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Local inequalities for some functionals in the class S

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Garabedian [1] proved that the local maximum of $|a_5|$ in the class S is assumed by the Koebe function $z/(1-z)^2$. The proof is rather complicated and therefore it would be desirable to find another way to obtain the required estimation. In this note I want to show the possibility of some estimations in the class S using formulas proved by Leja [4].

To each function $f(z)\in S$ there corresponds in one-to-one way the function $w = 1/f(1/\zeta)$ which maps the set $|\zeta| = 1/|z| > 1$ conformally onto the exterior of a certain continuum E with capacity 1; it carries the point $\zeta = \infty$ into $w = \infty$ and the origin O of the coordinate system belongs to E. By the rotation of the coordinate system round the origin through an angle a the coefficients a_n of the corresponding function $f(z)$ are multiplied by powers of $\exp(ia)$ so that the absolute value of $|a_n|$, $n = 2, \ldots,$ remains unchanged. On the other hand, if one changes the position of the point O on E one obtains a new function in the class S.

1. Let $\eta_1, \eta_2, \ldots, \eta_n$ be to n-th extremal system of points in E, i.e., a system of n points in E such that $\prod\limits_{1\leqslant j<k\leqslant n} |\eta_j-\eta_k| = \sup\limits_{\zeta\in E} \prod|\zeta_j-\zeta_k|$.

F. Leja proved the existence of the following limits:

$$s_k = \lim_{n\to\infty} (\eta_1^k+\eta_2^k+\ldots+\eta_n^k)/n, \quad k = 1, 2, \ldots$$

The point $s_1 = O\bar{O}$ is the center of gravity of the natural mass distribution on E and its position relative to E remains unchanged if one changes the coordinate system.

Let us consider the set of all continua E of capacity 1 on the w-plane situated so that the center of gravity $\bar{O}$ is common for all E. It is known that each E lies in the disc K of radius 2 and the center at $\bar{O}$. Among all considered continua E there exist segments of length 4 with their endpoints on the boundary of K. The other continua are contained the interior of K. If the coordinate system is chosen in such a way that the origin O lies at one of the endpoints of a segment, then the corresponding function is the Koebe function.

Leja [4] gave formulas which express the coefficients a_n of $f(z)\in S$ as polynomials in $s_1, s_2, \ldots, s_{n-1}$:

$$a_2 = -s_1, \quad a_3 = (3s_1^2 - s_2)/2,$$

If we compute the "moments" s_k relatively to the point $\bar{O}$ we obtain

$$s_k = s_k(\bar{O}) + \binom{k}{1} s_{k-1}(\bar{O}) s_1 + \binom{k}{2} s_{k-2}(\bar{O}) s_1^2 + \ldots + s_1^k$$

and the preceding formulas have the form

$$\begin{aligned} &a_2 = -s_1, \quad a_3 = s_1^2 - s_2(\bar{O})/2, \quad a_4 = -s_1^3 + s_1 s_2(\bar{O}) - s_3(\bar{O})/3, \\ &a_5 = s_1^4 - 3s_1^2 s_2(\bar{O})/2 + 2s_1 s_3(\bar{O})/3 - s_4(\bar{O})/4 + 5s_2^2(\bar{O})/8, \end{aligned} \tag{1}$$

The coefficients $a_3, a_4, \ldots$ of those functions $f(z)\in S$ which correspond to the same continuum E are polynomials in $a_2 = -s_1 = \bar{O}O$. As the rotation of the coordinate system round 0 does not change the modulus of a_n, we can choose it so that the positive real axis has the direction of $\bar{O}O$ and therefore $a_2 = x \geqslant 0$.

2. In the general case the origin 0 of the coordinate system lies on E and $a_2 = 2 - \varepsilon + i\varepsilon_1$, $\varepsilon > 0$. For the Koebe function we have $\varepsilon = 0$, $\varepsilon_1 = 0$, $s_2(\bar{o}) = 2$, $s_3(\bar{o}) = 0$, $s_4(\bar{o}) = 6$. On the other hand (see [2]), $|s_2(\bar{o})| \leqslant 2$, $|s_4(\bar{o})| \leqslant 6$ for all $f(z)\in S$, the equality holds only for the Koebe function. Therefore if

$$s_2(\bar{o}) = 2 - \delta + i\delta_1, \quad \delta > 0, \quad s_4(\bar{o}) = 6 - \eta + i\eta_1, \quad \eta > 0$$

then, according to (1),

$$\begin{aligned} a_3 &= (2-\varepsilon+i\varepsilon_1)^2 - (2-\delta+i\delta_1)/2 \\ &= 3 - 4\varepsilon + \delta/2 + \varepsilon^2 - \varepsilon_1^2 + i(4\varepsilon_1 - 2\varepsilon\varepsilon_1 - \delta_1/2). \end{aligned} \tag{1*}$$

Since $|a_3| \leqslant 3$ for all $f(z)\in S$, so $\mathrm{re}\, a_3 \leqslant 3$, i.e.

$$-4\varepsilon + \delta/2 - \varepsilon_1^2 + \varepsilon^2 < 0 \quad \text{for all } f(z)\in S,\ f(z) \neq z/(1-z)^2. \tag{2}$$

In a similar way it follows from the inequality $|a_4| \leqslant 4$ that

$$-10\varepsilon + 2\delta - \mathrm{re}\, s_3(\bar{o})/3 + 6\varepsilon^2 - 6\varepsilon_1^2 - \varepsilon^3 + 3\varepsilon\varepsilon_1^2 + \varepsilon_1\delta_1 - \varepsilon\delta < 0; \tag{3}$$

M. Schiffer proved the inequality

$$|a_4 - 2a_2a_3 + \tfrac{13}{12}a_2^3| \leqslant \tfrac{2}{3}.$$

If one substitutes (1) one obtains

$$\left|\frac{a_2^3}{12} - \frac{s_3(\bar{o})}{3}\right| \leqslant \tfrac{2}{3},$$

hence

$$(4)\qquad -\varepsilon-\frac{\operatorname{re}s_3(\bar{o})}{3}-\frac{\varepsilon_1^2}{2}+\frac{\varepsilon\varepsilon_1^2}{4}+\frac{\varepsilon^2}{2}-\frac{\varepsilon^3}{12}<0,\qquad f(z)\in S,\ f(z)\neq z/(1-z)^2.$$

The real part of a_5 has the value

$$(5)\qquad \begin{aligned}\operatorname{re}a_5 &= 5-20\varepsilon-\tfrac{4}{3}\operatorname{re}s_3(\bar{o})+\eta/4+\tfrac{7}{2}\delta+21\varepsilon^2-8\varepsilon^3+\varepsilon^4-\\ &\quad -21\varepsilon_1^2+24\varepsilon\varepsilon_1^2-6\varepsilon^2\varepsilon_1^2+\varepsilon_1^4-6\varepsilon\delta+\tfrac{3}{2}\varepsilon^2\delta-\tfrac{3}{2}\varepsilon_1^2\delta+\\ &\quad +6\varepsilon_1\delta_1-3\varepsilon\varepsilon_1\delta_1+\tfrac{2}{3}\varepsilon\operatorname{re}s_3(\bar{o})+\tfrac{5}{8}\delta^2-\tfrac{5}{8}\delta_1^2+\tfrac{2}{3}\varepsilon_1\operatorname{im}s_3(\bar{o}).\end{aligned}$$

Multiplying (3) by $\frac{7}{4}$ and (4) by $\frac{9}{4}$, we obtain

$$-\tfrac{35}{2}\varepsilon+\tfrac{7}{2}\delta-\tfrac{7}{12}\operatorname{re}s_3(\bar{o})+\tfrac{21}{2}\varepsilon^2-\tfrac{7}{4}\varepsilon^3-\tfrac{21}{2}\varepsilon_1^2+\tfrac{21}{4}\varepsilon\varepsilon_1^2-\tfrac{7}{4}\varepsilon\delta+\tfrac{7}{4}\varepsilon_1\delta_1<0,$$

$$-\tfrac{9}{4}\varepsilon-\tfrac{3}{4}\operatorname{re}s_3(\bar{o})-\tfrac{9}{8}\varepsilon_1^2+\tfrac{9}{16}\varepsilon\varepsilon_1^2+\tfrac{9}{8}\varepsilon^2-\tfrac{3}{16}\varepsilon^3<0.$$

Therefore

$$\begin{aligned}\operatorname{re}a_5 &< 5+\tfrac{1}{4}\eta-\tfrac{1}{4}\varepsilon+\tfrac{75}{8}\varepsilon^2-\tfrac{97}{16}\varepsilon^3+\varepsilon^4-\tfrac{75}{8}\varepsilon_1^2+\tfrac{291}{16}\varepsilon\varepsilon_1^2-6\varepsilon^2\varepsilon_1^2+\varepsilon_1^4-\tfrac{17}{4}\varepsilon\delta+\\ &\quad +\tfrac{3}{2}\varepsilon^2\delta-\tfrac{3}{2}\varepsilon_1^2\delta+\tfrac{17}{4}\varepsilon_1\delta_1-3\varepsilon\varepsilon_1\delta_1+\tfrac{2}{3}\varepsilon\operatorname{re}s_3(\bar{o})+\tfrac{5}{8}\delta^2-\tfrac{5}{8}\delta_1^2+\tfrac{2}{3}\varepsilon_1\operatorname{im}s_3(\bar{o}).\end{aligned}$$

Multiplying (2) by $\frac{5}{4}\delta$ we have $\frac{5}{8}\delta^2< -\frac{5}{4}\varepsilon^2\delta+5\varepsilon\delta+\frac{5}{4}\varepsilon_1^2\delta$. As $\frac{5}{4}\varepsilon_1\delta_1 = -\frac{5}{8}(\delta_1-\varepsilon_1)^2+\frac{5}{8}\delta_1^2+\frac{5}{8}\varepsilon_1^2$, it follows that

$$\begin{aligned}\operatorname{re}a_5 &< 5+\tfrac{1}{4}\eta+\varepsilon[-\tfrac{1}{4}+\tfrac{75}{8}\varepsilon-\tfrac{97}{16}\varepsilon^2+\varepsilon^3+\tfrac{291}{16}\varepsilon_1^2+\tfrac{1}{4}\varepsilon\delta-\tfrac{3}{2}\varepsilon_1\delta+\tfrac{2}{3}\operatorname{re}s_3(\bar{o})-\\ &\quad -6\varepsilon\varepsilon_1^2-3\varepsilon_1\delta_1+\tfrac{3}{4}\delta]+\varepsilon_1^2[-\tfrac{35}{4}-\tfrac{1}{4}\delta+\varepsilon_1^2]-\tfrac{5}{8}[\delta_1-\varepsilon_1]^2+\tfrac{2}{3}\varepsilon_1\operatorname{im}s_3(\bar{o})+3\varepsilon_1\delta_1.\end{aligned}$$

But $\operatorname{re}\frac{1}{4}s_4(\bar{o})=\frac{3}{2}-\frac{1}{4}\eta$, hence

$$(6)\qquad \begin{aligned}\operatorname{re}[a_5+\tfrac{1}{4}s_4(\bar{o})] &< 6\tfrac{1}{2}+\varepsilon[-\tfrac{1}{4}+\ldots]+\varepsilon_1^2[-\tfrac{35}{4}-\ldots]-\tfrac{5}{8}[\delta_1-\varepsilon_1]^2+\\ &\quad +3\varepsilon_1\delta_1+\tfrac{2}{3}\varepsilon_1\operatorname{im}s_3(\bar{o})\qquad \text{for all } f(z)\in S,\ f(z)\neq\frac{z}{(1-z)^2}.\end{aligned}$$

Let $A>0$, $\varepsilon_0>0$ be two fixed numbers. By the (A,ε_0)-neighbourhood of the Koebe function we understand the set of all functions $f(z)\in S$ for which $|\varepsilon_1|<A\varepsilon$, $0<\varepsilon<\varepsilon_0$ holds. The last inequality proves that:

For all functions $f(z)\in S$ which belong to the (A,ε_0) neighbourhood of the Koebe function with sufficiently small ε_0 the inequality ([1])

$$(6^*)\qquad \operatorname{re}[a_5+\tfrac{1}{4}s_4(\bar{o})]<6\tfrac{1}{2}$$

holds.

([1]) When $|a_2|=2$, $\varepsilon_1\neq 0$, then $\varepsilon_1^2=4\varepsilon-\varepsilon^2$, $s_3(\bar{o})=0$. For sufficiently small $\varepsilon>0$ inequality (6*) holds.

3. It is shown in [3] that

$$|a_5 - aa_2^4| \leqslant \begin{cases} 16a-5, & a \geqslant 1, \\ 5-16a, & a \leqslant -1 \end{cases} \tag{7}$$

for all $f(z) \in S$.

Now we can show that (7) is true for $a \leqslant -\frac{1}{2}$ and all $f(z)$ which belong to sufficiently small (A, ε_0) neighbourhood of $z/(1-z)^2$.

Proof. As shown in [3], $|\frac{3}{4}s_2^2(\bar{o}) - s_4(\bar{o})| \leqslant \frac{3}{2}$ for all $f(z) \in S$; hence, in our previous notation,

$$\tfrac{1}{4}\eta < 3\delta - \tfrac{3}{4}\delta^2 + \tfrac{3}{4}\delta_1^2. \tag{8}$$

From (5) and (8) we have

$$\begin{aligned} \operatorname{re} a_5 &= 5 - 20\varepsilon - \tfrac{4}{3}\operatorname{re} s_3(\bar{o}) + \tfrac{7}{2}\delta + \tfrac{1}{4}\eta + \tfrac{5}{8}\delta^2 - \tfrac{5}{8}\delta_1^2 + O_1 \\ &< 5 - 20\varepsilon - \tfrac{4}{3}\operatorname{re} s_3(\bar{o}) + \tfrac{13}{2}\delta - \tfrac{1}{8}\delta^2 + \tfrac{1}{8}\delta_1^2 + o_1, \end{aligned}$$

where

$$\begin{aligned} O_1 = {} & 21\varepsilon^2 - 8\varepsilon^3 + \varepsilon^4 - 21\varepsilon_1^2 + 24\varepsilon\varepsilon_1^2 - 6\varepsilon^2\varepsilon_1^2 + \varepsilon_1^4 - 6\varepsilon\delta + \tfrac{3}{2}\varepsilon^2\delta - \tfrac{3}{2}\varepsilon_1^2\delta + \\ & + 6\varepsilon_1\delta_1 - 3\varepsilon\varepsilon_1\delta_1 + \tfrac{2}{3}\varepsilon_1 \operatorname{im} s_3(\bar{o}) + \tfrac{2}{3}\varepsilon \operatorname{re} s_3(\bar{o}). \end{aligned}$$

Since $\delta_1^2 \leqslant 4\delta - \delta^2$, we get

$$\operatorname{re} a_5 < 5 - 20\varepsilon - \tfrac{4}{3}\operatorname{re} s_3(\bar{o}) + 7\delta - \tfrac{1}{4}\delta^2 + O_1 = 5 - 20\varepsilon - \tfrac{4}{3}\operatorname{re} s_3(\bar{o}) + 7\delta + O_2,$$
$$O_2 = O_1 - \tfrac{1}{4}\delta^2.$$

Using (3) we obtain

$$7\delta < \tfrac{70}{2}\varepsilon + \tfrac{7}{6}\operatorname{re} s_3(\bar{o}) - 21\varepsilon^2 + 21\varepsilon_1^2 + \tfrac{7}{2}\varepsilon^3 - \tfrac{21}{2}\varepsilon\varepsilon_1^2 - \tfrac{7}{2}\varepsilon_1\delta_1 + \tfrac{7}{2}\varepsilon\delta.$$

Therefore

$$\operatorname{re} a_5 < 5 + 15\varepsilon - \frac{\operatorname{re} s_3(\bar{o})}{6} + O_2 + O_3,$$
$$O_3 = -21\varepsilon^2 + 21\varepsilon_1^2 + \tfrac{7}{2}\varepsilon^3 - \tfrac{21}{2}\varepsilon\varepsilon_1^2 - \tfrac{7}{3}\varepsilon_1\delta_1 + \tfrac{7}{2}\varepsilon\delta.$$

Finally (see (4))

$$-\frac{\operatorname{re} s_3(\bar{o})}{6} < \tfrac{1}{2}\varepsilon + \tfrac{1}{4}\varepsilon_1^2 - \tfrac{1}{8}\varepsilon\varepsilon_1^2 - \tfrac{1}{4}\varepsilon^2 + \tfrac{1}{12}\varepsilon^3 = \tfrac{1}{2}\varepsilon + O_4,$$
$$O_4 = \tfrac{1}{4}\varepsilon_1^2 - \tfrac{1}{8}\varepsilon\varepsilon_1^2 - \tfrac{1}{4}\varepsilon^2 + \tfrac{1}{12}\varepsilon^3.$$

Hence

$$\operatorname{re} a_5 < 5+15\tfrac{1}{2}\varepsilon+O_2+O_3+O_4$$

and

$$\operatorname{re}[a_5+\tfrac{1}{2}a_2^4] < 13+15\tfrac{1}{2}\varepsilon-\tfrac{32}{2}\varepsilon+\tfrac{1}{2}(24\varepsilon^2-8\varepsilon^3+\varepsilon^4-24\varepsilon_1^2+24\varepsilon\varepsilon_1^2+\varepsilon_1^4-6\varepsilon^2\varepsilon_1^2)+$$
$$+O_2+O_3+O_4 = 13-\tfrac{1}{2}\varepsilon+O_2+O_3+O_4+O_5,$$
$$O_5 = 12\varepsilon^2-4\varepsilon^3+\tfrac{1}{2}\varepsilon^4-12\varepsilon_1^2+12\varepsilon\varepsilon_1^2+\tfrac{1}{2}\varepsilon_1^4-3\varepsilon^2\varepsilon_1^2.$$

For functions $f(z)$ which belong to a sufficiently small (A, ε_0)-neighbourhood of the Koebe function we have

$$\operatorname{re}[a_5+\tfrac{1}{2}a_2^4] < 13.$$

4. According to (1), if $a_2 = x \geqslant 0$, $x \leqslant 2$, then

$$\operatorname{re} a_5 = x^4-\tfrac{3}{2}x^2\operatorname{re} s_2(\bar{o})-\tfrac{2}{3}x\operatorname{re} s_3(\bar{o})-\frac{\operatorname{re} s_4(\bar{o})}{4}+\tfrac{5}{8}\operatorname{re} s_2^2(\bar{o}).$$

We can find global estimations for $\operatorname{re} a_5$. Indeed, as proved in [3],

$$\tfrac{3}{4}\operatorname{re} s_2^2(\bar{o})-\frac{\operatorname{re} s_4(\bar{o})}{4} \leqslant \tfrac{3}{2}$$

and

$$\left|\frac{x^3}{12}-\frac{s_3(\bar{o})}{3}\right| \leqslant \tfrac{2}{3}$$

holds for all $f(z) \in S$. From the last inequality follows

$$-\tfrac{2}{3}x\operatorname{re} s_3(\bar{o}) \leqslant \tfrac{4}{3}x-\tfrac{1}{6}x^4.$$

Hence

$$\operatorname{re} a_5 < \tfrac{5}{6}x^4-\tfrac{3}{2}x^2\operatorname{re} s_2(\bar{o})+\tfrac{4}{3}x+\tfrac{3}{2}-\frac{\operatorname{re} s_2^2(\bar{o})}{8}.$$

On the other hand,

$$\operatorname{re} a_5 = x^4+\tfrac{3}{2}\operatorname{re} s_2(\bar{o})\left[-\frac{x^2}{2}+\frac{\operatorname{re} s_2(\bar{o})}{4}\right]-\tfrac{3}{4}x^2\operatorname{re} s_2(\bar{o})-$$
$$-\tfrac{2}{3}x\operatorname{re} s_3(\bar{o})+\left[\frac{\operatorname{re} s_2^2(\bar{o})}{4}-\frac{\operatorname{re} s_4(\bar{o})}{4}\right]-\tfrac{3}{8}[\operatorname{re} s_2(\bar{o})]^2+\tfrac{3}{8}\operatorname{re} s_2^2(\bar{o}).$$

But

$$-\frac{x^2}{2}+\frac{\operatorname{re} s_2(\bar{o})}{4} = -\frac{\operatorname{re} a_3}{2} \geqslant -\tfrac{3}{2} \quad \text{and} \quad |s_4(\bar{o})-s_2^2(\bar{o})| \leqslant 2$$

(see [2]). Therefore

$$\frac{\operatorname{re} s_2^2(\bar{o})}{4} - \frac{\operatorname{re} s_4(\bar{o})}{4} \geqslant -\tfrac{1}{2},$$

$$-\tfrac{3}{8}[\operatorname{re} s_2(\bar{o})]^2 + \tfrac{3}{8}\operatorname{re} s_2^2(\bar{o}) = -\tfrac{3}{8}(2-\delta)^2 + \tfrac{3}{8}(4-4\delta+\delta^2-\delta_1^2) = -\tfrac{3}{8}\delta_1^2$$

and

$$\operatorname{re} a_5 \geqslant x^4 - \tfrac{3}{2}x^2 - \tfrac{2}{3}x \operatorname{re} s_3(\bar{o}) - 5 - \tfrac{3}{8}\delta_1^2.$$

References

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