UNBOUNDED MULTIPLIERS AND SUMMATION OF SERIES

BY

A. F. KLEINER (DES MOINES, IOWA)

Let $A$ be a (sequence to sequence) matrix method of summation. If the sequence of partial sums $\{\sum_{k=1}^{n} u_k\}$ is in the convergence domain of the matrix, then $A$ sums the infinite series $\sum_{k} u_k$.

We consider the question:

If a regular matrix sums the series $\sum_{k} u_k$, does there exist an unbounded sequence $\{\lambda_k\}$ such that $A$ sums $\sum_{k} \lambda_k u_k$?

Bryant showed that the answer is yes if $A$ is the $(C, 1)$ matrix ([1], lemma 2.1). More generally, the answer is yes if $\sum_{k} u_k$ has bounded partial sums. An example is also given which shows that such a sequence does not necessarily exist if the series does not have bounded partial sums. Finally, an application of theorem 1 is given.

If $A$ is a matrix, $e_A$ is the convergence domain of $A$. For a sequence $x$ define $Vx_k = x_k - x_{k-1}$ (where $x_0 = 0$) and $Vx = \{Vx_k\}$; then $x$ is the sequence of partial sums of the series whose terms form the sequence $Vx$. Since $\sum Vx$ is summed by $A$ if and only if $x \in e_A$, we define $Vc_A = \{Vx : x \in e_A\}$.

If $x$ is summed by $A$ to $a$, this is indicated by $\lim_{A} x = a$.

The spaces of bounded, convergent and null sequences are denoted by $m, c$ and $c_0$, respectively; and for $x \in m$,

$$||x|| = \sup \{|x_n| : n = 1, 2, \ldots\}.$$  

For a matrix $A$, $||A|| = \sup \{|\sum a_{nk}| : n = 1, 2, \ldots\}$.

An index sequence $Q = \{q_i\}$ is a strictly increasing sequence of integers such that $q_0 = 0$. If $Q$ is an index sequence we define

$$Q(j) = \{q_{j-1} + 1, \ldots, q_j\} \quad \text{for } j = 1, 2, \ldots$$

and denote the sum $\sum_{k \in Q(j)} x_k$ by $\sum_{k \in Q(j)} x_k (k \in Q(j))$. A sequence $x$ is said to satisfy condition $|Q|$ if $\sum_{k \in Q(j)} |Vx_k| = o(1)$ ($k \in Q(j)$).
We prove first the following theorem:

**Theorem 1.** Let $A$ be a regular matrix. If $\sum A x_k$ is a divergent series with bounded partial sums which is summed by $A$, then there exists an unbounded sequence $\{\lambda_k\}$ such that the series $\sum \lambda_k A x_k$ is summed by $A$.

**Proof.** Let $x$ be the sequence of partial sums of $\sum A x_k$. Notice first that it is sufficient to suppose that $A$ sums $x$ to zero, for if $\lim_{x = a \neq 0}$, the sequence $\{x_k - a\}$ is summable $A$ to zero and $A x_k = V(x_k - a)$ for $k > 1$. Now define $G = (g_{nk})$ by $g_{nk} = a_{nk} x_k$. Since $x$ is bounded,

$$\sup_{n,r} \left| \sum_{k=r}^{\infty} g_{nk} \right| = \sup_{n,r} \left| \sum_{k=r}^{\infty} a_{nk} x_k \right| \leq \sup_n \sum_{k} \left| a_{nk} \right| \cdot |x_k| \leq \|A\| \cdot \|x\|.$$

Moreover, $a_{nk} \to 0$ as $n \to \infty$ and $|g_{nk}| = |a_{nk}| \cdot |x_k| \leq |x| \cdot |a_{nk}|$ and so $\lim_{n} g_{nk} = 0$. Finally,

$$\lim_{n} \sum_{k} g_{nk} = \lim_{n} \sum_{k} a_{nk} x_k = \lim_{A} x = 0.$$

Thus $G$ is a coset, $v - c$ matrix and there exists an index sequence $Q$ such that if $y$ satisfies $|Q|$ then $y \in c_Q$ ([3], p. 532, and [4], theorem 3.2). Also since $x$ is a bounded divergent sequence there exists a sequence $y$ such that

$$0 \leq V y_n \leq |V x_n|, \quad y_n \uparrow \infty, \; y \text{ satisfies } |Q|.$$

Hence $\{x_k y_k\} \in c_A$.

Let $\{\lambda_k\}$ be any sequence such that $\lambda_k V x_k = V(x_k y_k)$. Since $x$ is a bounded divergent sequence, $V x_k \neq 0$ for infinitely many $k$, and for these $k$

$$\lambda_k = V(x_k y_k)/V x_k = y_k(V x_k)/(V x_k) + x_k^{-1}(V y_k)/(V x_k)$$

and thus

$$|\lambda_k| \geq |y_k| - |x_k^{-1}| \geq |y_k| - |x|.$$

Since $y_k \uparrow \infty$, the sequence $\{\lambda_k\}$ is unbounded. This completes the proof of the theorem.

Let $x$ be an unbounded sequence and $A$ be a regular matrix such that $c_A = \{a x + e : a \text{ is complex, } e \in c\}$ ([2], theorem 2). Suppose $y$ is a divergent sequence and $\{\lambda_k\}$ is a sequence such that both $V y$ and $\{\lambda_k V y_k\}$ belong to $V c_A$. Then there exist complex numbers $a, \beta (a \neq 0)$ and sequences $e, f \in c$ such that

$$V y_k = a V x_k + V e_k, \quad \lambda_k V y_k = \beta V x_k + V f_k.$$

Thus, for $k$ such that $V x_k \neq 0$,

$$\lambda_k = (\beta V x_k + V f_k)/(a V x_k + V e_k)$$

$$= (\beta/a) + [a V f_k - \beta V e_k]/a \cdot (a V x_k + V e_k)^{-1}.$$
If $x$ is chosen so that $Vx$ is bounded away from zero, then $Vy$ is bounded away from zero and, since $Ve, Vf \in c_0$, we have $\lim_{k} \lambda_k = \beta/\alpha$. In particular, there exists a regular (normal) matrix $A$ such that if $A$ sums the divergent series $\sum_{k} V y_k$ and $A$ sums the series $\sum_{k} \lambda_k V y_k$, then $\{\lambda_k\} \in c_0$.

We now give an application of theorem 1.

**Theorem 2.** Let $A$ be a regular matrix which sums a bounded divergent sequence $x$, such that $Vx \notin c_0$. Then $A$ sums a series with unbounded terms.

**Proof.** Since $x$ is a bounded divergent sequence summed by $A$, there exists, by theorem 1, an unbounded sequence $\{\lambda_k\}$ such that the series $\sum_{k} \lambda_k V x_k$ is summed by $A$. We note, from the proof of theorem 1, that there is a $y$ such that $y_k \uparrow \infty$ and $|\lambda_k| \geq |y_k| - ||x||$ if $V x_k \neq 0$ and that $\lambda_k$ is arbitrary if $V x_k = 0$. If, for $k$ such that $V x_k = 0$ we define $\lambda_k = y_k$, then $|\lambda_k| \geq |y_k| - ||x||$ for each $k$ and $\lim_{k} |\lambda_k| = \infty$. Since $V x \notin c_0$, $\{\lambda_k V x_k\}$ has an unbounded subsequence and the proof is complete.

An extension of theorem 2 and several related examples will appear elsewhere \(^{(1)}\).

**REFERENCES**


DRAKE UNIVERSITY
DES MOINES, IOWA

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\(^{(1)}\) Some of these results were contained in the author’s dissertation, written at Texas A. & M. University under the direction of Jack Bryant.