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# *ON PROJECTIONS OF $L^\infty(G)$ ONTO TRANSLATION-INVARIANT SUBSPACES*

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**1. Introduction.** Let $G$ be a locally compact Abelian group with the Haar measure $\mu$. Let $\Phi$ be a translation-invariant *-weakly closed subspace of $L^\infty(G)$. Regard $L^\infty(G)$ and $\Phi$ as Banach spaces with the norm topology of $L^\infty(G)$. The present note is a contribution to solution of the following problem (**P 1042**): when is $\Phi$ complemented in $L^\infty(G)$? Let $\hat{G}$ be the dual group of $G$. Denote by $\sigma(\Phi)$ the spectrum of $\Phi$, i.e., the set $\{\chi \in \hat{G}: \chi \in \Phi\}$. If $\Phi \neq \{0\}$, then $\sigma(\Phi) \neq \emptyset$. The main result of this note is Theorem 2 which gives a sufficient condition for $\Phi$ to be uncomplemented in $L^\infty(G)$ expressed in terms of $\sigma(\Phi)$.

**2. Some examples.** The first example of a complemented subspace is the most general. Let $\Phi$ have a finite spectrum. Then $\Phi$ is the *-weak closure of the linear space spanned by $\sigma(\Phi)$. Hence $\Phi$ is finite-dimensional and complemented in $L^\infty(G)$.

The second example involves a group $G$ of a special type. Let $G = G_1 \oplus G_2$, where $G_1$ and $G_2$ are locally compact Abelian groups. Let $\mu_1$ be the Haar measure on $G_1$. Let

$$I = \Big\{f \in L^1(G): \int_{G_1} f(s,t)\mu_1(ds) = 0 \text{ for almost all } t \in G_2\Big\}.$$

$I$ is a *closed ideal* in the group algebra $L^1(G)$, i.e., a translation-invariant closed subspace of $L^1(G)$. Let $g \in L^1(G_1)$ and

$$\int_{G_1} g\,d\mu_1 = 1.$$

Write

$$Pf(s,t) = f(s,t) - g(s)\int_{G_1} f(u,t)\mu_1(du), \qquad s \in G_1,\ t \in G_2.$$

$P$ is a continuous projection of $L^1(G)$ onto $I$. Denote by $\operatorname{An} I$ the *annihilator of $I$*, i.e., the set

$$\{\varphi \in L^\infty(G): \langle f, \varphi\rangle = 0 \text{ for every } f \in I\},$$

where

$$\langle f, \varphi\rangle = \int_G f\varphi\, d\mu, \quad f \in L^1(G),\ \varphi \in L^\infty(G).$$

An $I$ is a translation-invariant *-weakly closed subspace of $L^\infty(G)$, and $Q = E - P'$ is a continuous projection of $L^\infty(G)$ onto An $I$, where $E$ denotes the identity operator on $L^\infty(G)$. The spectrum of An $I$ may be identified with $\hat{G}_2$. In particular, if $\hat{G}_2$ is infinite, then An $I$ provides an example of a translation-invariant *-weakly closed subspace of $L^\infty(G)$ which is complemented in $L^\infty(G)$ and has an infinite spectrum.

**3. Main results.** We introduce the following notation:

$C_c(G)$ = {all continuous functions on $G$ with compact support},

$C_0(G)$ = {all continuous functions on $G$ vanishing at infinity},

$C_u(G)$ = {all bounded uniformly continuous functions on $G$},

$B(G)$ = {all bounded complex functions on $G$},

$M(G)$ = {all bounded regular Borel measures on the field of Borel subsets of $G$}.

THEOREM 1. *Let $\Phi$ be a translation-invariant *-weakly closed subspace of $L^\infty(G)$ which is complemented in $L^\infty(G)$. Let $\hat{G}$ be connected. Then $\Phi = L^\infty(G)$ or $\Phi \cap C_0(G) = \{0\}$.*

Proof. We use argumentation based on ideas which go back to K. De Leeuw and are contained in [1], Theorem 4.1.

Suppose that $Q$ is a continuous projection of $L^\infty(G)$ onto $\Phi$. At the beginning we prove that there exists a continuous projection $R$ of $L^\infty(G)$ onto $\Phi$ such that

$$T_s R = R T_s \tag{1}$$

for every $s \in G$, where $T_s h(x) = h(x+s)$, $x \in G$, $h$ — any function on $G$.

Let $\mathscr{M}$ denote a *Banach mean*, i.e., a linear functional on $B(G)$ satisfying the following conditions:

(i) $|\mathscr{M}\psi| \leqslant \|\psi\|$, $\|\psi\| = \sup_{s \in G} |\psi(s)|$,

(ii) $\mathscr{M} T_s \psi = \mathscr{M}\psi$ for every $s \in G$,

(iii) $\mathscr{M}c = c$ for any function $c$ constant on $G$.

The proof of the existence of a Banach mean may be found in [2], Theorem 1.2.1, p. 5.

Consider the function

$$\psi(f, \varphi)(s) = \langle f, T_{-s} Q T_s \varphi\rangle, \quad f \in L^1(G), \varphi \in L^\infty(G),\ s \in G.$$

For an arbitrarily fixed $s \in G$ we have

$$|\psi(f, \varphi)(s)| \leqslant \|f\|_1 \|T_{-s}QT_s\varphi\|_\infty \leqslant \|Q\| \|f\|_1 \|\varphi\|_\infty.$$

Thus $\psi(f, \varphi) \in B(G)$ and by (i) we have

$$(2) \qquad |\mathscr{M}\psi(f, \varphi)| \leqslant \|Q\| \|f\|_1 \|\varphi\|_\infty,$$

whence the mapping $f \to \mathscr{M}\psi(f, \varphi)$ with fixed $\varphi$ is a linear continuous functional on $L^1(G)$. It is represented in the form

$$(3) \qquad \mathscr{M}\psi(f, \varphi) = \langle f, R\varphi\rangle$$

for some $R\varphi \in L^\infty(G)$. From (3) and (2) it follows that $R$ is a linear continuous operator and $\|R\| \leqslant \|Q\|$. Let $\mathrm{An}\,\Phi$ denote the *annihilator of* $\Phi$, i.e., the set

$$\{f \in L^1(G)\colon \langle f, \varphi\rangle = 0 \text{ for every } \varphi \in \Phi\}.$$

If $\varphi \in L^\infty(G)$ and $f \in \mathrm{An}\,\Phi$, then $\psi(f, \varphi) = 0$ and, by (3), $R\varphi \in \mathrm{An}\,\mathrm{An}\,\Phi = \Phi$. If $\varphi \in \Phi$, then $T_{-s}QT_s\varphi = \varphi$, and by (iii) and (3) we obtain $R\varphi = \varphi$. Actually, we have proved that $R$ is a continuous projection of $L^\infty(G)$ onto $\Phi$. In order to prove (1) notice that for every $t \in G$ we have

$$\begin{aligned}\psi(f, T_s\varphi)(t) &= \langle f, T_{-t}QT_tT_s\varphi\rangle = \langle f, T_sT_{-(t+s)}QT_{t+s}\varphi\rangle \\ &= \langle T_{-s}f, T_{-(t+s)}QT_{t+s}\varphi\rangle = \psi(T_{-s}f, \varphi)(t+s) \\ &= T_s\psi(T_{-s}f, \varphi)(t).\end{aligned}$$

and (1) now follows from (ii), namely

$$\begin{aligned}\langle f, RT_s\varphi\rangle &= \mathscr{M}\psi(f, T_s\varphi) = \mathscr{M}T_s\psi(T_{-s}f, \varphi) = \mathscr{M}\psi(T_{-s}f, \varphi) \\ &= \langle T_{-s}f, R\varphi\rangle = \langle f, T_sR\varphi\rangle.\end{aligned}$$

Now we show that $C_u(G)$ is an invariant subspace of $R$. For $\varphi \in C_u(G)$ and an arbitrary $\varepsilon > 0$ there exists a symmetric neighbourhood of zero $V_\varepsilon$ such that

$$\|T_s\varphi - \varphi\|_\infty \leqslant \frac{\varepsilon}{\|R\|}, \qquad s \in V_\varepsilon.$$

Hence by (1) we obtain

$$\|T_sR\varphi - R\varphi\|_\infty = \|R(T_s\varphi - \varphi)\|_\infty \leqslant \|R\| \|T_s\varphi - \varphi\|_\infty \leqslant \varepsilon,$$

whence for a non-negative continuous function $\eta_\varepsilon$ on $G$ such that

$$\operatorname{supp}\eta_\varepsilon \subset V_\varepsilon \quad \text{and} \quad \int_G \eta_\varepsilon d\mu = 1$$

we have

$$\|R\varphi * \eta_\varepsilon - R\varphi\|_\infty \leqslant \varepsilon.$$

Evidently, $R\varphi * \eta_s \in C_u(G)$, whence there exists a function $\xi \in C_u(G)$ such that $\xi(x) = R\varphi(x)$ for almost every $x$. After modification (if necessary) on a set of the Haar measure zero we may assume that $R\varphi \in C_u(G)$.

Next we show that $C_0(G)$ is an invariant subspace of $R$. The mapping $\varphi \to R\varphi(0)$ is a linear continuous functional on $C_0(G)$. By the Riesz theorem, it is represented in the form

$$R\varphi(0) = \int_G \varphi d\nu$$

for some $\nu \in M(G)$. Hence by (1) we obtain

$$R\varphi(s) = (T_s R\varphi)(0) = (RT_s\varphi)(0) = \int_G \varphi(s+x)\nu(dx).$$

Consequently, if $\varphi \in C_c(G)$ and $K$ is compact with

$$|\nu|(G\setminus K) \leqslant \frac{\varepsilon}{\|\varphi\|},$$

then for $s \notin \operatorname{supp}\varphi - K$ we have $|R\varphi(s)| \leqslant \varepsilon$. Hence $R\varphi \in C_0(G)$. For $\varphi \in C_0(G)$ choose a sequence $\varphi_n \in C_c(G)$ such that

$$\lim_{n\to\infty} \|\varphi - \varphi_n\| = 0.$$

Then

$$\lim_{n\to\infty} \|R\varphi - R\varphi_n\| = 0 \quad \text{and} \quad R\varphi \in C_0(G),$$

so $C_0(G)$ is an invariant subspace of $R$. Now for $\varphi \in C_0(G)$ we can write

$$\int_G \varphi(s)\nu(ds) = R\varphi(0) = R^2\varphi(0) = \int_G R\varphi(s)\nu(ds) = \int_{G\times G} \varphi(s+t)\nu(dt)\nu(ds),$$

whence $\nu * \nu = \nu$. Thus $\hat{\nu}(\chi) = 1$ or $0$ for every $\chi \in \hat{G}$, where $\hat{\nu}$ denotes the Fourier transform of $\nu$. Since the mapping $\chi \to \hat{\nu}(\chi)$ is continuous and $\hat{G}$ is connected, we have $\hat{\nu} \equiv 1$ or $\hat{\nu} \equiv 0$ and $R = E$ or $R = 0$, respectively, on $C_0(G)$. In the case $R = E$ on $C_0(G)$ we have $C_0(G) \subset \Phi$, whence $\Phi = L^\infty(G)$. In the case $R = 0$ on $C_0(G)$ we have $\Phi \cap C_0(G) = \{0\}$. Thus the proof is completed.

Let $M_0(G)$ denote the class of all closed subsets of $G$ which contain a support of a non-zero measure belonging to $M(G)$ with the Fourier transform vanishing at infinity. Every closed subset $F$ of $G$ such that $\mu(F) > 0$ belongs to $M_0(G)$. In a non-discrete $G$ there may exist $F \in M_0(G)$ such that $\mu(F) = 0$ (cf. [3]).

THEOREM 2. *Let $\Phi$ be a translation-invariant *-weakly closed proper subspace of $L^\infty(G)$ and let $\hat{G}$ be connected. If $\sigma(\Phi) \in M_0(\hat{G})$, then $\Phi$ is uncomplemented in $L^\infty(G)$.*

Proof. Suppose that $\Phi$ is complemented in $L^{\infty}(G)$. By Theorem 1, $\Phi \cap C_0(G) = \{0\}$. Let $\nu$ be a non-zero measure supported by $\sigma(\Phi)$ such that $\hat{\nu} \in C_0(G)$. For every $f \in \operatorname{An}\Phi$ we have $\langle f, \hat{\nu}\rangle = \langle \hat{f}, \nu\rangle = 0$, whence $\hat{\nu} \in \operatorname{An}\operatorname{An}\Phi = \Phi$. Thus $\hat{\nu}$ is a non-zero element of $\Phi \cap C_0(G)$. This contradiction proves our theorem.

THEOREM 3. *Let $I$ be a non-trivial ideal in $L^1(G)$ and let $\hat{G}$ be connected. If $\sigma(\operatorname{An} I) \in M_0(\hat{G})$, then $I$ is uncomplemented in $L^1(G)$.*

Proof. If $P$ were a continuous projection of $L^1(G)$ onto $I$, then $Q = E - P'$ would be a continuous projection of $L^{\infty}(G)$ onto $\operatorname{An} I$, contrary to Theorem 2.

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