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M. M. SYSŁO* (Wrocław)

A SOLVABLE CASE OF THE SET-PARTITIONING PROBLEM

Abstract. We show in this paper that the travelling salesman problem on Halin graphs, that can be solved by a polynomial-time algorithm due to Cornuéjols, Naddef, and Pulleyblank, gives rise to a solvable case of another NP-hard problem, the set-partitioning problem. This special case of the latter problem is defined by a family of vertex sets of certain paths in a plane tree. We present also a complete characterization of such set families that can be implemented in polynomial time.

1. Introduction. Halin graphs constitute a family of minimally 3-connected planar graphs and have many interesting properties related to their cycles. A *Halin graph* H can be defined by taking a plane tree T that has no vertices of degree two and adding a new edge between every pair of consecutive leaves of T to form a cycle C containing all the leaves of T (the order of leaves of T is induced by the embedding of T). Thus, we can write $H = T \cup C$ (see also Fig. 1), and in the sequel we assume that a Halin graph is a plane graph. Alternatively, Halin graphs can be introduced as a family of plane graphs whose intersection graphs of the interior faces over the set of edges, called *cycle graphs*, are outerplanar (see [7]). Precisely, if $G = (V, E)$ is a plane graph and $\mathscr{C} = \{C_i\}_I$ denotes the set of the interior faces (cycles) of G, then the *cycle graph* $B(G, \mathscr{C})$ *of* G *with respect to* $\mathscr{C}$ has the vertex set corresponding to $\mathscr{C}$ and two vertices are adjacent in $B(G, \mathscr{C})$ if and only if the corresponding cycles share an edge. It is evident that the cycle graph of a Halin graph is a 2-connected outerplanar graph, although the converse does not hold in general.

Much of the research devoted to Halin graphs has been focused on the structure of their cycles. In particular, we know that Halin graphs are Hamiltonian, 1-Hamiltonian (i.e., H and $H-v$ are Hamiltonian for every vertex v), and almost pancyclic (see [4] for details). Moreover, Cornuéjols *et al.* [1] (see also [2]) proved that the travelling salesman (TS) problem on

* Fellow of the Alexander von Humboldt-Stiftung; on leave from the Institute of Computer Science, University of Wrocław, 51-151 Wrocław, Poland.

Halin graphs can be solved by a polynomial-time algorithm. This result follows from a general theory of the TS problem on graphs which contain a 3-edge cut. The algorithm proposed in [1] is a recursive procedure which transforms a given instance of the TS problem to another instance of the problem in a reduced Halin graph. The reduction depends on shrinking a certain subgraph, called a fan, to a vertex. A *fan* in a Halin graph $H = T \cup C$ is induced by a non-leaf vertex v of T, which is adjacent to exactly one other non-leaf of T, and the leaves adjacent to v. For example, the graph in Fig. 1 has three fans. The reduction process redefines also the weight function. A sequence of reductions leads to a wheel for which the TS problem can be easily solved.

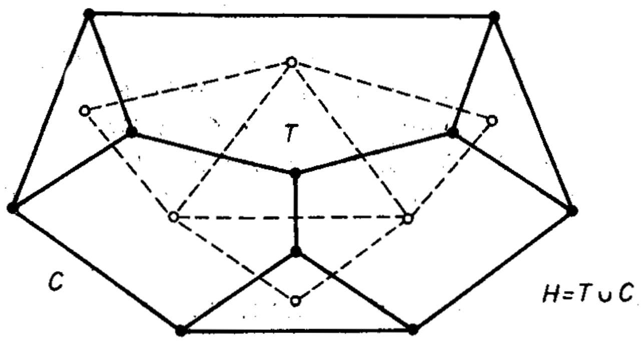

Fig. 1. A Halin graph and its intersection graph (in dashed lines)

The purpose of this paper is: (1) to provide transformations between the special case of the TS problem on Halin graphs and a special case of the set-partitioning (SP) problem and (2) to show that these special instances of the latter problem can be recognized by a polynomial-time algorithm. This will prove that the special case of the set-partitioning problem can be solved by a polynomial-time algorithm.

In what follows, an instance of the TS problem is defined by specifying a pair (G, x), where G is a graph and x is a weight function described on the edge set of G. The problem is to find a minimum-weight Hamiltonian cycle of G. Similarly, an instance of the SP problem is defined by $(S, \mathcal{S}, y)$, where $\mathcal{S}$ is a set family on the ground set S, and y is a weight function described on $\mathcal{S}$. A *partition* of S is a collection $\{S_1, S_2, \ldots, S_m\} \subset \mathcal{S}$ such that

$$S_i \cap S_j = \emptyset \ (i \neq j) \quad \text{and} \quad \bigcup_{i=1}^{m} S_i = S.$$

The problem is to find a minimum-weight partition of S.

Hamiltonian cycles in Halin graphs can be also characterized by certain vertex covers in their intersection graphs. This leads to a new vertex-covering problem on planar graphs (see [6]). The results of this paper have been announced in a preliminary form in [6].

Graph-theoretic terms not defined here can be found in [3].

2. The TS problem on Halin graphs as an SP problem. In this section we define a class of instances of the SP problem generated by the TS problem on Halin graphs and demonstrate the equivalence of both problems.

Let $H = T \cup C = (V, E)$ be a Halin graph and let x be a weight function defined on the edge set of H. Let us put $C = (e_1, e_2, \ldots, e_m)$, where $e_i \in E$, $i = 1, 2, \ldots, m$. By the MacLane characterization of planar graphs, the set $\mathscr{C} = \{C_1, C_2, \ldots, C_m\} = \{C_i\}_I$ of the interior faces of H constitutes a cycle basis of H. Therefore, every cycle of H can be expressed as a combination of basic cycles. Let c be a cycle of H and let $I(c)$ denote the subset of $I = \{1, 2, \ldots, m\}$ such that c is generated by the cycles in $I - I(c)$. Hence

$$c = C \oplus \bigoplus_{i \in I(c)} C_i,$$

since $C = \bigoplus_{i \in I} C_i$. In other words, a cycle c of H can be cut out of C by using the basic cycles which are in $I(c)$. Our first proposition identifies those subsets J of I which generate cycles in H.

PROPOSITION 1. *Let $J \subset I$. Then $\bigoplus_{j \in J} C_j$ is a cycle in H if and only if the subgraph of the intersection graph $B(H, \mathscr{C})$ of H generated by J is connected.*

Proof. It is clear that if J generates a disconnected graph in $B(H, \mathscr{C})$, then $\bigoplus_{j \in J} C_j$ is a union of edge-disjoint cycles. On the other hand, let F denote the subgraph of $B(H, \mathscr{C})$ induced by J. The graph F is outerplane and, moreover, it follows from the relation between H and $B(H, \mathscr{C})$ that the vertices of F can be ordered in such a way $(v_1, v_2, \ldots, v_l)$ that the subgraph F_i of F generated by $v_1, v_2, \ldots, v_i$ $(i = 1, 2, \ldots, l)$ is connected and the basic cycle C_i corresponding to v_i has exactly one path in common with F_{i-1}. Hence F generates a cycle.

Some general results about the relations between cycles in planar graphs and their intersection graphs are discussed in [5]. Hamiltonian cycles are characterized as follows (see Fig. 2 for illustration).

PROPOSITION 2. *Let $J \subset I$. Then $c = C \oplus \bigoplus_{j \in J} C_j$ is a Hamiltonian cycle of H if and only if no two cycles in J have a vertex in common and $\{C_j\}_J$ cover all interior vertices of H.*

Proof. Let c be a Hamiltonian cycle in H and let C_g and C_h be two cycles in J which share at least a vertex. In the former case, if (g, h) is an interior edge, then $I - J$ induces a disconnected subgraph contradicting Proposition 1. If (g, h) is an exterior edge, then the exterior vertex of H shared by C_g and C_h does not belong to c, so c is not a Hamiltonian cycle. In the latter case, $I - J$ induces also a disconnected subgraph since g and h belong to the same face of the outerplane graph $B(H, \mathscr{C})$. This shows that no two cycles of J share a vertex in H. Moreover, if there is an interior vertex of

H which is not covered by a cycle of J, then c does not pass through this vertex; therefore c is not a Hamiltonian cycle.

On the other hand, if the cycles in J are vertex-disjoint and cover all interior vertices of H, then c is a cycle (by Proposition 1) and every vertex of H is covered by c. Hence c is a Hamiltonian cycle.

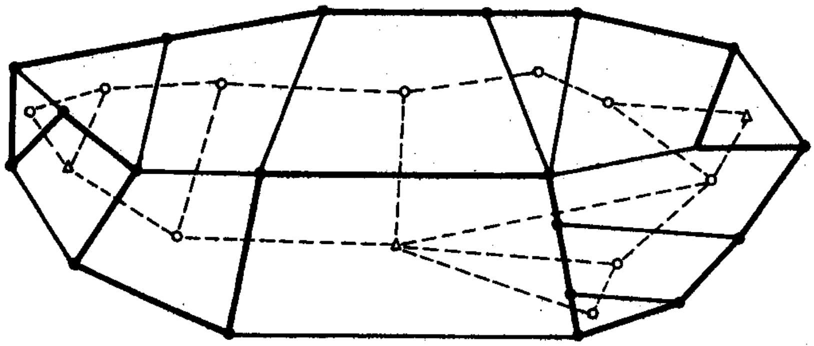

Fig. 2. A Halin graph, one of its Hamiltonian cycles (in heavy lines), and the corresponding cycle cover

By Proposition 2, there exists a correspondence between Hamiltonian cycles of a Halin graph H and certain cycle partitions of the interior vertices of H. This leads to the following special case of the SP problem.

Let $H = (V, E)$ be a Halin graph and let $S \subset V$ denote the set of interior vertices of H. We now define the set family $\mathscr{S} = \{S_i\}_I$, where S_i consists of the interior vertices of the basic cycle C_i. Note that the sets in $\mathscr{S}$ correspond to maximal sections of non-leaves of the plane tree T in the cyclic order of all vertices of T generated by the embedding of T. A set family which can be constructed in such a way is referred to as an *interior plane-tree family*. Proposition 2 can be now reformulated as follows:

COROLLARY 1. *There exists a one-to-one correspondence between Hamiltonian cycles of a Halin graph $H = T \cup C$ and the set partitions of the interior plane-tree family $(S, \mathscr{S})$ of T.*

Let us now consider the weighted problems. Let $(S, \mathscr{S}, y)$ be an instance of the SP problem defined on the interior plane-tree family $(S, \mathscr{S})$ of a Halin graph H. The solution to $(S, \mathscr{S}, y)$ can be found by solving the TS problem (H, x), where $x(e) = 0$ if e is an interior edge of H, and $x(e) = y(S_i)$ if e is an exterior edge and S_i corresponds to the face containing e.

On the other hand, let (H, x) be an instance of the TS problem and let $\mathscr{S}$ be the interior plane-tree family generated by H. The corresponding instance $(S, \mathscr{S}, y)$ of the SP problem is defined by

$$y(S_i) = \sum_{f \in C_i} z(f) - 2x(e_i),$$

where e_i is the exterior edge of C_i. Let us consider the objective function

$$z = x(C) + \sum_{j \in J} y(S_j),$$

where $\{S_j\}_J$ is a partition of $\mathscr{S}$. The first term in z, the length of the exterior cycle C of H, is a constant. The corresponding SP problem is to minimize z over all partitions $\{S_j\}_J$ of $\mathscr{S}$. We have

$$z = \sum_{i \in I} x(e_i) + \sum_{j \in J} y(S_j) = \sum_{i \in I} x(e_i) + \sum_{j \in J} \Big[\sum_{f \in C_j} x(f) - 2x(e_j) \Big]$$

$$= \sum_{i \in I - J} x(e_i) + \sum_{j \in J} \Big[\sum_{f \in C_j} x(f) - x(e_j) \Big].$$

Therefore, z is the weight of the Hamiltonian cycle c which is generated by a given partition $\{S_j\}_J$ of $\mathscr{S}$. Now the first term in z is the weight of that portion of c which belongs to the cycle C and the second term is the weight of the interior portion of C. Thus, we proved the following theorem:

THEOREM 1. *The TS problem on a Halin graph can be transformed to an instance of the SP problem on the corresponding interior plane-tree family, and the converse transformation also exists.*

In the next section, we present a characterization of interior plane-tree families that can be easily implemented in polynomial time. Thus, the total time spent on recognizing and solving the special case of the SP problem defined above is bounded by a polynomial function in the problem size.

3. Characterizations of plane-tree set families. The purpose of this section is to present a characterization of interior plane-tree families and discuss the complexity of its implementation. The characterization is derived from a characterization of plane-tree families which are also defined on plane trees, however the sets are taken over the whole vertex sets. Let $H = T \cup C$ be a Halin graph and let $\{C_i\}_I$ denote the set of its interior faces. Then the *plane-tree family* $\mathscr{T}$ of H (or of T) is defined as $\mathscr{T} = \{T_i\}_I$, where T_i is the vertex set of C_i. We remind the reader that for the sake of applications of our results to Halin graphs we assume that trees have no vertices of degree two, although the results of this section can be easily extended to arbitrary plane trees.

The following theorem gives necessary and sufficient conditions for a set family $\mathscr{T} = \{T_i\}_I$ of subsets of V to be reprezentable by the vertex sets of the interior faces of a Halin graph.

THEOREM 2. *A family $\mathscr{T} = \{T_i\}_I$ of subsets of V is a plane-tree family if and only if $\mathscr{T}$ and V satisfy the following conditions:*

1. $|V| \geq 3$, $|I| \geq 3$, *and* $|T_i| \geq 3$ *for every* $i \in I$.
2. $\bigcup_{i \in I} T_i = V$.
3. $|T_k \cap T_l| \leq 2$ *for every* $k, l \in I$, $k \neq l$.
4. *Let us define the following edge set*:

$$E = \{\{u, v\} : u, v \in V,\ u, v \in T_k \cap T_l \text{ for certain } k, l \in I,\ k \neq l\}.$$

(i) $|E| = |V|-1$.

(ii) *For every* $\{u, v\}$ *in* E *there exist exactly two sets* T_k *and* T_l $(k \neq l)$ *such that* $\{u, v\} \subset T_k \cap T_l$.

(iii) T_i *has* $|T_i|-1$ *pairs in E that form a path whose vertices belong to exactly two sets of* $\mathcal{T}$.

Let s_i *and* t_i *denote the endvertices of* T_i, *let* $W = \{s_i, t_i\colon i \in I\}$, *and let us define the following edge set on* W:

$$C = \{e_i = \{s_i, t_i\}\colon i \in I\}.$$

(iv) *C is a cycle.*

Before proving the theorem, we first illustrate that all the conditions are necessary and independent. Conditions 1 and 2 are evident. Condition 3 follows from the assumption that we consider set families on trees with no vertices of degree two.

Figure 3(a) shows a set family on a unicyclic graph which satisfies all the conditions except 4(i). In this case we define

$$V = \{u_1, u_2, u_3, u_4, v_1, v_2, v_3, v_4\}$$

and

$$T_1 = \{u_1, v_1, v_2, v_3, u_3\}, \quad T_2 = \{u_3, v_3, v_4, v_1, u_1\},$$

$$T_3 = \{u_2, v_2, v_3, v_4, u_4\}, \quad T_4 = \{u_4, v_4, v_1, v_2, u_2\}$$

A family which satisfies all the conditions except 4(ii) can be defined as follows:

$$V = \{u_i, v_i\colon i = 1, 2, 3, 4, 5\}$$

and

$$T_1 = \{u_1, u_5, v_5, v_1\}, \quad T_2 = \{u_2, u_5, v_5, v_2\}, \quad T_3 = \{u_1, u_5, u_2\},$$

$$T_4 = \{v_1, v_5, v_2\}, \quad T_5 = \{u_3, u_5, v_5, v_3\}, \quad T_6 = \{u_4, u_5, v_5, v_4\},$$

$$T_7 = \{u_3, u_5, u_4\}, \quad T_8 = \{v_3, v_5, v_4\}.$$

A family which satisfies all the conditions except 4(iii) is illustrated in Fig. 3(b) and is defined as follows:

$$V = \{u_j\colon j = 1, \ldots, 7\},$$

$$T_1 = \{u_1, u_2, u_3\}, \quad T_2 = \{u_1, u_2, u_4\}, \quad T_3 = \{u_3, u_2, u_4, u_6, u_7\},$$

$$T_4 = \{u_4, u_6, u_5\}, \quad T_5 = \{u_5, u_6, u_7\}.$$

Finally, let us consider the family $\mathcal{T}$ on $V = \{v_j\colon j = 1, \ldots, 7\}$ defined by

$$T_1 = \{v_1, v_4, v_2\}, \quad T_2 = \{v_2, v_4, v_3\}, \quad T_3 = \{v_3, v_4, v_1\},$$

$$T_4 = \{v_5, v_4, v_6\}, \quad T_5 = \{v_6, v_4, v_7\}, \quad T_6 = \{v_7, v_4, v_5\}.$$

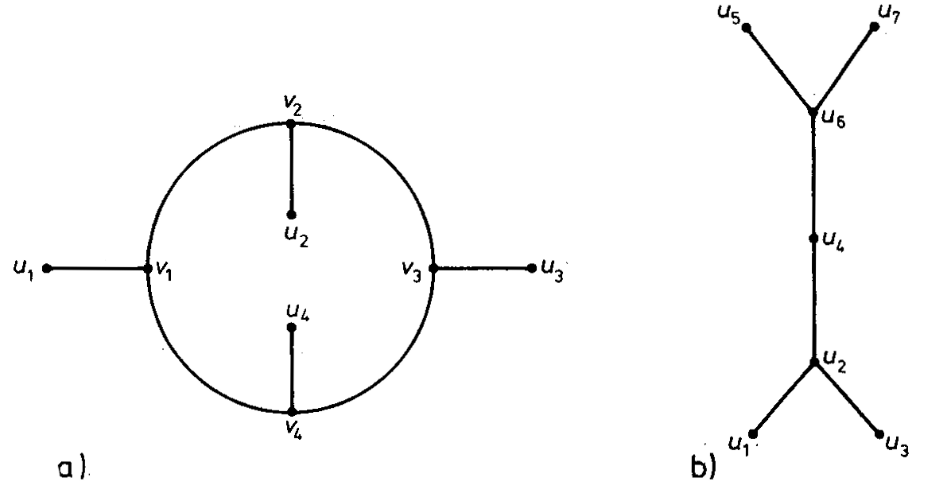

Fig. 3. Non plane-tree families

$\mathscr{T}$ satisfies Conditions 1–3 and 4(i)–(iii). However,

$$W = \{v_1, v_4, v_5, v_7, v_8\},$$

$$C = \{\{v_1, v_4\}, \{v_4, v_5\}, \{v_5, v_1\}, \{v_7, v_8\}, \{v_8, v_7\}\}$$

and C consists of two cycles. It is easy to check that $\mathscr{T}$ has no realization on a plane tree.

Proof of Theorem 2. The necessity follows by the examples above.

To prove the sufficiency, we construct a Halin graph such that $\mathscr{T} = \{T_i\}_I$ is the family of vertex sets of its interior faces. By Conditions 1 and 4, every set T_i is a path over E with the ends contained in exactly two such paths. Let us consider the graph $F = (V, E')$ on the set V at its vertex set and with the edge set $E' = E \cup C$. The elements of each set T_i can be arranged in a cycle C_i of F and every such cycle contains an edge e_i which does not belong to any other cycle in $\mathscr{C} = \{C_i\}_I$. By Condition 4(iii), every vertex of W belongs to exactly two paths of $\mathscr{T}$. Hence C induces a 2-regular graph on W and, by Condition 4(iv), C is a cycle. Thus, F is a connected graph. The cyclomatic number of F is equal to $\mu(F) = |E'| - |V| + 1$. Moreover, by Condition 4(i) and the definition of E', we have $\mu(F) = |E| + |I| - |V| + 1 = |I|$. Therefore, the set of cycles $\mathscr{C}$ constitutes a cycle basis of F, which is a fundamental cycle set, since every cycle in $\mathscr{C}$ contains the edge that does not belong to any other cycle of $\mathscr{C}$. Let us consider the set of cycles $\mathscr{C}' = \mathscr{C} \cup \{C\}$, which consists of $\mu(F)+1$ cycles, $\mu(F)$ of which form a cycle basis, and every edge of F belongs to exactly two cycles in $\mathscr{C}'$. Hence, by the MacLane theorem, F is a planar graph, $\mathscr{C}$ is the set of interior faces of F, and C is the exterior cycle of F corresponding to $\mathscr{C}$. Since $\mathscr{C}$ is a fundamental cycle set, removal of the edges of C from F results in a spanning tree T of F. By Condition 3, T has no vertices of degree two. Thus, we may conclude that F can be expressed as $T \cup C$, where T is a plane tree with no vertices of degree two and C is a cycle going through pendant vertices of T.

Therefore, F is a Halin graph and $\mathscr{T}$ is the family of vertex sets of interior faces of F.

It is easy to show that Condition 4(iv) can be replaced by the following one:

4(iv′) *$\mathscr{T}$ is a minimal family satisfying the other conditions, that is, there is no proper subset $U \subset V$ such that the family $\mathscr{T}' = \{T_i \in \mathscr{T}: T_i \subset U\}$ satisfies Conditions* 1–3 *and* 4(i)–(iii).

However, for the sake of effectiveness we choose the former as easier for testing. Note also that the Conditions 1, 2, and 4 characterize set families on plane trees which may contain vertices of degree two.

We now proceed to characterization of the set families which define the special cases of the SP problem on Halin graphs. Let $H = T \cup C$ be a Halin graph and let S be the set of its interior vertices. The interior plane-tree family $\mathscr{S} = \{S_i\}_I$ of T (or of H) may be considered as the set family on the tree T' obtained from T by removing all the leaves of T. In this case, every set S_i which is not a singleton also forms a path (over the set of pairs of elements which belong to exactly two sets), however its endvertices may belong to more than two sets. A characterization of $\mathscr{S}$ must guarantee the possibility of extending the sets $\{S_i\}_I$ to the family $\{T_i\}_I$ which satisfies the conditions of Theorem 2.

Let us discuss necessary modifications of the conditions of Theorem 2. The set of interior vertices of a Halin graph may consist of one element; hence $|S| \geqslant 1$ and, therefore, $|S_i| \geqslant 1$ for every $i \in I$. Conditions 2, 3, and 4(i)–(ii) remain unchanged for $\mathscr{S}$. Let us define D as the set of pairs which belong to at least two sets of $\mathscr{S}$. We refer to the elements of D as to *edges*. Condition 4(iii) has to be modified since $\mathscr{S}$ may contain one-element sets and sets in $\mathscr{S}$ may terminate with vertices which belong to several other sets. Thus, we require only that if S_i has at least two elements, then it has $|S_i|-1$ (pairs) edges in D which form a path. Endvertices of paths in $\mathscr{S}$ are closely related to the one-element sets of $\mathscr{S}$. Let us first define the following sets:

$$S' = \{u \in S: S_i = \{u\} \text{ for certain } i \in I\},$$

$$U = \{p_i, q_i: p_i, q_i \text{ are the endvertices of } S_i\ (i \in I), \text{ where } |S_i| \geqslant 2\},$$

$$U' = \{u \in U: u \text{ belongs to exactly two sets of } \mathscr{S}\}.$$

Figure 4 illustrates all possible configurations of pendant edges (up to their number in fans) in a plane tree. An element of S' corresponds to a pair of pendant edges in T incident with a common vertex. Let u denote a vertex in T' adjacent to a pendant vertex of T. Then either $T' = \{u\}$ or T' is a tree on at least two vertices. In the former case, T is a star; hence $S = S' = \{u\}$ and $\mathscr{S}$ is the family of one-element subsets of S. If T' has at least two vertices, then U consists of neighbours of pendant vertices of T and every vertex $u \in S'$

corresponds to a bunch of at least two pendant edges of T; therefore, u terminates some paths of $\mathcal{S}$ in T'. Hence $S' \subseteq U$. Moreover, if u is pendant in T', then $u \in U'$ and u has at least two pendant neighbours in T, since T has no vertices of degree two. Hence $U' \subseteq S'$.

If u is non-pendant in T', then it may or may not belong to S' (see Figs. 4(b) and (c)). Observe that if we remove all pendant edges of T incident with a common vertex u and $T' \neq \{u\}$, then u becomes an endvertex of an even number of sets in $\mathcal{S}$. However, if u is not an endvertex of a set in $\mathcal{S}$, then $\mathcal{S}$ must contain at least a pair of sets which terminate at u and have no other vertices in common (see Fig. 4(b)).

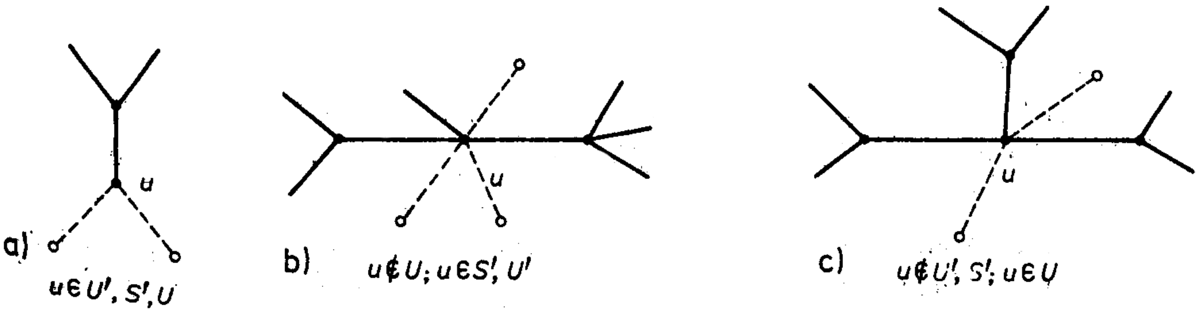

Fig. 4. Configurations of pendant edges (dashed lines) in plane trees

The properties of $\mathcal{S}$ discussed so far guarantee that $\mathcal{S}$ can be represented on a tree. Additionally, we have to assure that $\mathcal{S}$ corresponds to a certain embedding of the tree in the plane. In the terms of the tree T, we would like to augment $\mathcal{S}$ on T' to $\mathcal{T}$ on T in such a way that the endvertices of sets in $\mathcal{T}$ combined into pairs form exactly one cycle (see Condition 4(iv) in Theorem 2). Since the elements of this cycle are not present in $\mathcal{S}$, we formulate a corresponding condition (Condition 6 in Theorem 3) using subfamilies of $\mathcal{S}$ which would correspond to subcycles when $\mathcal{S}$ is extended to $\mathcal{T}$. Note here that we must prevent decomposition only at a non-endvertex of some subfamily since, otherwise, using pendant edges of T we can always combine the trees of subfamilies sharing an endvertex into T.

Thus, we have demonstrated the necessity of the following conditions.

THEOREM 3. *A family $\mathcal{S} = \{S_i\}_I$ of subsets of S is an interior plane-tree family if and only if $\mathcal{S}$ and S satisfy the following conditions:*

1. $|S| \geqslant 1$, $|I| \geqslant 3$, *and* $|S_i| \geqslant 1$ *for every* $i \in I$.
2. $\bigcup_{i \in I} S_i = S$.
3. $|S_k \cap S_l| \leqslant 2$ *for every* $k, l \in I$, $k \neq l$.
4. *Let us define the following edge set:*

$$D = \{\{u, v\}: u, v \in S,\ u, v \in S_k \cap S_l \text{ for certain } k, l \in I, k \neq l\}.$$

(i) $|D| = |S|-1$.

(ii) *For every* $\{u, v\} \in D$ *there exist exactly two sets* S_k *and* S_l $(k \neq l)$ *such that* $\{u, v\} \subseteq S_k \cap S_l$.

(iii) *If* S_i *has at least two elements, then it has* $|S_i|-1$ *pairs in* D *which form a path.*

5 (i) *If* $|S| > 1$, *then* $U' \subseteq S' \subseteq U$.

(ii) *If* $u \in U$ *and* u *is a non-endvertex of some set in* $\mathscr{S}$, *then there are two sets* $S_k, S_l \in \mathscr{S}$ *for which* u *is an endvertex and* $|S_k \cap S_l| = 1$.

6. *The family* $\mathscr{S}$ *is a minimal family satisfying Conditions* 1–5, *that is,* S *cannot be expressed as a union of non-empty subsets* $S = W_1 \cup W_2 \cup \ldots \cup W_t$ $(t \geq 2)$ *such that the subfamily* $\mathscr{S}_j = \{S_i \in \mathscr{S}: S_i \subseteq W_j\}$ *satisfies Conditions* 1–5 *for every* $j = 1, 2, \ldots, t$, *and two subfamilies have at most one common vertex which is a non-endvertex in at least one of them.*

Proof. The necessity has been demonstrated before stating the theorem.

For the proof of the sufficiency, let $\mathscr{S} = \{S_i\}_I$ be a family of subsets of S satisfying the theorem conditions. We construct a family $\mathscr{T} = \{T_i\}_I$ which is a plane-tree family on the tree that generates $\mathscr{S}$ as an interior plane-tree family. At the beginning we assume $\mathscr{T} = \mathscr{S}$ (i.e., $T_i = S_i$, $i \in I$), $V = S$, and then we proceed with regard to the status of the elements of S. Let $u \in S$.

If $|S| = 1$, then $S = S'$ and $\mathscr{S}$ consists only of one-element sets. In this case we augment V with $\{r_1, r_2, \ldots, r_m\}$, where $|I| = m$, and every set $S_i = \{u\}$ for $i \in I$ is augumented to $\{r_i, u, r_{i+1}\}$, where addition is modulo m.

Assume that S has at least two elements. Then, by Conditions 4(i)–(iii), $D \neq \emptyset$ and $U \neq \emptyset$.

If $u \notin U$, that is, u is not an endvertex of any set in $\mathscr{S}$, then we do not alter $\mathscr{T}$ and V. Let now $u \in U$. Let us put

$$\mathscr{S}_u = \{S_j \in \mathscr{S}: u \in S_j, |S_j| \geq 2\}$$

and

$$\mathscr{S}''_u = \{S_j \in \mathscr{S}: u \text{ is an endvertex of } S_j\}.$$

We first show that $|\mathscr{S}''_u|$ is even. To this end, let us consider the set D_u of those edges in D which contain u. Every edge in D_u is a subset of exactly two sets of $\mathscr{S}_u$. Since a set which has u as an interior vertex contains two edges of D_u, the number of sets which have u as an endvertex is even. Next, we claim that the family $\mathscr{S}_u$ can be decomposed into non-empty and disjoint subfamilies each of which contains exactly two sets of $\mathscr{S}'_u$. First we match every two sets $S_j, S_k \in \mathscr{S}''_u$ that share a pair, that is, $|S_j \cap S_k| = 2$. Then, let R_0 be a member of $\mathscr{S}''_u$ (if there is any) which has no mate assigned yet. There is $R_1 \in \mathscr{S}_u$ such that $|R_0 \cap R_1| = 2$. Since $R_1 \notin \mathscr{S}'_u$, there is $R_2 \in \mathscr{S}_u$, $R_2 \neq R_1$, such that $|R_1 \cap R_2| = 2$. If $R_2 \in \mathscr{S}''_u$, then R_2 is assigned to R_0. Otherwise, we continue and find $R_3, R_4, \ldots$ in $\mathscr{S}_u$ such that $|R_i \cap R_{i+1}| = 2$. Since $\mathscr{S}_u$ is finite, there must exist a set R_k which terminates this sequence, that is

$R_k \in \mathscr{S}'_u$. The set R_k becomes the mate of R_0. Thus, we can exhaust the family $\mathscr{S}'_u$ and combine its members into pairs.

We claim that this process exhausts also the subfamily $\mathscr{S}_u$. Otherwise, starting with a set which is left in $\mathscr{S}_u$ and using Condition 4(ii), we could augment it first to a subfamily of $\mathscr{S}_u$ and then to a subfamily of $\mathscr{S}$ which satisfies Conditions 1–5. Moreover, the vertex u would not terminate any set of this subfamily; hence the subfamily would generate a partition of $\mathscr{S}$ described in Condition 6, a contradiction.

Therefore, we can assume that there exists a partition

$$\mathscr{S}_u = \mathscr{S}_u^1 \cup \mathscr{S}_u^2 \cup \ldots \cup \mathscr{S}_u^r$$

into non-empty and disjoint subfamilies such that every subfamily $\mathscr{S}_u^j$ contains exactly two sets of $\mathscr{S}'_u$. It is clear from the construction of the subfamilies that all sets of $\mathscr{S}_u$ (or, equivalently, all edges of D_u) can be ordered around u as $R_1, R_2, \ldots, R_q$ in such a way that if $|R_k \cap R_l| = 2$, then $k = l+1 \pmod q$. We augment now the sets of $\mathscr{S}_u$ and one-element sets $\{u\}$ of $\mathscr{S}$ to the form they should have in $\mathscr{T}$. If $u \notin S'$ (i.e., $\mathscr{S}$ has no one-element set of the form $\{u\}$), then for every j such that $|R_j \cap R_{j+1}| = 1$ we add a new element v_j to V and augment R_j and R_{j+1} with v_j. If $u \in S'$ and there are p one-element sets $\{u\}$ in $\mathscr{S}$, then, additionally, instead of one of the v_j's, say v_h, we add $w_0, w_1, \ldots, w_p$ to V, augment R_h with w_0 and R_{h+1} with w_p (instead of augmenting both with v_h) and add to $\mathscr{T}$ new sets $\{w_{k-1}, u, w_k\}$ for $k = 1, 2, \ldots, p$.

Now, we show that, after considering all elements of S, the set family $\mathscr{T}$ over the ground set V satisfies the conditions of Theorem 2. First, if $|S| = 1$, then $|S'| \geqslant 3$, and hence $|V| \geqslant 3$. If $|S| > 1$, then $U \neq \emptyset$ and again $|V| \geqslant 3$. Moreover, if $|S_j| = 1$, then, by Condition 5(i), $S_j \subset U$ and $|T_j| = 3$ in $\mathscr{T}$. If $|S_j| \geqslant 2$, then S_j is extended in $\mathscr{T}$ from both its ends, and hence $|T_j| \geqslant 4$ in $\mathscr{T}$. Conditions 2–3 and 4(i)–(iii) of Theorem 2 follow easily from the corresponding conditions of Theorem 3 and from the construction of $\mathscr{T}$. Finally, we have to show that the pairs of endvertices of sets in $\mathscr{T}$ form exactly one cycle. To this end, it suffices to observe that for every set T_i considered as a path (over E, the augmented set of edges) it is always possible to order the edges around its vertices in the plane in such a way that its two endedges (pendant) are adjacent to the rest of T_i (or to itself) in clockwise and counterclockwise orders of all the edges adjacent to them (see Fig. 5).

Such an ordering of all edges of E can be done iteratively by fixing an order at one of the vertices of S. Therefore, the pairs of endvertices of the sets in $\mathscr{T}$ form exactly one cycle in the plane embedding of the sets of $\mathscr{T}$.

Thus, the family $\mathscr{T}$ satisfies the conditions of Theorem 2. Hence $\mathscr{T}$ is a plane-tree family on V and $\mathscr{S}$ is an interior plane-tree family on $\mathscr{S}$. This proves the theorem.

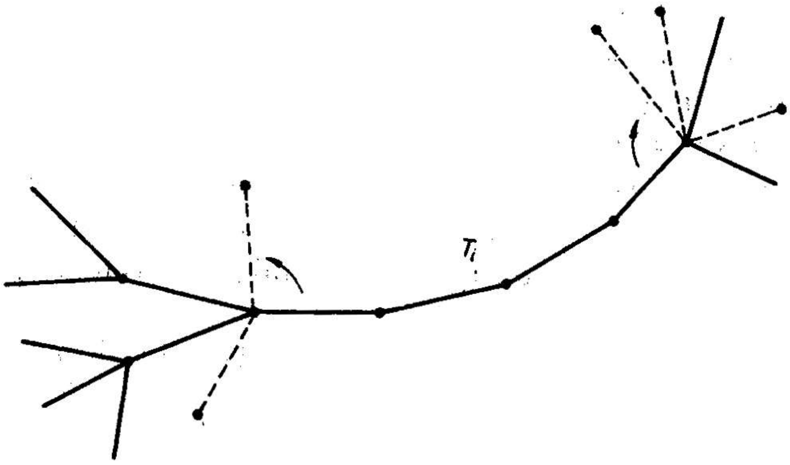

Fig. 5. Embedding of a path T_i

It is easy to see that the conditions of Theorem 2 can be tested in time that is bounded by a polynomial function in the size of the ground set V and the family $\mathcal{T}$. Similarly, all conditions of Theorem 3, except Condition 6, can be easily verified by a polynomial-time algorithm. Condition 6 can be tested in a way described in the proof of the Theorem. Namely, for every $u \in U$, we form the partition $\mathcal{S}_u^1 \cup \mathcal{S}_u^2 \cup \ldots \cup \mathcal{S}_u^r$ of $\mathcal{S}_u$ such that every subfamily $\mathcal{S}_u^j$ contains exactly two sets of $\mathcal{S}'_u$. If such a partition does not exist for some $u \in U$, the family $\mathcal{S}$ can be decomposed as described in Condition 6, so $\mathcal{S}$ is not an interior plane-tree family. Once all vertices $u \in U$ are successfully tested, we can construct a plane representation of the augmented family $\mathcal{T}$ as described in the last paragraph of the proof. This shows that T satisfies the condition of Theorem 2, and therefore $\mathcal{S}$ is an interior plane-tree family. The total amount of work is clearly bounded by a polynomial function in the size of S and $\mathcal{S}$.

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INSTITUTE OF COMPUTER SCIENCE
UNIVERSITY OF WROCŁAW
51-151 WROCŁAW

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