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On the geometric definition of the quasi-conformality

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To the memory of Professor Stefan Bergman

Abstract. This note contains the quasi-conformality criteria I and II: given n linearly independent $(n-1)$-planes in R^n, a sense-preserving homeomorphism $f\colon G \to G^*$ (G and G^* — domains in R^n) is a quasi-conformal one iff it verifies Grötzsch's inequality (5) $M \leqslant QM^*$, where M designates the module of an arbitrary rectangular n-parallelotope $\Omega_n \subset G$ with the base parallel to one of these $(n-1)$-planes and M^* the module of $\Omega_n^* = f(\Omega_n)$. The proof for the ACL_n-property (Theorem 2) is detailed, by using results of J. Väisälä and S. Agard.

The geometric definition of the qcty (quasi-conformality) asserts that a homeomorphism $f\colon G \to G^*$, where G and $G^* = f(G)$ are domains in R^n, is a K-qc (K-quasi-conformal) mapping for a constant $K \geqslant 1$ if the module $M(\Gamma)$ of every path family Γ in G and the module $M(\Gamma^*)$ of the image family $\Gamma^* = f(\Gamma)$ satisfy the inequalities of Grötzsch's type:

$$K^{-1} M(\Gamma) \leqslant M(\Gamma^*) \leqslant KM(\Gamma) \tag{1}$$

([7], 13.1). Various simplifications were given to this definition, proving that it suffices to suppose inequalities (1) to be verified only for certain path families, for instance for the families which define the module of a ring in G or of a cylinder, or even of a right cylinder in G ([7], 36, 34.8.7). In [3] we showed that it is sufficient to consider only families Γ which define the module of a rectangular n-parallelotope or n-parallelipiped (which is a particular case of right cylinder) and only for the parallelotopes whose height is parallel to one of n linearly independent given directions Π_1^k, $k = 1, \ldots, n$ in R^n.

DEFINITION 1. Let us consider an n-segment

$$I_n = \{x \in R^n\colon x = (x_1, \ldots, x_n),\ 0 \leqslant x_j \leqslant l_j,\ j = 1, \ldots, n\}$$

as the direct product $I_n = I_{n-1} \times I_1$, where $l_j > 0$ are arbitrary,

$$I_{n-1} = \{x \in I_n\colon x = (x_1, \ldots, x_{n-1}, 0)\} \quad \text{and} \quad I_1 = \{x \in I_n\colon x = (0, \ldots, 0, x_n)\}.$$

We call I_{n-1} the base of I_n and I_1 its height. We designate by $\Sigma(I_n, I_{n-1})$ the family of all the 1-segments in I_n which are parallel to I_1 and by $\mathcal{S}(I_n, I_{n-1})$ the family of all the paths which join in I_n the base I_{n-1} with the parallel face to I_{n-1}.

The module $M(I_n, I_{n-1})$ of I_n with respect to the base I_{n-1} is defined by the module of the family $\mathcal{S}(I_n, I_{n-1})$ and is equal to the module of $\Sigma(I_n, I_{n-1})$:

(2) $$M(I_n, I_{n-1}) = M[\mathcal{S}(I_n, I_{n-1})] = M[\Sigma(I_n, I_{n-1})].$$

All homeomorphisms considered in this paper are supposed to be sense-preserving.

DEFINITION 2. A topological cube P_n will be the image of I_n by a homeomorphism $\varphi\colon I_n \to P_n$ which gives also the vertices and the faces of P_n. We take $P_{n-1} = \varphi(I_{n-1})$ as the base of P_n, and define the module of P_n with respect to P_{n-1} by

(3) $$M(P_n, P_{n-1}) = M[\mathcal{S}(P_n, P_{n-1})],$$

where $\mathcal{S}(P_n, P_{n-1}) = \varphi[\mathcal{S}(I_n, I_{n-1})]$.

DEFINITION 3. A rectangular n-parallelotope Ω_n will be the image of an n-segment I_n by a translation and a rotation φ. In this case we put

$$\Sigma(\Omega_n, \Omega_{n-1}) = \varphi[\Sigma(I_n, I_{n-1})], \quad \Omega_{n-1} = \varphi(I_{n-1}), \quad \Omega_1 = \varphi(I_1),$$

and we have

(4) $$M(\Omega_n, \Omega_{n-1}) = M[\Sigma(\Omega_n, \Omega_{n-1})] = M[\mathcal{S}(\Omega_n, \Omega_{n-1})].$$

Ω_{n-1} will be the base and Ω_1 the height of Ω_n.

DEFINITION 4. Let $f\colon G \to G^*$ be a homeomorphism and Π_{n-1} an $(n-1)$-plane. We say that f *verifies the Grötzsch inequality*

(5) $$M \leqslant QM^* \quad (Q \text{ — a positive constant})$$

for the $(n-1)$-plane Π_{n-1}, if (5) holds when M designates the module of an arbitrary rectangular n-parallelotope Ω_n with the base Ω_{n-1} parallel to Π_{n-1}, $M = M(\Omega_n, \Omega_{n-1})$, $\Omega_n \subset G$, and $M^* = M(\Omega_n^*, \Omega_{n-1}^*)$ the module of the topological cube $\Omega_n^* = f\circ\varphi(I_n) = f(\Omega_n)$ with respect to the base $\Omega_{n-1}^* = f(\Omega_{n-1})$.

DEFINITION 5. Let $x_0 \in G$ be an A-point (point of regularity) of the homeomorphism $f\colon G \to G^*$ and Π_{n-1} an $(n-1)$-plane through x_0. We say that f *verifies the Grötzsch inequality* (5) for the $(n-1)$-plane Π_{n-1} at the point x_0 if (5) holds for every $\Omega_n \subset G$, where φ is the translation $0 \mapsto x_0$ and a rotation such that $\Omega_{n-1} = \varphi(I_{n-1})$ be contained in Π_{n-1}.

The family which plays in (5) the role of Γ in (1) is evidently

$\mathcal{S}(\Omega_n, \Omega_{n-1})$. In what follows it will be useful to consider the orthogonal direction to Π_{n-1}, which will be designated by Π_1.

By means of Rengel's inequalities we established in [2], see also [3], § 1, 1.4, under the hypothesis of Definition 5:

THEOREM 1. *Grötzsch's inequality* (5) *for f at x_0 with respect to Π_{n-1} implies*

$$J_1^n \leqslant Q J_n, \tag{6}$$

where J_n is the Jacobian of f at x_0 and J_1 the norm of the derivative of f on the direction Π_1 at x_0, i.e. the Jacobian of $f|_{\Pi_1}$ at x_0.

Inequality (6) remains valid even if x_0 is only a differentiability point of f and $J_n = 0$ at x_0, i.e. in this case $J_1 = 0$ too.

Combining this theorem and its consequences ([3], § 2) with classical devices given by Strebel, Pfluger, Väisälä and Agard, we prove the following *geometric qcty criteria* for $Q \geqslant 1$ (1):

I. A homeomorphism $f\colon G \to G^*$ which verifies Grötzsch's inequality (5) for all the n coordinate $(n-1)$-planes is $Q^{n-1} n^{n(n-1)/2}$-qc.

II. Let Π_{n-1}^k, $k = 1, \dots, n$, be n linearly independent $(n-1)$-planes in R^n. A homeomorphism $f\colon G \to G^*$ which verifies Grötzsch's inequality (5) for these $(n-1)$-planes is a qc mapping.

The proof of these criteria is based on the reduction to the analytic definition of the qcty in Väisälä's sense ([7], 34.6). Thus one has to demonstrate that f is ACL_n (hence, according to a result of Väisälä [6], n-a.e. differentiable) and then, one obtains by means of the consequences of Theorem 1 ([3], § 2, 2.4–2.5) in the case of the first criterium the inequality

$$|f'(x)|^n \leqslant Q n^{n/2} J_n, \tag{7}$$

n-a.e. in G, and in the second case an analogue inequality

$$|f'(x)|^n \leqslant C J_n, \tag{8}$$

where the constant C depends on the $(n-1)$-planes for which f verifies the Grötzsch inequality (5) ([3], § 3, 3.2).

The fact that f is ACL_n is a consequence of the following

THEOREM 2 ([3], § 3, 3.1). *A homeomorphism $f\colon G \to G^*$ which verifies Grötzsch's inequality* (5) *for an $(n-1)$-plane Π_{n-1} has the properties*:

1° *f is* AC *on the orthogonal direction Π_1 to Π_{n-1}, i.e. if Ω_n is an arbitrary rectangular n-parallelotope in G with the base Ω_{n-1} parallel to Π_{n-1}, f is* AC *on $(n-1)$-a. every 1-segment in $\Sigma(\Omega_n, \Omega_{n-1})$, where $(n-1)$-a.e. refers to the Lebesgue measure in Ω_{n-1}.*

(1) The case $Q < 1$ need supplementary conditions on Q ([3], § 2, § 3).

2° The derivative of f on the direction Π_1, which exists n-a.e. in Ω_n and is measurable, is L^n-integrable on Ω_n. Its norm is equal to J_1, the Jacobian of $f|_{\Pi_1}$.

3° At every differentiability point of f it holds (6), where J_n is the Jacobian of f.

The purpose of this note is to detail the proof of assertions 1° and 2° of Theorem 2, assertion 3° being already contained in Theorem 1.

In this aim we use for 1° a method of Väisälä (Lemma 2 in [6]) which is based on [5] and [4], and for 2° a method of Agard (Lemma 4.3 in [1]). The same dealing is indicated in [7], 34.8.6.

Proof. Let us suppose that Π_{n-1} is the $(n-1)$-plane $x_n = 0$, hence that Π_1 is the direction of the axis $0x_n$. We write every point $x \in R^n$ in the form $x = (\xi, x_n)$ with $\xi = (x_1, \dots, x_{n-1})$. The n-parallelotopes Ω_n with the base parallel to Π_{n-1} will be now n-segments and we shall generally denote one of them by $I_n = I_{n-1} \times I_1$. (We change the notation in Definition 1 in the sense that by I_n, I_{n-1}, I_1 we understand now the image of the segments in Definition 1 by an arbitrary translation.) Evidently we suppose $I_n \subset G$.

Fix an arbitrary n-segment $\mathscr{I}_n = \mathscr{I}_{n-1} \times \mathscr{I}_1$ in G.

The measure function

$$\Phi(I_{n-1}, I_1) = m_n[f(I_n)], \qquad I_n = I_{n-1} \times I_1, \tag{9}$$

is a positive additive segment function of the variable $I_{n-1} \subset \mathscr{I}_{n-1}$ as well as of the variable $I_1 \subset \mathscr{I}_1$.

As a function of I_{n-1} (I_1 being fixed), Φ admits a derivative

$$D\Phi(\xi, I_1) = \lim_{\varrho \to 0} \frac{\Phi[I_{n-1}(\xi, \varrho), I_1]}{\varrho^{n-1}}$$

for $(n-1)$-a.e. $\xi \in \mathscr{I}_{n-1}$, where $I_{n-1}(\xi, \varrho)$ is an $(n-1)$-cube with the center at ξ and the side of length ϱ. It holds

$$\int_{I_{n-1}} D\Phi(\xi, I_1)\, d\sigma_{n-1} \leqslant \Phi(I_{n-1}, I_1). \tag{10}$$

As a function of I_1 (I_{n-1} being this time fixed), Φ admits the derivative

$$\Phi'(I_{n-1}, x_n) = \lim_{\varrho \to 0} \frac{\Phi[I_{n-1}, I_1(x_n, \varrho)]}{\varrho}$$

for a.e. $x_n \in \mathscr{I}_1$, where $I_1(x_n, \varrho) = [x_n - \frac{1}{2}\varrho, x_n + \frac{1}{2}\varrho]$. It follows also

$$\int_{I_1} \Phi'(I_{n-1}, x_n)\, d\sigma_1 \leqslant \Phi(I_{n-1}, I_1). \tag{11}$$

Taking into account that for $I_1 = [x_n', x_n'']$, $x_n'' - x_n' = a$, and $I_{n-1} = I_{n-1}(\xi, \varrho)$ the module $M = M(I_n, I_{n-1}) = (\varrho/a)^{n-1}$ and that according to Rengel's inequality the module $M^* = M[f(I_n), f(I_{n-1}\times\{x_n'\})] \leqslant \Phi(I_{n-1}, I_1)/\delta(I_{n-1}, I_1)^n$, where $\delta(I_{n-1}, I_1)$ is the distance between $f(I_{n-1}\times\{x_n'\})$ and $f(I_{n-1}\times\{x_n''\})$ and $\lim_{\varrho\to 0}\delta(I_{n-1}, I_1) = \delta(\xi, I_1) = |f(\xi, x_n'')-f(\xi, x_n')|$, Grötzsch's inequality (5) implies

$$\left(\frac{\delta(I_{n-1}, I_1)}{a}\right)^n \leqslant \frac{Q}{a}\,\frac{\Phi(I_{n-1}, I_1)}{\varrho^{n-1}}. \tag{12}$$

1° Following the Strebel–Pfluger–Väisälä device one proves that f is AC on each segment $\mathscr{I}_{1\xi} = \{\xi\}\times\mathscr{I}_1$, where ξ is an arbitrary point in $\mathscr{I}_{n-1}$ at which $D\Phi(\xi, \mathscr{I}_1)$ exists. Let us choose for every $\varepsilon > 0$ a finite number N of segments $I_1^k = [x_{nk}', x_{nk}'']$ in $\mathscr{I}_1$ with disjoint interiors and with $\sum_{k=1}^{N} a_k < \varepsilon$, $a_k = x_{nk}''-x_{nk}'$, write (12) for each $I_n^k = I_{n-1}\times I_1^k$, sum these inequalities, and by Hölder's inequality one obtains

$$\left(\sum_{k=1}^{n}\delta(I_{n-1}, I_1^k)\right)^n \leqslant \varepsilon^{n-1}\,Q\,\frac{\Phi(I_{n-1}, \mathscr{I}_1)}{\varrho^{n-1}}.$$

By letting here ϱ tend to zero the deduced inequality shows that f is AC on $\mathscr{I}_{1\xi}$.

2° In order to prove now that $\partial f/\partial x_n$, which exists n-a.e. in G and is measurable, is locally L^n-integrable in G, we adapt Agard's device. Fix for the moment an arbitrary I_1 in $\mathscr{I}_1$ and choose ξ such that $D\Phi(\xi, I_1)$ exists. Further write (12) for $I_{n-1}\times I_1$, pass to the limit $\varrho\to 0$ and integrate over an arbitrary but fixed $(n-1)$-segment $\tilde{I}_{n-1}\subset\mathscr{I}_{n-1}$. By (10) one has

$$\int_{\tilde{I}_{n-1}}\left(\frac{\delta(\xi, I_1)}{a}\right)^n d\sigma_{n-1} \leqslant \frac{Q}{a}\,\Phi(\tilde{I}_{n-1}, I_1). \tag{13}$$

Now choose $x_n\in\mathscr{I}_1$ such that $\Phi'(\tilde{I}_{n-1}, x_n)$ exists, take $I_1 = I_1(x_n, 1/\nu)$, and by Fatou's Lemma inequality (13) gives

$$\int_{\tilde{I}_{n-1}}\varliminf_{\nu\to\infty}\left(\frac{\delta(\xi, I_1)}{1/\nu}\right)^n d\sigma_{n-1} \leqslant Q\Phi'(\tilde{I}_{n-1}, x_n).$$

From here, applying (11) to an arbitrary segment $\tilde{I}_1\subset\mathscr{I}_1$ one deduces

$$\int_{\tilde{I}_n}\varliminf_{\nu\to\infty}\left(\frac{\delta(\xi, I_1)}{1/\nu}\right)^n d\sigma_n \leqslant Q\Phi(\tilde{I}_{n-1}, \tilde{I}_1)$$

for each $\tilde{I}_n = \tilde{I}_{n-1}\times\tilde{I}_1$ in $\mathscr{I}_n$.

But n-a.e. in $\mathscr{I}_n$ one has

$$\lim_{\nu\to\infty}\frac{\delta(\xi, I_1)}{1/\nu} = \left|\frac{\partial f}{\partial x_n}(x)\right|,$$

what ends the proof of 2°.

It is natural to look for a qcty criterium in the case when the homeomorphism f satisfies n-a.e. in G the local form of Grötzsch's inequality given in Definition 5, assuming that the n linearly independent $(n-1)$-planes through a point $x_0 \in G$ depend on x_0. Let us denote these $(n-1)$-planes by $\Pi^k_{n-1}(x_0)$ and the corresponding orthogonal directions by $\Pi^k_1(x_0)$, $k = 1, \ldots, n$. If we suppose that the angles between these directions are minorized independent of x_0 by positive constants and further that f is ACL_n, then it will result (again from Theorem 1) that f is qc.

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