\varrho(x, y) = \prod_{i=1}^{n} (y^i - x^i)^{a_i}, \quad a_i > 0, \quad \sum a_i = 1.$$ 

For we conclude from (6.3) that $F_\infty \cup C_\infty$ is a simplex in $H_\infty$. Therefore we can choose affine coordinates in $A^n$ such that $F(0)$ is the set $x^i > 0$ ($i = 1, \ldots, n$). Put $S_\Delta = S(0, \Delta)$. Since $S_\Delta$ is almost everywhere differentiable and the stability group of 0 is transitive on $S_\Delta$, the sphere $S_\Delta$ is everywhere differentiable and being convex it is of class $C^1$ (see [3, p. 6]). For $y \in S_\Delta$ the tangent hyperplane of $S_\Delta$ is with $\partial f / \partial x^i = f_i$ given by

$$\sum x^i f_i(y) = f(y).$$

Denote its intersection with the $x^i$-axis by $\bar{a}_i$ and let $a^i$ be the $i$-th coordinate of $\bar{a}_i$. Then $\bar{a}^i = f(y)/f_i(y)$. The line through $\bar{a}_i$ and $y$ intersects $x^i = 0$ at a point $\mu_i \bar{a}_i + (1 - \mu_i)y$ with $\mu_i < 0$ and

$$\frac{\mu_i}{\mu_i - 1} \cdot \frac{f(y)}{f_i(y)} = y^i.$$ 

Then $0 < a_i = \mu_i(\mu_i - 1)^{-1} < 1$ and $a_i$ is invariant under the given motions. Therefore $f(x)$ satisfies the differential equations $f_i(x)|f(x) = a_i/x^i$ which yields

$$f(x) = k \prod_{i=1}^{n} (x^i)^{a_i} \quad \text{with} \quad \sum a_i = 1,$$

because $f(\lambda x) = \lambda f(x)$ for $\lambda > 0$. The factor $k$ has no significance since different $k$ lead to isometric spaces. This proves (6). We observe that

(7) $$x^i = \beta^i x^i + \Delta^i, \quad \beta^i > 0, \quad \prod (\beta^i)^{a_i} = 1.$$
is the identity component of the group of motions. This may not be the entire group, which may contain elements interchanging the coordinate axis depending on the values of the \( d^i \). The Lorentz space is for \( n > 2 \) not a special case because \( C_\infty \) is not an ellipsoid.

If a connected metric space possesses a group of motions which has on orbit containing an open set (\( \neq \emptyset \)) then the group is transitive on \( R \). The group (7) shows that the corresponding statement is not true for timelike spaces. It contains the subgroup

\[
\overline{x}^h = \beta^h x^h + \sigma^h, \quad \overline{x}^j = \beta^j x^j, \quad 1 \leq h \leq k \leq n, \quad k + 1 \leq j \leq n
\]

which is transitive on \( \{ x : x^j > 0 \} \).

Since, according to (6), the existence of a pairwise transitive group of motions does not characterize Lorentz spaces we now prove

(8) Theorem. A timelike Minkowski space with a triplewise transitive group of motions is a Lorentz space.

The hypothesis implies that for any two rays in \( H_\infty \) issuing from a given point \( r \) in \( F_\infty \) a projectivity of \( F_\infty \) induced by a centro-affinity of \( S_A \) exists which leaves \( r \) fixed and sends the first ray into the second. This implies that \( C_\infty \) possesses a transitive group of projectivities and hence is, by (6.1), an ellipsoid.

This alone does not imply that the space is a Lorentz space, for \( C_\infty \) is an ellipsoid for all the spaces \( L^n_a \) of Section 4 (see below). Consider a point \( q \in S_A \). The tangent hyperplane \( T_q \) of \( K_A \) at \( q \) intersects \( C(0) \) in an ellipsoid \( E_q \) with a center \( c_q \). If \( q = c_q \) for some \( q \) on \( S_A \), then this holds for all \( q \) on \( S_A \) and \( S_A \) is a branch of a hyperboloid, so that the space is Lorentzian. We prove that \( c_q \neq q \) is impossible.

If \( q_\infty \) and \( c_\infty \) are the projections of \( q \) and \( c_q \) form 0 on \( H_\infty \), then a projectivity of \( F_\infty \) corresponding to an element of the stability group of 0 which leaves \( q_\infty \) fixed also leaves \( c_\infty \) fixed, which contradicts the transitivity of the group of projectivities of \( F_\infty \) on the rays with origin \( q_\infty \).

As gauge function of \( L^n_a \) we may take

\[
f(x) = \left[ (x^n)^{\alpha} - \left( \sum_{i=1}^{n-1} (x^i)^2 \right)^{\alpha/2} \right]^{1/\alpha}, \quad x^n > \left[ \sum_{i=1}^{n-1} (x^i)^2 \right]^{1/2}.
\]

The light cone is the same for all \( \alpha \) and \( f(x) \) is invariant under the affinities

\[
\overline{x}^i = \sum_{k=1}^{n-1} a^i_k x_k \quad (i = 1, \ldots, n-1), \quad (a^i_k) \text{ orthogonal}, \quad \overline{x}^n = x^n.
\]

Therefore the stability group of 0 is (to a high degree) transitive on the generators of \( C(0) \), but the group of motions is not pairwise transitive.
As an immediate consequence of (6.2, 7) we have:

(9) **Theorem.** The time cone \( C(0) \) of a timelike Minkowski space is elliptic if one of the following conditions a), b) is satisfied.

a) The stability group of 0 is transitive on the generators of \( C(0) \).

b) The group of motions is pairwise transitive and a cross-section of \( C(0) \) by a hyperplane not through 0 possesses an Euler point with non-vanishing Gauss curvature.

Under either condition the space need not be Lorentzian.

We know the last assertion to be true for a) and establish it for b) as follows. Let \( C = C(0) \) be given by \( (x^n)^2 - \sum_{i=1}^{n-1} (x^i)^2 = 0, \ x \geq 0 \) and consider the group \( \Gamma^* \) of central affinities which maps \( C \) on itself and leaves the generator \( G: x_n = x_1, x_2 = \ldots = x_{n-1} = 0 \) of \( C \) fixed. A sphere \( S_A = \{ x, f(x) = \Delta > 0, x_n > 0 \} \) must have the property that a variable segment tangent to \( S_A \), cut out by \( C \) and with one endpoint on \( G \) be divided in a constant ratio by the point of contact with \( S_A \). In each plane through \( G \) the metric must be of the type described in (5) with the same \( \mu \) for all planes. A simple calculation shows that \( S_A \) must have the form

\[
(x^n - x_1)^{2\mu - 1} \left( (x^n)^2 - \sum_{i=1}^{n-1} (x^i)^2 \right)^{1-n} = c, \quad 0 < \mu < 1.
\]

The group \( \Gamma^* \) considered as group of projectivities of \( H_\infty \) is the group of those hyperbolic motions of \( \sum_{i=1}^{n-1} (x^i)^2 < (x^n)^2 \) which leave the point \( x^n = x^1 = 1, x^2 = \ldots = x^{n-1} = 0 \) fixed. They take the hypersurfaces (10) into themselves. For \( n = 3 \) they are found in Lie [10, p. 221] in the form obtained from (10) by the coordinate transformation \( x^n - x^1 = u_1, \ x^n + x^1 = u_n, \ x^i = u_i \) for \( i = 2, \ldots, n-1 \), i.e. \( u_1^{2\mu - 1} (u_n u_1 - \sum_{i=2}^{n-1} (u_i)^2)^{1-\mu} = c \)

or, for \( n = 3, u_3 = u_2^2 u_1^{-1} + c^1 u_1^{(\mu - 1) - 1} \) where \( c^1 > 0 \).

One checks readily that the Gauss curvature is positive. The fact that \( u_2, \ldots, u_{n-1} \) occur for \( n > 3 \) only in the combination \( \sum_{i=2}^{n-1} u_i^2 \) then shows that (10) is a convex hypersurface in \( x^n > [\sum_{i=2}^{n-1} (x^i)^2]^{1/2} \).

For \( n = 3 \) one easily sees that the metrics given by (5) and (10) exhaust the timelike Minkowski spaces with pairwise transitive groups of motions either by examining Mostow [12] for the groups acting as a plane which can be groups of motions of a Hilbert geometry or by proving that the triangle and the ellipse are the only closed convex curves in the plane whose interiors possess transitive groups of affinities, compare also [G, p. 370]. Thus
The three dimensional timelike Minkowski spaces with pairwise transitive groups of motions are in suitable affine coordinates given by the gauge functions

$$\prod_{i=1}^{3} (x^i)^a_i, \quad a_i > 0, \quad \sum a_i = 1, \quad x^i > 0$$

or

$$(x^3 - x^1)2^{\mu - 1}(x^3)^2 - (x^1)^2 - (x^2)^2\mu, \quad 0 < \mu < 1, \quad x^3 > [(x^1)^2 + (x^2)^2]^{1/2}.$$ 

Finally we observe that the group $I^*$ as subgroup of the group $I$ of the motions of a Lorentz space is transitive on the side containing $F(0)$ of the hyperplane tangent to $C(0)$ along $G_s$ so that we have again a subgroup whose orbit is a proper open subset of the space.

8. TIMELIKE HILBERT GEOMETRIES

The analogues to the metric Hilbert geometries encountered in Section 6 are of special interest because they provide a test for our axioms. Two possibilities present themselves, one for a timelike space, the other for a locally timelike space which cannot be consistently ordered. The latter seems more natural, because it includes as special case the hyperbolic geometry defined in the exterior of an ellipsoid. In fact, in the timelike case our axiom $G'_2$ cannot be satisfied if all other axioms are.

Let $K_1, K_2$ be complete convex hypersurfaces in $A^n$ such that the closed convex sets $K_1^0, K_2^0$ bounded by $K_1$ and $K_2$ are disjoint. (If $K_1$ or $K_2$ is a hyperplane this condition defines $K_1^0$ or $K_2^0$.) We consider all open oriented affine segments $L(a_1, a_2)$ where $a_i \in K_i$ and $a_0 = (1 - \theta)a_1 + a_2 \in K_1^0 \cup K_2^0$ for $0 < \theta < 1$.

The space $R$ is the union of all $L(a_1, a_2)$ with the topology of $A^n$. The relation $p < q$ is defined to mean that $p$ and $q$ lie on an $L(a_1, a_2)$ and $q$ follows $p$. Then $p < q$ and $q < r$ imply $p < r$. For $p < q$ on $L(a_1, a_2)$ we define

$$h(p, q) = \frac{1}{2}k\log R(p, q, a_1, a_2) = \frac{1}{2}k\log R(p, q, a_2, a_1).$$  

(For the following compare [G, Section 18.]) With this distance $L(a_1, a_2)$ is isometric to the real axis.

Assume that $p < q < r$ are not collinear (see Figure). Let $p, q \in L(a_1, a_2); q, r \in L(b_1, b_2); p, r \in L(c_1, c_2)$ the affine lines through $c_i$ and $b_i$ ($i = 1, 2$) intersect at a point $t$ (possibly at $\infty$) and the line through $t$ and $q$ intersects the segment from $p$ to $r$ in a point $s$. Finally, the line containing $L(c_1, c_2)$ intersects the line through $a_i$ and $b_i$ in a point $c_i'$. Then

$$R(p, q, a_2, a_1) = R(p, q, c_2', c_1'), \quad R(q, r, b_2, b_1) = R(s, r, c_2', c_1'),$$
\[ R(p, q, a_2, a_1) = R(q, r, b_2, b_1) = R(p, r, c_2, c_1) \leq R(p, r, c_2, c_1) \] with equality only when \( c_1 = c'_1 \) and \( c_2 = c'_2 \). Therefore \( h(p, q) + h(q, r) < h(p, r) \) for non-collinear \( p < q < r \), if and only if no proper affine segments \( T_i \subset K_i \cap \overline{R} \) exist which are coplanar. We will assume that this condition is satisfied.

The sphere \( h(p, x) = \sigma > 0 \) lies on a strictly convex hypersurface; it possesses at a given point \( r \) a supporting hyperplane \( H_r \) touching it only at \( r \); and \( H_r \) separates \( p \) from the sphere (except for \( r \)).

To see this let \( H_i \) be a supporting hyperplane of \( K_i \) at \( c_i \) and \( H_r \) the hyperplane through \( r \) and the \((n-2)\)-flat \( F = H_1 \cap H_2 \) (possibly at \( \infty \)). For convenience consider the point \( x = H_r \cap L(a_1, a_2) \). Then \( H_i \) intersects \( L(a_1, a_2) \) in \( a'_i \) and according to our assumption either \( a_1 \neq a'_1 \) or \( a_2 \neq a'_2 \) or both. Therefore

\[ R(p, x, a_2, a_1) < R(p, x, a'_2, a'_1) = R(p, r, c_2, c_1). \]

Examining the space with respect to our axioms \( T_1, T_2, T_3 \) one finds that \((<)\) is open in \( R \times R \) if and only if no \( L(a_1, a_2) \) lies on a supporting line (or a supporting hyperplane) of \( K_1 \) or \( K_2 \). It is clear that \((<)\) cannot be open if \( L(a_1, a_2) \) exist which lie on supporting lines of \( K_1 \) or \( K_2 \) (or both) which contain interior points of \( R \). If \( L(b_1, b_2) \) lies on the boundary of \( R \) then an \( L(a_1, a_2) \) of the previous type exists. Thus for \((<)\) to be open it is necessary and sufficient that no \( L(a_1, a_2) \) lying on a supporting of \( K_1 \) or \( K_2 \) exists.
If we assume this, then $G_2'$ is not satisfied. For then there are points $p \in R$ with a sequence $L(a_i^*, a_2^*)$ containing $p$ and such that one of the sequences $\{a_i^*\}$, say $\{a_2^*\}$, tends to a point $a_2$, whereas $\{a_i^*\}$ diverges, so that $L(a_i^*, a_2^*)$ tends to a ray through $p$ with $a_2$ as endpoint. (Actually the points $p$ for which such $L(a_i^*, a_2^*)$ do not exist are exceptional.) If we choose $x_\sigma$ on $L(a_i^*, a_2^*)$ such that $p < x_\sigma$ and $h(p, x_\sigma) = \sigma > 0$, then the definition of $R(p, x_\sigma, a_2^*, a_1^*)$ yields that $x_\sigma$ tends to a point $x$ on $R$ different from $p$ and $a_2$. But $p < x$ is not true since $R$ does not intersect $K_1$. This with $h(p, x_\sigma) = \sigma$ contradicts $C_2'$.

All other axioms for a timelike $G$-space are satisfied. The space is geodesically complete and all geodesics are lines.

We summarize our results without repeating all details:

(2) The Hilbert metric outside two convex hypersurfaces $K_1, K_2$ satisfies all axioms for a timelike $G$-space with the exception of $G_2'$ if and only if no proper segments $T_i \subset K_i \cap \overline{R}$ exist which are coplanar and no $L(a_1, a_2)$ lies on a supporting line of $K_1$ or $K_2$. Under these conditions the axiom $G_2'$ does not hold.

The $L(a_1, a_2)$ are isometric to the real axis and are the geodesics, so that the space is geodesically complete.

The case where $K_1$ is a hyperplane and $K_2$ is a strictly convex hypersurface is the timelike analogue to Funk's "Geometrie der spezifischen Massbestimmung", see [7]. If $K_1$ is considered as hyperplane at infinity of a euclidean space with distance $e(x, y)$ then (1) becomes

$$h(p, q) = \frac{1}{2} k \log \frac{e(p, a_2)}{e(q, a_2)}.$$

Notice that the spheres $h(p, x) = \sigma > 0$ are homothetic to $K_2$. But $(<)$ is not open and $G_2'$ does not hold.

To define the locally timelike analogue to Hilbert's Geometry let $K$ be a closed convex hypersurface in the $n$-dimensional projective space $P^n$ and $I$ its interior. This means that hyperplanes not intersecting $K$ exist and that $K$ is convex with interior $I$ relative to such a hyperplane considered as locus at infinity. Put $K^o = K \cap I$.

For $p \notin K^o$ take the union $C(p)$ of all supporting lines of $K^o$ through $p$, i.e. the lines through $p$ intersecting $K$ but not $I$ Then $C(p) \cap K$ has two nappes $C_1(p), C_2(p)$, and $C(p)$ bounds together with $K$ an open set $F_i(p)$. Each point $p$ has a neighborhood $U(p)$ such that with a proper choice of the notations $F_1(q), F_2(q)$ the set $F_i(q)$ depends continuously (in an obvious sense) on $q$ for $q \in U(p)$. Such a choice of notation is not possible in all of $R = P^n - K^o$, which will be our space.

In $U(p)$ define $q <_p r$ by $r \in F_1(q)$. Then $q <_p r$ and $r <_p s$ imply $q <_p s$. Any projective line which intersects $I$ intersects $K$ in two points
\(a_1, a_2\) and for any two points \(q, r\) in \(R\) on this line we define

\[h(q, r) = \frac{1}{2}k|\log R(p, q, a_1, a_2)|.\]

If \(q <_p r\) then the line through \(q\) and \(r\) intersects \(I\) and we define

\[e_p(q, r) = h(p, q).\]

Then \(T_4\) is satisfied because \(r \in F_1(q)\) is equivalent to \(q \in F_2(r)\). The time inequality

\[e_p(q, r) + e_p(q, s) \leq e_p(p, s)\]

for \(q <_p r <_p s\) follows from the previous discussion, which also shows that equality holds only if \(q, r, s\) are collinear, unless certain pairs of segments lie on \(\overline{K}\). This need not be made precise because \(G_2\) is valid if and only if no segment of a projective line lies on \(K\).

For, if \(L\) is a supporting line of \(K\) containing a proper segment with endpoints \(b_1, b_2\), let \(p \in L - L \cap K\) and \(L_v\) a line through \(p\) intersecting \(I\) and whose intersections with \(K\) tend to \(b_1\) and \(b_2\). If \(q\) is a point of \(L\) in \(U(p)\) and different from \(p\) and \(q, eL_v\) tends to \(q\), then

\[\lim h(p, q_v) = \frac{1}{2}k|\log R(p, q, b_1, b_2)| > 0\]

although neither \(p <_p q\) nor \(q <_p p\) which contradicts \(G_2\).

On the other hand \(h(p, q_v) \to 0\) if \(L \cap K\) is a point.

The remaining axioms are easily verified. So we have:

(3) **Theorem.** Let \(K\) be a strictly convex hypersurface in \(P^n\) with interior \(I\). If \(R = P^n - K \cap I\) and \(U(p), e_p(q, r)\) are defined as above, then \(R\) is a geodesically complete locally timelike \(G\)-space which cannot be consistently ordered.

The geodesics lie on the projective lines intersecting \(I\). The length of the arc of a geodesic from \(p\) to \(q\) is \(h(p, q)\).

The motions of \(R\) are the restrictions to \(R\) of the projectivities of \(P^n\) which map \(K\) on itself.

That such projectivities are motions is evident and that every motion has this form is seen in a similar way as (7.1).

For a motion \(\varphi\) of \(R\) we have \(\varphi C(p) = C(\varphi p)\). Therefore, if the group of motions is transitive on \(R\), any \(C(p)\) can be moved into any \(C(q)\). We note that in this case the strict convexity of \(K\) can be proved:

(4) **If for a closed convex surface \(K\) with interior \(I\) in \(P^n\) the group of projectivities mapping \(K\) on itself is transitive on \(R = P^n - K \cap I\) then \(K\) is strictly convex.**

According to a theorem by Ewald and Rogers [6], if \(H\) is a hyperplane not intersecting the closed convex hypersurface \(K\) then for almost
all $C(p)$ with $p \in H$ the generators of $C(p)$ touch $K$ in one point only. Our hypothesis implies that this is true for every $C(q)$.

If the group $\Gamma$ of motions of $R$ is transitive on $R$ then for no $x \in K$ the orbit $\Gamma(x)$ can consist of $x$ alone since $\Gamma$ would then map the union of all supporting hyperplanes of $K$ at $x$ on itself and hence would not be transitive on $R$.

Let $y \in \Gamma(x) - \{x\}$. Give a point $z$ on $K$ and a neighborhood $V(z)$ of $z$ on $K$. Choose $q$ in $R$ so close to $z$ that one of the sets on $K$ bounded by $C(q) \cap K$ lies in $V(z)$. For any point $p$ in $R$ collinear with $x$ and $y$ let $\varphi \in \Gamma$ take $p$ into a point of $F_1(q) \cup F_2(q)$. Then one of the points $x, y$ goes into a point of $V(z)$. Therefore $\Gamma(x) = \Gamma(x) \cup \Gamma(y)$ is dense in $K$. If $\Gamma(x)$ contains an open subset of $K$ no other orbit $\Gamma(u) \ (u \in K)$ can be dense in $K$, hence $\Gamma(x) = K$ and $K$ is an ellipsoid by (6.2).

The only other alternative is that $\dim \Gamma(x) < n - 1$, [9, p. 46]. Since $\Gamma(x)$ is the countable union of compact sets, the number of distinct orbits $\Gamma(x) \ (x \in K)$ must be non-countable. So we see: Unless $K$ is an ellipsoid it is the union of a non-countable number of distinct orbits $\Gamma(x)$ each of which is dense on $K$. It is very probable that this cannot happen, but the absence of information on locally compact transformation groups seems to make a proof difficult at the present time except for $n = 3$. For, $\dim \Gamma \geq n$ and $\dim \Gamma(x) \leq n - 2$ for $x \in K$ imply that the stability group of $x \in K$ has at least dimension 2 and $\Gamma$ acts effectively on $K$.

A group of motions $\Gamma$ of a locally timelike $G$-space $R$ is transitive on the line elements of $R$ if for two given line elements $L_\mu$ and $L_\nu$, a motion in $\Gamma$ exists which maps $p$ on $p'$ and the geodesic containing $L_\mu$ on the geodesic containing $L_\nu$ (or equivalently, which maps all sufficiently small segments in $L_\mu$ on elements of $L_\nu$).

It is clear that: in a locally timelike Hilbert space $K$ must be an ellipsoid if the group of motions is transitive on the line elements for then $\Gamma(x) \ (x \in K)$ is open on $K$.

If $K$ is an ellipsoid we speak of a locally timelike hyperbolic geometry. It is the restriction to the pairs $x \leq \nu y$ of the so-called exterior hyperbolic geometry, see [15], which is an indefinite metric defined for pairs in any position (compare footnote (2) on p. 28). Thus we may say

(5) A locally timelike Hilbert geometry is hyperbolic if its group of motions is transitive on the line elements.

We repeat that transitivity on the points suffices for $n = 2, 3$ and probably for all $n$.

For $n = 2$ this geometry gives rise to a second locally timelike $G$-space. The geodesics through a point $p$ are the projective lines which do not intersect or touch the ellipse $K$ (see [15]). Whether there is analogue to this for arbitrary strictly and/or differentiable closed convex $K$. 
is not known. The standard definition of distance when $K$ is an ellipse uses the conjugate complex intersections of the projective lines with $K$ and cannot be generalized.

If in (1) we choose $K_1$ and $K_2$ as the branches of the hyperboloid 
\[(x^n)^2 - \sum_{i=1}^{n-1} (x^i)^2 = 1,\] the resulting timelike space is, essentially, a restriction of the locally timelike hyperbolic geometry. By changing coordinates it can be given a form analogous to the Poincaré model of hyperbolic geometry in $y^n > 0$, namely

\[ds = \left[ (dy^n)^2 - \sum_{i=1}^{n-1} (dy^i)^2 \right]^{1/2} (ky^n)^{-1}\]

and $x < y$ means $y^n - x^n > \left[ \sum_{i=1}^{n-1} (y^i - x^i)^2 \right]^{1/2}$. From this form it would be hard to guess that the space, which is not geodesically complete, is part of a geodesically complete space. We have again the phenomenon of a subgroup of the group of all motions (of the locally timelike hyperbolic geometry) which has an open orbit.
REFERENCES

English translation: