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Mapping properties of a class of univalent functions with pre-assigned zero and pole

by Z. A. Lewandowski (Lublin), R. J. Libera* (Newark) and E. J. Złotkiewicz (Lublin)

1. Introduction. We will use the symbol U_p to represent the class of functions meromorphic and univalent in the open unit disk Δ, $\Delta = \{z\colon z \in C, |z| < 1\}$, normalized by the conditions

$$f(0) = 0, \quad f'(0) = 1 \quad \text{and} \quad f(p) = \infty, \tag{1.1}$$

for a fixed p, $0 < p < 1$. Functions satisfying these conditions have been studied by several authors (see [5], [6], for example); and by using variational methods Złotkiewicz [8] characterized the region $\left\{\frac{\zeta f'(\zeta)}{f(\zeta)}\colon f(z) \in U_p\right\}$, for ζ fixed in Δ.

Our purpose here is to determine some common properties of the set $f[\Delta]$, the image of Δ under an arbitrary $f(z)$ in U_p. The principal result is the theorem which follows.

Theorem 1. *For a given R, $R \geqslant 0$, and an $f(z)$ in U_p let $m(R; f(z))$ be the Lebesgue measure of the arc of the circle $C_R = \{\omega\colon |\omega| = R\}$ left uncovered by $f[\Delta]$, i.e.*

$$m(R; f(z)) = \operatorname{meas}(C_R \cap \mathscr{C} f[\Delta]), \tag{1.2}$$

where $\mathscr{C}$ is used to denote the complement with respect to the complex plane; and let

$$\Phi(R) = \sup\{m(R; f(z))\colon f(z) \in U_p\}; \tag{1.3}$$

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then

$$(1.4)\qquad \Phi(R) = \begin{cases} 4R \arcsin\left[(1+p)\sqrt{\dfrac{R}{p}} - 1\right], & \dfrac{p}{(1+p)^2} \leqslant R \leqslant p, \\ 4R \arcsin\left[1-(1-p)\sqrt{\dfrac{R}{p}}\right], & p \leqslant R \leqslant \dfrac{p}{(1-p)^2}, \\ 0, & \textit{otherwise}. \end{cases}$$

This conclusion cannot be improved.

Similar questions have been considered by others. Jenkins [2] obtained the corresponding result for the class $\mathscr{S}$ (of functions $f(z)$ regular and univalent in Δ normalized so that $f'(0)-1 = f(0) = 0$) and Reade and Złotkiewicz [7] considered the same question for regular univalent functions with Montel's normalization. In both these papers the main tool is circular symmetrization and this is the case in the proof of Theorem 1. However, the normalization (1.1) together with the boundedness of $\mathscr{C}f[\Delta]$ for $f(z)$ in U_p cause technical difficulties in our proof which were not experienced in either case cited. The proof of Theorem 1 is given in Sections 3 and 4 below.

In Section 2 we given an explicit representation for the Koebe set of U_p; this is the set

$$(1.5)\qquad \mathscr{K}(U_p) = \bigcap_{f(z)\in U_p} f[\Delta].$$

$\mathscr{K}(U_p)$ contains all points common to each $f[\Delta]$ and no others.

2. The Koebe set of U_p. If $f(z)$ is in U_p and $f(z)$ does not assume the value $-c$, then

$$(2.1)\qquad g(z) = \frac{cf(z)}{c+f(z)}$$

is in $\mathscr{S}$. Taking limits we may write $g(p) = c$. It follows from a well-known bound of Grunsky [1], p. 117; that

$$(2.2)\qquad \left|\log c - \log\left(\frac{p}{1-p^2}\right)\right| \leqslant \log\left(\frac{1+p}{1-p}\right).$$

Conversely, if c satisfies (2.2), then there is an $f(z)$ in $\mathscr{S}$ such that $f(p) = c$. Now $\varphi(z) = f(p)f(z)/(f(p)-f(z))$ is in U_p and $\varphi(z)$ does not assume the value $-c$. Rearrangement and exponentiation of (2.2) gives the result which follows:

THEOREM 2. *For z in Δ we let*

$$(2.3)\qquad F(z) = \frac{p}{p^2-1}\left(\frac{1+p}{1-p}\right)^z,$$

where the principal branch is chosen. Then the region of values omitted by some function in U_p is the image of the closure of the disk under $F(z)$; i.e.,

$$\mathscr{C}\mathscr{K}(U_p) = F[\bar{\Delta}]. \tag{2.4}$$

COROLLARY 1. *If for z in Δ,*

$$|z| \log\left(\frac{1+p}{1-p}\right) \leqslant \pi, \tag{2.5}$$

then the set of omitted values is simply-connected; otherwise it is the closure of a ring domain.

COROLLARY 2. *If d is a value not assumed by an $f(z)$ in U_p, then*

$$\frac{p}{(1+p)^2} \leqslant |d| \leqslant \frac{p}{(1-p)^2}. \tag{2.6}$$

The last bounds can also be obtained directly from (2.1) and known bounds for $\mathscr{S}$.

3. A special subclass of U_p. Here we construct and examine the functions of U_p which map Δ onto the complement of the union of a segment of the negative real axis and a circular arc symmetric with respect to that axis; the cases where either the arc or the segment degenerate to a point will be included.

The Zhukovskii function,

$$Z(\omega) = \omega\varrho \frac{1-\varrho\omega}{\varrho-\omega}, \tag{3.1}$$

maps $\partial\Delta$ onto an arc of the circle of radius ϱ centered at the origin which is symmetric with respect to the real axis. This makes sense for $\varrho < 1$ and in this case the tips of the arc are the points $\varrho \pm i\sqrt{1-\varrho^2}$.

For $-1 < t < 1$, we let

$$h(z) = \frac{z}{1-2tz+z^2}, \tag{3.2}$$

for z in Δ. $h(z)$ is typically-real and starlike with respect to the origin and maps Δ onto the complement of the plane slit from ∞ along the real axis. The ends of the slit are the points $-1/2(1+t)$ and $1/2(1-t)$ which correspond to the boundary points -1 and 1, respectively.

Now the function given by $\check{h}(kh(z))$, where $\check{h}$ denotes the inverse of h and $0 < k < 1$, maps Δ onto itself but equipped with radial slits issuing from $\partial\Delta$ along the real axis. The image of this set under $Z(\omega)$ is a transformation of the type we seek. To guarantee normalization and that an arc of the circle of radius R be left uncovered we introduce the function

$$\omega(z) = M\check{h}\left(\frac{h\left(\frac{R}{M}\right)}{h(p)}h(z)\right), \tag{3.3}$$

with the parameters satisfying the conditions $M \geqslant 1$ and $h\left(\frac{R}{M}\right) = kh(p)$. Prior restrictions on k and properties of $h(z)$ insure that $h\left(\frac{R}{M}\right) < h(p)$, $R < Mp$ and $\omega(p) = R$.

Modifying (3.1) to correspond to the disk of radius M we form the composition

$$F(z) = \frac{R}{M}\omega(z)\frac{1-\left(\frac{R}{M}\right)\left(\frac{\omega(z)}{M}\right)}{\left(\frac{R}{M}\right)-\left(\frac{\omega(z)}{M}\right)} = \frac{R}{M^2}\omega(z)\frac{M^2-R\omega(z)}{R-\omega(z)}. \tag{3.4}$$

Normalizing $F(z)$ so that its interior mapping radius is 1 gives the additional condition that

$$Mh\left(\frac{R}{M}\right) = h(p). \tag{3.5}$$

We denote the class of all functions $F(z)$ defined by (3.1) through (3.5) by K_p; K_p is a subclass of U_p with the property that if $F(z)$ is in K_p, then $F[\Delta]$ is its own circular symmetrization with respect to the negative real axis.

We now proceed to give a solution to the problem of Theorem 1 over the subclass K_p.

LEMMA. $\sup\{m(R; F(z)) : F(z) \in K_p\} = \Phi(R)$, *with* $\Phi(R)$ *given as in* (1.4).

To justify the lemma we further analyze the mapping properties of $F(z)$ in K_p, for R satisfying (2.6), i.e., when

$$\frac{p}{(1+p)^2} \leqslant R \leqslant \frac{p}{(1-p)^2}. \tag{3.6}$$

If R is at either end point of (3.6), then $\Phi(R)$ takes on the value 0 and the corresponding function $F(z)$ in K_p has the property that $F[\partial\Delta]$ is simply a segment of the real axis. In all other cases $F[\partial\Delta]$ includes a circular arc. Therefore we need consider only the cases where R is inside (3.6).

For such an R, fixed, the end points of the circular slit are among points on $\partial\Delta$ where $h'(z) = 0$. A calculation shows that $h'(z) = 0$ when $z = \pm 1$ and when $\omega = \omega(z)$ satisfies

$$\omega^2 - 2R\omega + M^2 = 0. \tag{3.7}$$

The first two zeros correspond to the end points of the segment on the real axis, hence we are interested in the solutions of (3.7). We write these as

$$\omega = R \pm i\sqrt{M^2-R^2} = Me^{\pm i\alpha} \tag{3.8}$$

and assume that $h(z_0) = Me^{i\alpha}$, $\alpha > 0$.

The angular measure of the arc of the circle of radius R contained in $h[\Delta]$ is

$$2\arg h(z_0) = 2\arg\left(\frac{M-Re^{i\alpha}}{Re^{i\alpha}-M}\right) = 2[\pi+2\arg(M^2-RMe^{i\alpha})] \tag{3.9}$$

$$= 2\left[\pi+2\left(\frac{-\pi}{2}+\alpha\right)\right] = 4\arccos\frac{R}{M}$$

and the angular measure of the arc of the same circle subtended by $h[\partial\Delta]$ is

$$2\pi - 4\arccos\frac{R}{M} = 4\arcsin\frac{R}{M}. \tag{3.10}$$

To complete the proof of the lemma we maximize the quotient R/M. Because of (3.5), M is not arbitrary. Indeed, (3.2) together with (3.5) give the relation

$$\left(\frac{R}{M}-t\right)^2 = (t-R)^2-\left[R^2-\left(p+\frac{1}{p}\right)R+1\right]. \tag{3.11}$$

Now writing $\frac{R}{M} = s(t)$, solving (3.11) for $s(t)$ and maximizing $s(t)$ over the interval $-1 \leqslant t \leqslant 1$, shows that for $\frac{p}{(1+p)^2} < R \leqslant p$, $s(t)$ assumes its maximum $t = -1$ and for $p \leqslant R < \frac{p}{(1-p)^2}$, $s(t)$ assumes its maximum at $t = 1$. This together with (3.9) and (1.4) concludes our discussion of the lemma.

Finally, to ease discussion in the next section we will speak of domains of "type K_p". Such a domain will consist of the complement of a segment of the negative real axis met by a circular arc symmetric with respect to that axis and with its center at the origin. Furthermore we will refer to the components of the complement of such a domain as the "left segment", "right segment" and "the arc". Clearly, for each $f(z)$ in K_p, $f[\Delta]$ is a domain of type K_p, however, the converse does not follow.

For each R the function $f(z)$ in K_p corresponding to the value $m(R; f(z)) = \Phi(R)$ is unique. If $p < R < \frac{p}{(1-p)^2}$, then $\mathscr{C}f[\Delta]$ is the arc of C_R together with the right segment; if $R = p$, $\mathscr{C}f[\Delta]$ is only the arc of C_R; if $\frac{p}{(1+p)^2} < R < p$, then $\mathscr{C}f[\Delta]$ is the arc of C_R and the left segment; and when R assumes either end point $\frac{p}{(1+p)^2}$, or $\frac{p}{(1-p)^2}$, then $\mathscr{C}f[\Delta]$ is the segment $\left[\frac{p}{(1+p)^2}, \frac{p}{(1-p)^2}\right]$.

4. Proof of Theorem 1. U_p is a compact family, hence there is a function $f(z)$ in the class which is extremal for the problem of Theorem 1. This means that the set of omitted values on $|z| = R$ is maximal for $f(z)$.

Let $D = f[\Delta]$ and let $D^{(1)}$ be the circular symmetrization of D with respect to the positive real axis. Let $r(D; 0)$ be the inner conformal radius of D with respect to zero and let $g(0, z; D)$ be the corresponding Green's function. Then it follows from well-known properties of symmetrization [3] that

$$1 = r(D; 0) \leqslant r(D^{(1)}; 0) \tag{4.1}$$

and from the work of Krzyż [4] that

$$g(0, \infty;\ D) \leqslant g(0, \infty;\ D^{(1)}) \quad \text{and} \quad p^{(1)} \leqslant p, \tag{4.2}$$

where $p^{(1)}$ is the pole of the function mapping Δ onto $D^{(1)}$.

We now place $D^{(1)}$ inside a domain $D^{(2)}$ of "type K_p" chosen so that the arc of the complement of $D^{(2)}$ coincides with the circular arc of radius R of the complement of $D^{(1)}$. (We have essentially cut the "fat" from $D^{(1)}$ to obtain a domain of type K_p.) This operation gives

$$r(0, D^{(1)}) \leqslant r(0, D^{(2)}) \quad \text{and} \quad g(0, \infty; D^{(1)}) \leqslant g(0, \infty; D^{(2)}). \tag{4.3}$$

Next we remove the left or the right segment of the complement of $D^{(2)}$ and again obtain a domain $D^{(3)}$ of type K_p whose complement has at most one segment. Which segment is removed in this operation depends on R and the corresponding extremal function in K_p as discussed in the preceding section. We then have the inclusion relation $D^{(2)} \subset D^{(3)}$. Now, if $r(0, D^{(3)})$ and $p^{(3)}$ carry meanings like those above we have, from (4.1), (4.2), and (4.3), that

$$1 \leqslant r(0, D^{(3)}) \quad \text{and} \quad p^{(3)} \leqslant p. \tag{4.4}$$

Equality can occur in (4.4) only if $D \equiv D^{(3)}$ (see [3], p. 136, for a discussion of this point).

Suppose $1 < r(0, D^{(3)})$. We modify domain $D^{(3)}$ by prolonging its segment and obtaining a domain $D^{(4)}$ still of type K_p such that $D^{(4)} \subset D^{(3)}$. This operation decreases the inner conformal radius and shifts the pole to the right. We prolong the segment until we obtain one of the following:

$$\begin{array}{llll} 1^\circ & p^{(4)} = p & \text{and} & r(0, D^{(4)}) = 1, \\ 2^\circ & p^{(4)} < p & \text{and} & r(0, D^{(4)}) = 1, \\ 3^\circ & p^{(4)} = p & \text{and} & r(0, D^{(4)}) > 1. \end{array} \tag{4.5}$$

We will show that 2° and 3° are untenable; we dispense with 3° first.

It follows from the construction of the family K_p given above that a function mapping Δ onto a domain of type K_q has the form

$$\varphi(z) = A\left\{F\left(\frac{z+Q}{1+Qz}\right)-F(Q)\right\} \quad \text{for } Q = \frac{p-q}{1-pq},$$

$f(z)$ in K_p and $A > 0$. Now, if we assume furthermore that $\varphi[\Delta] = D^{(4)}$ and $p = q$, i.e., we let $q \to p$, then $\varphi(z) = AF(z)$ and $A = r(0, D^{(4)}) > 0$. An examination of $\varphi[\Delta]$ and $F(z)$ reveals that $F(-1) = \varphi(-1) = -R$ for the case when $p \leqslant R \leqslant \frac{p}{(1-p)^2}$ and $F(1) = \varphi(1) = R$ otherwise. In either case we conclude that $A = 1$, i.e., $r(0, D^{(4)}) = 1$. Consequently, 3° cannot hold.

Now we suppose $\varphi(z)$ and $F(z)$ have meanings like those above and that 2° holds. Then, with $q = p^{(4)}$, we have

$$\varphi(z) = \frac{F\left(\frac{z+Q}{1+Qz}\right)-F(Q)}{(1-Q^2)F'(Q)}.$$

Condition 2° requires that $\varphi(-1) = f(-1) = -R$, or $(1-Q^2)F'(Q)R = R+F(Q)$, and that $q < p$. But if this equation has a solution for q, with $q < p$, then it follows from the lemma that to this smaller value q, i.e., $p^{(4)}$ smaller than p, there corresponds a smaller value to the measure of the uncovered circular arc. But this contradicts our choice $f(z)$ as being the extremal function for the problem.

We conclude that 1° of (4.5) is the exclusively valid condition and consequently that the unique extremal function given in the lemma gives the solution for whole class U_p as well.

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DEPARTMENT OF APPLIED MATHEMATICS
MARIAE CURIE-SKŁODOWSKA UNIVERSITY
LUBLIN, POLAND

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF DELAWARE
NEWARK, DELAWARE, U.S.A.

INSTITUTE OF MATHEMATICS
MARIAE CURIE-SKŁODOWSKA UNIVERSITY
LUBLIN, POLAND

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