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CONCERNING POINT SETS WITH A SPECIAL CONNECTEDNESS PROPERTY, II

BY

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1. Introduction. The author introduced [2] a connectedness property which was used to obtain some theorems related to arcs and arcwise connectedness. In this paper certain related properties will be used to obtain similar results concerning simple closed curves and cyclic connectedness. The context of this discussion will be the same as that of [2] with some notions being repeated here for completeness. Some of the arguments in this paper parallel those of Whyburn [1].

2. Definitions and notations. If M is a point set, then:

(i) M is said to have *Property* C if and only if it is true that M has at least three points, and if x, y and z are three points of M, then M contains a continuum K which contains x and contains only one of the points y and z. M has Property C *hereditarily* (denoted CH) if and only if each non-degenerate subcontinuum of M has Property C.

(ii) The non-degenerate point set K is said to be a *simple inner limiting set* in M if and only if K is a subcontinuum of M or there is a subcontinuum K' of M and a non-cut point p of K' such that $K = K'-p$.

(iii) M is said to have Property C *strongly* (denoted CS) if and only if each simple inner limiting set in M has Property C.

(iv) M is said to have *Property* C′ if and only if it is true that M has at least three points, and if x, y and z are three points of M, then M contains a continuum K containing x and y but not z.

3. The properties involved. It was shown in [2] that every locally compact continuum with Property C which is irreducible between two of its points is an arc and hence that every compact continuum with Property CH is arcwise connected. Every point set with Property C′ has Property C, whereas an arc is a compact continuum with Property CS which fails to have Property C′. Property CS is stronger than Property CH. Property C′ is a generalization of cyclic connectedness.

4. Properties CH and C′. Throughout this section let M denote a compact continuum which has Properties CH and C′.

LEMMA 1. *M contains a simple closed curve. Moreover, if L is an arc in M and p is a cut point of L, then M contains a simple closed curve K such that $K-L$ is an arc containing p as a cut point.*

Proof. M is non-degenerate and arcwise connected. Let L be an arc in M from a to b and let p be a cut point of L. Then M contains a continuum N containing a and b but not p. Since M has Property CH, N is arcwise connected and contains an arc L' from a to b. Since p is not in L', there exist points x and y in L such that the interval $[x, y]$ in L contains p as a cut point and has only the points x and y in common with L'. Then $K = [x, y]$ in $L+[x, y]$ in L' is such a simple closed curve.

COROLLARY 1. *The point set N is a simple closed curve if and only if it is a compact continuum with Property* CH *which is irreducible with respect to having Property* C′.

LEMMA 2. *If L is an arc in M and x and y are cut points of L, then M contains a simple closed curve containing x and y.*

Proof. Let K denote the interval $[x, y]$ in L. By Lemma 1, for each point p of K there is a simple closed curve C_p in M such that $C_p \cdot L$ is an arc Q_p containing p as a cut point. For each point p of K let S_p denote the arc Q_p less its end points and let G denote the collection of all such sets S_p. Then G is a collection of segments (in the order topology on L) covering the closed interval K of L, so some finite subcollection $G' = \{S_1, S_2, \ldots, S_n\}$ of G is irreducible with respect to covering K. Let $C_1, C_2, \ldots, C_n$ be the simple closed curves corresponding to $S_1, S_2, \ldots, S_n$ and let $N = C_1+\ldots+C_n$. Then N is the sum of a finite number of compact locally connected continua each intersecting the arc K, and K is a subset of N, so N is a compact locally connected continuum. Let $N' = N \cdot L$. Then N' is an arc. Let p be a point of N. Suppose p is not in N' or is an end point of N'. Then for each i, C_i-p is a connected set intersecting the connected set $N'-p$ and $N-p = (C_1-p)+(C_2-p)+\ldots+(C_n-p)$, so $N-p$ is connected. Suppose p is a cut point of N'. Then for some j, p is a cut point of $C_j \cdot N'$. Then $N'-p$ is the sum of two connected sets each of which intersects the connected set C_j-p, so $N'' = (N'-p)+(C_j-p)$ is connected. Then for each i, C_i-p is connected and intersects N'', so $N''+(C_1-p)+\ldots+(C_n-p) = N-p$ is connected. Hence N has no cut point and N contains a simple closed curve containing x and y ([1], p. 79).

THEOREM 1. *If x and y are points of M each of which is a cut point of some arc in M, then M contains a simple closed curve containing x and y.*

Proof. There exist in M mutually exclusive arcs L_1 and L_2 containing x and y respectively as cut points. Since M contains an arc from

x to y, there is an arc L_3 in M whose common part with L_1 and L_2 respectively is degenerate. Then $N = L_1 + L_2 + L_3$ is a compact dendron and x and y are not end points of N, so N contains an arc L containing x and y as cut points. By Lemma 2, M contains a simple closed curve containing x and y.

Remark. In Lemma 3 and Theorem 2, let A and B denote non-degenerate, closed and mutually exclusive subsets of M.

LEMMA 3. *Let G denote the set to which x belongs if and only if there exist mutually exclusive arcs L_1 and L_2 in M such that L_1 joins x to A and L_2 joins A and B. If $G + A \neq M$, then there is a point y_0 in $M - (G + A)$ and an arc T from y_0 to a point x_0 of G such that y_0 is a limit point of $G \cdot T$.*

Proof. Suppose that $G + A \neq M$. Let y be a point of $M - (G + A)$, let x_0 be a point of G and let T' be an arc from y to x_0. If y is a limit point of $G \cdot T'$ let $y_0 = y$ and $T = T'$. Suppose y is not a limit point of $G \cdot T'$. Let y_0 denote the last point in the ordering from y to x_0 in T' such that the half-open interval $[y, y_0)$ does not intersect $G \cdot T'$. Suppose y_0 belongs to A. Since A and B are non-degenerate, by Properties C' and CH there is an arc L_2 joining A and B which does not contain y_0. Let R be a region containing y_0 and not intersecting L_2. Then R contains an interval $[z, y_0] = L_1$ of the arc T', where $y \leqslant z < y_0$ in T'. Then z belongs to G and hence to $G \cdot T'$, contradicting the choice of y_0. Thus y_0 is not in A. Suppose y_0 is in G. Then there exist mutually exclusive arcs L_1 and L_2 such that L_1 joins y_0 to a point p of A and L_2 joins A and B. Let R be a region containing y_0 and not intersecting L_2. Let z be a point of T' such that the interval $K = [z, y_0]$ in T' is a subset of R. Then $K + L_1$ contains an arc from z to p which does not intesect L_2 and z belongs to G, again contradicting the choice of y_0. Thus y_0 is in $M - (G + A)$ and letting T be the interval $[y_0, x_0]$ in T', the conclusion follows.

THEOREM 2. *There exist mutually exclusive arcs in M joining A and B.*

Proof. Suppose no such arcs exist. Let G be defined as in Lemma 3. Then $G + A \neq M$. Let y_0, x_0 and T be as in the conclusion of Lemma 3. By Properties C′ and CH there is an arc not containing y_0 and joining A and B. Let N be such an arc whose common part with A and B respectively is degenerate. Let R be a region containing y_0 whose closure does not intersect $N + A$. Then R contains an interval $[y_0, z]$ of T, where $y_0 < z < x_0$ in T. This interval contains a point x of $G \cdot T$, so R contains an arc L joining y_0 to a point x of G. There exist mutually exclusive arcs L_1 and L_2 such that L_1 joins x to a point p of A and L_2 joins A and B at the points a and b, respectively. L must intersect L_2, for if not, $L + L_1$ contains an arc from y_0 to p which does not intersect L_2 and y_0 would belong to G. Then L intersects both L_1 and L_2.

Let r denote the last point of $L \cdot L_1$ in order from x to p in L_1, and let w denote the last point of $L \cdot L_2$ in order from b to a in L_2. Let $L_1' = [r, p]$ in L_1 and let $L_2' = [w, a]$ in L_2. Then the interval $[y_0, r]$ in L has only the point r in common with L_1', and $[y_0, w]$ in L has only the point w in common with L_2'. Let r' and w' be points of L_1' and L_2' respectively such that $r < r' < p$ in L_1', $w < w' < a$ in L_2' and both intervals $[r, r']$ and $[w, w']$ are subsets of R. Let $L_1'' = [r', p]$ in L_1' and let $L_2'' = [w', a]$ in L_2'. Then $[y_0, r]$ in $L + [r, r']$ in L_1' is an arc in R from y_0 to r' which does not intersect L_2'' and $[y_0, w]$ in $L + [w, w']$ in L_2' is an arc in R from y_0 to w' which does not intersect L_1''.

Let $H = A + L_1'' + L_2''$. Let a' and b' denote the points of $A \cdot N$ and $B \cdot N$ respectively. Now b' is not in A and b' is not in L_1'' for if so, L_1 and L_2 contain mutually exclusive arcs joining A and B. Suppose b' is in L_2''. Then L_2'' contains an arc joining A and B. But y_0 can be joined to r' by an arc in R which does not intersect L_2'', so y_0 is in G, a contradiction. Then b' is not in L_2'' and hence is not in H. Then N contains an arc N' such that $B \cdot N'$ and $H \cdot N'$ are degenerate. Let u denote the point of $H \cdot N'$.

Suppose that u is not in L_1''. Then N' does not intersect L_1''. Then $N' + L_2''$ contains an arc V joining A and B. But y_0 can be joined in R to r' by an arc T' which does not intersect L_2'' and since R does not intersect N', $T' + L_1''$ contains an arc from y_0 to p which does not intersect V. Then y_0 is in G, a contradiction. Hence u is in L_1''.

Similarly, if u is not in L_2'', $N' + L_1''$ contains an arc V joining A and B, and y_0 can be joined to a by an arc not intersecting V, resulting in the contradiction that y_0 is in G. Hence u is in L_2''.

Since L_1 and L_2 are mutually exclusive, u cannot belong to both L_1'' and L_2'' and a contradiction results. Hence there exist mutually exclusive arcs in M joining A and B.

5. Properties CS and C′. Throughout this section let M denote a compact continuum which has Properties CS and C′.

THEOREM 3. *If x is a point of M which is not a cut point of any subcontinuum of M containing it and R is a region containing x, then R contains a continuum K' containing x such that K' has Property* C′.

Proof. Let R' be a region containing x whose closure is a proper subset of R and let K be the component of $\bar{R}'$ containing x. Let K' denote the subset of K to which the point z belongs if and only if there does not exist a point r of $K - \{x, z\}$ such that every subcontinuum of K containing x and z contains r. Suppose that K' contains no point other than x. Now either K is semi-locally connected at x or x belongs to a non-degenerate continuum of convergence of K ([1], p. 19).

Suppose first that K is semi-locally connected at x. Let g be a region in the space K containing x and not intersecting the boundary in M of R'.

Since x is not a cut point of K, there is an open set V in K containing x whose closure is a subset of g such that $K-V$ is connected. Let H be the component of V containing x and let q be a limit point of H on the boundary of V. Since q is not in K', there is a point r of $K-\{x, q\}$ such that every subcontinuum of K containing x and q contains r. Then r is in $\bar{H}$. Suppose r is not in H. Then r is a limit point of H on the boundary of V. Then $\bar{H}-r$ is connected and is a simple inner limiting set. Then it has Property C. Let T be an arc in $\bar{H}$ containing q but not r, and let s be a point of T different from q. Then x, s and q are three points of $\bar{H}-r$ and it contains a continuum N containing x and one of the points s and q. In either case, $N+T$ is a subcontinuum of K containing x and q but not r, which is impossible. Thus r is in H and hence in V. Let e be a point of M not in the closure of R'. By Properties CH and C', M contains an arc T from e to x which does not contain r. Some subarc T' of T joins x to a point b on the bundary of R' such that $T'-b$ is contained in R'. Then T' is a subset of K. Since b is on the boundary of R', b is not in V. Then $(K-V)+T'$ is a subcontinuum of K containing x and q but not r, a contradiction.

Now suppose that x belongs to a continuum H of convergence of K, say $H=\lim[H_i]$. Let b be a point of $H-x$. Then b is not in K', so there is a point p of $K-\{x, b\}$ such that every subcontinuum of K containing x and b contains p. Then p is in H. Now x and b belong to different components of $K-p$, for if not, the component of $K-p$ containing x and b is a simple inner limiting set and as in the previous section contains a continuum containing x and b. Then no component of $K-p$ contains infinitely many of the continua $H_1, H_2, \ldots$ Let $H_{n_1}, H_{n_2}, \ldots$ be a subsequence of these continua such that if $i \neq j$, H_{n_i} is contained in a different component of $K-p$ than H_{n_j} and such that no H_{n_i} is contained in the component C_x of $K-p$ containing x. For each i, let C_i denote the component of $K-p$ containing H_{n_i}. Then some subsequence C_{k_i} of the sequence C_i has a sequential limiting set N which is a continuum containing x, p and b. Since p is a limit point of each C_{k_i}, $p+C_{k_1}+C_{k_2}+\ldots$ is connected and $N'=N+C_{k_1}+\ldots$ is a subcontinuum of K. Now N contains an arc L from x to p. Since p is not in K', there is a point r of $K-\{x, p\}$ such that every subcontinuum of K containing x and p contains r. Then r is in L. Since $L-p$ is a connected subset of $K-p$ containing x and r, r is in C_x and hence is not in C_{k_i} for any i. Then $p+C_{k_1}+\ldots$ is a connected dense subset of $N'-r$, so $N'-r$ is connected and is a simple inner limiting set. But then $N'-r$ contains a continuum containing x and p which is a contradiction.

Hence K' contains a point z different from x. Now suppose that p is a limit point of K' but that p is not in K'. Then there is a point q of $K-\{x, p\}$ such that every subcontinuum of K containing x and

p contains q. Let C_x be the component of $K-q$ containing x. If z is a point of $K'-q$, there is a subcontinuum of K containing x and z but not q, so z is in C_x. Then $K'-q$ is a subset of C_x and p is a limit point of $K'-q$, so p is in C_x. But C_x is a simple inner limiting set and contains a continuum containing x and p, which is impossible. Thus K' is closed. Now suppose that p and q are points of K'. K contains an arc from p to q. Call one such arc L. Suppose that some point y of L is not in K'. Then there is a point r of $K-\{x, y\}$ such that every subcontinuum of K containing x and y contains r. Now one of the intervals $[p, y]$ and $[y, q]$ of L does not contain r. Call it N and its endpoint t. Then t is in K', so a subcontinuum T of K contains x and t but not r. Then $N+T$ is a subcontinuum of K containing x and y but not r, a contradiction. Hence L is a subset of K' and K' contains every arc in K from p to q. Thus K' is a non-degenerate subcontinuum of K containing x.

Finally, suppose that r, s and t are three points of K'. Since x is not a cut point of K', $K'-x$ is a simple inner limiting set. Then if $t = x$, K' contains a continuum containing r and s but not t. Suppose $t \neq x$. Then K contains a continuum N_1 containing x and r but not t, and a continuum N_2 containing x and s but not t. Then N_1+N_2 is a subcontinuum of K containing r and s but not t, and it contains an arc L from r to s. Then L is a subset of K' and hence of $K'-t$. Hence K' contains a continuum containing r and s but not t, and K' has Property C$'$.

6. A cyclic connectedness theorem.

THEOREM 4. *Let M be a compact continuum having Property* CS. *Then M is cyclicly connected if and only if it has Property* C$'$.

Proof. If M is cyclicly connected it has Property C$'$. Suppose that M has Property C$'$. Suppose the point x of M is not a cut point of any arc in M. Then x is not a cut point of any subcontinuum of M containing it. By applying Theorem 3, there is a sequence $K_1, K_2, \ldots$ such that for each n, K_{n+1} is a non-degenerate proper subcontinuum of K_n, K_n has Property C$'$, and x is the only point common to every K_n. Let a_1 and b_1 be points of K_1-K_2. Let $n_1 = 1$. Using Theorem 2, there exist in K_1 mutually exclusive arcs A_1 and B_1 not containing x, and joining a_1 and b_1 respectively to K_2 such that $A_1 \cdot K_2$ and $B_1 \cdot K_2$ are degenerate. Let a_2 and b_2 denote the points of $A_1 \cdot K_2$ and $B_1 \cdot K_2$, respectively. There is a positive interger $n_2 > n_1$ such that neither a_2 nor b_2 belongs to K_{n_2}. Again using Theorem 2, there exist in the space K_2 mutually exclusive arcs A_2 and B_2 not containing x and joining a_2 and b_2 to K_{n_2} such that $A_2 \cdot K_{n_2}$ and $B_2 \cdot K_{n_2}$ are degenerate. Continue this process to obtain a subsequence $\{K_{n_i}\}$, with mutually exclusive arcs A_i and B_i in $K_{n_{i-1}}$ each having only one point in common with K_{n_i} and that point being an end point of both A_i and A_{i+1} or both B_i and B_{i+1} respectively. Then the point set $N = x+A_1+A_2+\ldots+B_1+B_2+\ldots$ is an arc in M containing x as a cut point, contradicting the original assumption. Hence every point of M is a cut point of some arc in M. By Theorem 1, M is cyclicly connected.

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