On meromorphic solutions of a functional equation

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We consider the problem of the existence and uniqueness of meromorphic solutions in a domain $U$ for the equation

\[ \varphi[f(z)] - g(z)\varphi(z) = h(z), \]

where $\varphi(z)$ is the unknown function and $f(z)$, $g(z)$ and $h(z)$ are known functions of one complex variable $z$.

We assume that:

(I) The function $f(z)$ is analytic in a domain $U$, $f(U) \subset U$ and the boundary of the domain $U$ contains at least two finite points.

(II) $f(z) = z$ for $z \in U$ if and only if $z = a$, $a \in U$, and

\[ 0 < |c| < 1, \quad \text{where } c = f'(a); \]

(III) $g(z)$ and $h(z)$ are meromorphic functions in the domain $U$.

This problem was investigated by Raclis [3] in the case where $g(z) \equiv -1$ and by Pranger [2] in the case where $g(z) = \text{const}$ under a little more restrictive assumptions concerning the function $f(z)$.

Our considerations will be based on the following theorem, which has been proved by Smajdor [4]:

**Theorem 1.** Let us suppose that the function $f(z)$ is analytic at the point $z = a$

\[ f(z) = a + c(z-a) + \sum_{k=2}^{\infty} b_k(z-a)^k \quad \text{for } |z-a| < r, \]

where $|c| < 1$, and $h(z, w)$ is an analytic function of two complex variables $z$ and $w$ at the point $(a, b)$ with the expansion

\[ h(z, w) = \sum_{n, m=0}^{\infty} a_{nm}(z-a)^n(w-b)^m, \quad a_{00} = b \]

valid for $|z-a| \leq r_0$, $|w-b| \leq R_0$. Moreover, let $1-c\alpha_{01} \neq 0$ for every $n$ ($n = 1, 2, \ldots$).
Then there exists a unique solution \( \varphi \) of the equation
\[
\varphi(z) = h(z, \varphi[f(z)])
\]
algebraic at the point \( z = a \) and such that \( \varphi(a) = b \).

Let us denote by \( f^n(z) \) the \( n \)-th iteration of the function \( f(z) \):
\[
f^0(z) = z, \quad f^{n+1}(z) = f[f^n(z)].
\]

Now we prove the following

**Lemma 1.** Let hypotheses (I), (II) be fulfilled. Then the sequence \( \{f^n(z)\} \)
of the iterates of the function \( f(z) \) tends to a uniformly on every compact \( K \subset U \).

**Proof.** Since \( f^n(U) \subset U \) for every positive integer \( n \), the sequence \( f^n(z) \) in \( U \) omits at least two values, namely those lying on the boundary of \( U \). On account of Montel’s theorem [1] the sequence \( \{f^n(z)\} \) is a normal family in \( U \). We shall show that the sequence \( \{f^n(z)\} \) tends to a constant function equal to \( a \) in a neighbourhood of the point \( z = a \). There exists a number \( \theta < 1 \) such that \( |f^n(a)| < \theta \) and there exists a number \( r > 0 \) such that the inequality
\[
|f(z) - a| = |f(z) - f(a)| < \theta |z - a|
\]
holds for \( z \in U \) and \( |z - a| < r \). By induction we obtain the inequality
\[
|f^n(z) - a| < \theta^n |z - a|
\]
for \( z \in U \) and \( |z - a| < r \). From (2) it follows that \( \lim_{n \to \infty} f^n(z) = a \) for \( z \in U \)
and \( |z - a| < r \). Since \( f^n(z) \) is a normal family, this completes the proof of Lemma 1.

**Lemma 2.** Let the function \( f(z) \) fulfill hypotheses (I), (II) and let the functions \( g_1(z) \) and \( h_1(z) \) be a meromorphic in the domain \( U \). Moreover, let us suppose that the functions \( g_1(z) \) and \( h_1(z) \) are analytic at the point \( z = a \) and \( g_1(a) \neq 0 \).

If there exists an analytic function \( \varphi(z) \) in the neighbourhood \( U_a \) of the point \( z = a \) such that
\[
(z - a)^r \varphi[f(z)] - g_1(z) \varphi(z) = h_1(z) \quad \text{for } z \in U_a \ (r \in N),
\]
then there exists a unique meromorphic function \( \psi(z) \) such that
\[
(z - a)^r \varphi[f(z)] - g_1(z) \psi(z) = h_1(z) \quad \text{for } z \in U,
\]
\[
\psi(z) = \varphi(z) \quad \text{for } z \in U_a.
\]

**Proof.** First we suppose that \( g_1(z) \) and \( h_1(z) \) are analytic functions in the domain \( U \) and \( g(z) \neq 0 \) for \( z \in U \). Let \( K \subset U \) be any compact. Let
us put \( K_0 = K, K_1 = f(K), K_n = f(K_{n-1}) = f^n(K) \). On account of Lemma 1 it follows that there exists a positive integer \( n_0 \) such that \( f^n(K) \subseteq U_a \) for \( n \geq n_0 \). Since \( f(z) \in K_n \subseteq U_a \) for \( z \in K_{n-1} \), we can define the function \( \varphi_1(z) \) in the following manner:

\[
\varphi_1(z) = \frac{(z-a)^r \varphi[f(z)] - h_1(z)}{g_1(z)} \quad \text{for } z \in K_{n-1}.
\]

Similarly we define the function \( \varphi_2(z) \):

\[
\varphi_2(z) = \frac{(z-a)^r \varphi_1[f(z)] - h_1(z)}{g_1(z)} \quad \text{for } z \in K_{n-2},
\]

\[\vdots\]

\[
\varphi_n(z) = \frac{(z-a)^r \varphi_{n-1}[f(z)] - h_1(z)}{g_1(z)} \quad \text{for } z \in K_0 = K.
\]

Now we put

\[
\varphi_K(z) = \varphi_n(z) \quad \text{for } z \in K.
\]

We shall prove that \( \varphi_K(z) \) is independent of the choice of \( n \). Let \( m \) be a positive integer, \( m \geq n_0 \) and let \( \varphi_1(z), \ldots, \varphi_m(z) \) be functions defined on \( K \) in the same manner as \( \varphi_1(z), \ldots, \varphi_n(z) \). We can suppose that \( m < n \). Hence and from (3) we obtain

\[
\varphi_{n-m}(z) = \varphi(z) \quad \text{for } z \in K_m = f^m(K)
\]
or

\[
\varphi_{n-m}[f^m(z)] = \varphi[f^m(z)] \quad \text{for } z \in K.
\]

It follows that

\[
\frac{(z-a)^r \varphi_{n-m}[f^m(z)] - h_1(z)}{g_1(z)} = \frac{(z-a)^r \varphi[f^m(z)] - h_1(z)}{g_1(z)}
\]

thus

\[
\varphi_{n-m+1}[f^{m-1}(z)] = \varphi_1[f^{m-1}(z)].
\]

Hence in the same manner we obtain

\[
\varphi_{n-m+2}[f^{m-2}(z)] = \varphi_2[f^{m-2}(z)].
\]

Finally \( \varphi_n(z) = \varphi_m(z) \) for \( z \in K \).

Now we take a compact set \( K^* \subseteq U \) and the function \( \varphi_{K^*}(z) \). We must prove that \( \varphi_K(z) = \varphi_{K^*}(z) \) for \( z \in K \cap K^* \). Let \( p \) be a positive integer such that \( f^p(K \cap K^*) = f^p(K) \cap f^p(K^*) \subseteq U_a \) and take the function \( \varphi_{K \cap K^*}(z) \).

Since we may take \( n = p \), it follows that \( \varphi_K(z) = \varphi_{K \cap K^*}(z) \) for \( z \in K \). Similarly we get \( \varphi_{K^*}(z) = \varphi_{K \cap K^*}(z) \) for \( z \in K^* \), so \( \varphi_K(z) = \varphi_{K^*}(z) \) for \( z \in K \cap K^* \).

Let \( \{ K_n \} \) be a sequence of compact sets such that \( K_n \subseteq K_{n+1} \) for every \( n \) and \( \bigcup_{n=1}^{\infty} K_n = U \). We put

\[
\psi(z) = \varphi_{K_n}(z) \quad \text{for } z \in K_n.
\]
Evidently, \( \psi(z) \) is an analytic function. It remains to prove that (4) and (5) hold. If \( z \in U \), then there exist positive integers \( p \) and \( q \) such that \( z \in K = K_p \) and \( f(K) \subset K_q \) and we obtain

\[
\psi(z) = \psi_K(z) = \frac{(z-a)^p \psi_{q-1}[f(z)] - h_1(z)}{g_1(z)}
\]

\[
= \frac{(z-a)^p \psi_{K_p}[f(z)] - h_1(z)}{g_1(z)}
\]

\[
= \frac{(z-a)^p \psi_{K_q}[f(z)] - h_1(z)}{g_1(z)}
\]

Taking \( K \subset U^a \) it is easy to verify that (5) holds.

Now we assume that \( g_1(z) \) and \( h_1(z) \) are meromorphic functions and their poles are different from \( a \). Then, as in the preceding case, we shall obtain meromorphic function \( \psi(z) \) in the domain \( U \). In any compact \( K \subset U \) the function \( \psi(z) \) has a finite number of poles as the quotient of meromorphic functions. This completes the proof of Lemma 2.

By \( N \) we denote the set of positive integers. For a function \( a(z) \) we denote by \( Z(a) \) the order of the zero of \( a(z) \) at the point \( z = a \) and by \( P(a) \) the order of the pole of \( a(z) \) at the point \( z = a \).

Now we shall prove the following

**Theorem 1.** Let hypotheses (I)-(III) be fulfilled. Let us suppose that the functions \( g(z) \) and \( h(z) \) are analytic at the point \( z = a \) and, moreover, that

\[
g(a) \neq 0, 1, e^k \quad \text{for} \quad k = \pm 1, \pm 2, \ldots
\]

Then there exists exactly one meromorphic solution of equation (1) in the domain \( U \). This solution is analytic at the point \( z = a \) and \( \varphi(a) = \frac{h(a)}{1-g(a)} \).

**Proof.** It is easy to verify that the function

\[
h(z, w) = \frac{w - h(z)}{g(z)}
\]

is analytic at the point \((a, b)\), where \( b = \frac{h(a)}{1-g(a)} \) and \( h(a, b) = b \). Since \( g(a) \neq e^n \), we have

\[
1 - a_0 e^n = 1 - \frac{e^n}{g(a)} \neq 0 \quad \text{for} \quad n \in N.
\]
According to Theorem 1 there exists exactly one analytic solution of equation (1) in a neighbourhood of the point \( z = a \). By Lemma 2 we may extend this solution onto the whole domain \( U \). This is a meromorphic solution of equation (1) in the domain \( U \).

Now suppose that there exists another solution. It must have the form

\[
\varphi(z) = \frac{\varphi_1(z)}{(z-a)^r},
\]

where \( \varphi_1(z) \) is analytic at the point \( z = a \), \( \varphi_1(a) \neq 0 \) and \( r \in \mathbb{N} \). We have also

\[
f(z) = a + (z-a)f_1(z),
\]

where \( f_1(z) \) is an analytic function, and \( f_1(a) = c \). Putting (6) and (7) in equation (1) we obtain

\[
\frac{\varphi_1[f(z)]}{(z-a)^r[f_1(z)]^r} - g(z) \frac{\varphi_1(z)}{(z-a)^r} = h(z).
\]

Since \( g(a) \neq c^{-r} \), the left-hand side of (8) has at \( z = a \) a pole of order \( r \), whereas the right-hand side is analytic at the point \( z = a \), which is impossible. This completes the proof.

**Theorem 1′.** Let hypotheses (1)-(III) be fulfilled. Let us suppose that \( g(z) \) and \( h(z) \) are analytic at \( z = a \) and,

\[
g(a) = c^{-p}, \quad \text{where} \quad p \in \mathbb{N}.
\]

Then there exists in the domain \( U \):

(a) exactly one meromorphic solution of equation (1) that is analytic at \( z = a \);

(b) a one-parameter family \( \mathcal{F} \) of meromorphic solutions of equation (1) such that for every \( \varphi \in \mathcal{F} \) we have \( P(\varphi) = p \).

There are no other meromorphic solutions of equation (1) in \( U \).

**Proof.** (a) The proof is analogous to that of Theorem 1.

(b) Suppose that there exists another meromorphic solution of equation (1). It follows from (a) that it must have form (6), \( \varphi_1(a) \neq 0 \). Putting (6) in equation (1) we obtain equation (8).

Hence we conclude that necessarily \( r = p \). Putting \( r = p \) in (8), we get the equation

\[
\varphi_1(z) = \frac{\varphi_1[f(z)] - (z-a)^p[f_1(z)]^p h(z)}{g(z)[f_1(z)]^p}
\]

\((g(a)[f_1(a)]^p = 1, \text{ and } 1 - e^a a_1 = 1 - e^a \neq 0 \text{ for } n \in \mathbb{N}).\)
Theorem I says that there exists a one-parameter family of local analytic solutions of equation (9) (here \( b \) may be arbitrary). We can extend these solutions onto the whole domain \( U \). If \( \varphi(z) \) is a meromorphic solution of equation (9), then \( \varphi(z) = \varphi(z)/(z-a)^q \) is a meromorphic solution of equation (1). Evidently these are all the meromorphic solutions of equation (1).

**Theorem 2.** Let hypotheses (I)-(III) be fulfilled. Let us suppose that \( g(z) \) is analytic at the point \( z = a \), and \( h(z) = h_1(z)/(z-a)^q \), where \( h_1(z) \) is analytic at the point \( z = a \), \( h_1(a) \neq 0 \) and \( q \in N \). Moreover, let

\[
g(a) \neq 0, \quad a^k \quad \text{for} \quad k = 0, \pm 1, \pm 2, ...
\]

Then there exists exactly one meromorphic solution \( \varphi \) of equation (1) in \( U \), and \( P(\varphi) = q \).

**Proof.** If a function \( \varphi(z) \) is a meromorphic solution of equation (1), then it must have form (6), where \( \varphi_1(a) \neq 0 \) and \( r \in N \) (1).

Putting (6) and (7) into equation (1), we obtain

\[
\frac{\varphi_1[f(z)]}{[f(z)]^q} - g(z) \frac{\varphi_1(z)}{(z-a)^r} = \frac{h_1(z)}{(z-a)^q}.
\]

Since \( g(a) \neq [f_1(a)]^{-r} = c^{-r} \), we conclude that \( r = q \). Putting \( r = q \) in (10), we obtain for \( \varphi_1(z) \) the equation

\[
\varphi_1(z) = \frac{\varphi_1[f(z)]-[f_1(z)]^q h_1(z)}{g(z)[f(z)]^q}.
\]

Here \( 1-c^q a_0 = 1-\frac{c^q-q}{g(a)} \neq 0 \). By Theorem I we get the existence of exactly one solution \( \varphi_1 \) of equation (11) that is analytic at the point \( z = a \). By Lemma 2 we can extend this solution onto the whole domain \( U \). The function \( \varphi(z) = \varphi_1(z)/(z-a)^q \) is the only meromorphic solution of equation (1).

**Remark.** The example of the equation

\[
\varphi(z^a) - \varphi(z) = 1/z
\]

shows that in the case where \( c = 0 \) there may be no meromorphic solution of equation (1). Indeed, if \( \varphi(z) \) is a meromorphic solution of this equation, then \( \varphi(z) \) has a form \( \varphi(z) = \varphi_1(z)/z^r \), where \( \varphi_1(0) \neq 0 \) and \( r \in N \). But then

(1) Every meromorphic solution of equation (1) must have a pole at \( z = a \), for otherwise the left-hand side of (1) would be analytic at \( z = a \), whereas the right-hand side has a pole at \( z = a \).
the left-hand side has a pole of order \(2r\) and the right-hand side has a pole of order one.

Theorem 2'. Let hypotheses (I)-(III) be fulfilled. Let us suppose that \(g(z)\) is analytic at \(z = a\), and \(h(z) = h_1(z)(z-a)^q\), where \(h_1(z)\) is analytic at \(z = a\), \(h_1(a) \neq 0\) and \(q \in \mathbb{N}\).

Then (a) in the case where

\[
g(a) = c^{-q}
\]

there exists no meromorphic solution of equation (1).

(b) in the case where

\[
g(a) = c^{-p}, \quad \text{where} \quad p > q,
\]

there exists exactly one meromorphic solution of equation (1) with \(P(q) = q\) and there exists a one-parameter family \(F\) of meromorphic solutions such that we have \(P(q) = p\) for \(q \in F\).

Proof. The same argument as in the proof of Theorem 2 shows that if \(\varphi(z)\) is a solution of equation (1), then \(\varphi(z)\) must have form (6), and for \(\varphi_1(z)\) we obtain the following equation:

\[
(12) \quad \varphi_1(z) = \frac{f_1^*[f(z)] - h_1(z)[f_1(z)]^q}{g(z)[f_1(z)]^p}.
\]

Putting \(z = a\) we obtain \(\varphi_1(a) = \varphi_1(a) = c^{-q}h_1(a)\), which is impossible. This completes the proof of case (a).

(b) Evidently, every meromorphic solution of equation (1) has form (6). For \(\varphi_1(z)\) we obtain equation (8). It is easy to see that a necessary condition of the existence of an analytic solution \(\varphi_1(z)\) of equation (8) is \(r = q\) or \(r = p\).

Let \(r = q\); putting \(h(z) = h_1(z)/(z-a)^q\) and \(r = q\) in (8) we get for \(\varphi_1(z)\) equation (12).

Here \(1 - c^n a_{qt} = 1 - e^{nq}e^{-q} \neq 0\) for \(n = 1, 2, 3, \ldots\) By Theorem I we get exactly one analytic solution of equation (1). By Lemma 2 we may extend this solution onto the whole domain \(U\). Let \(\varphi_1(z)\) be a meromorphic solution of equation (12). Then \(\varphi(z) = \varphi_1(z)/(z-a)^q\) is a meromorphic solution of equation (1) and, evidently, \(P(q) = q\). Putting \(h(z) = h_1(z)/(z-a)^q\) and \(r = p\) in (8) we get the equation

\[
(13) \quad \varphi_1(z) = \frac{f_1^*[f(z)] - h_1(z)[f_1(z)]^p(z-a)^{p-q}}{g(z)[f_1(z)]^p}.
\]

As in case (b) of Theorem 1, we get a one-parameter family of solutions of equation (13). By Lemma 2 we may extend these solutions onto
the whole domain $U$. Let $\varphi_2(z)$ be a meromorphic solution of equation (13). Then $\varphi(z) = \varphi_2(z)/(z-a)^p$ is a meromorphic solution of equation (1). This completes the proof.

**Theorem 3.** Let hypotheses (I)-(III) be fulfilled. Suppose that $g(z)$ has a pole and $h(z)$ is analytic at $z = a$. Then there exists exactly one meromorphic solution $\varphi$ of equation (1). This solution is analytic at $z = a$ and fulfills the condition $Z(\varphi) = P(g) + Z(h)$.

**Proof.** Suppose that $\varphi(z)$ is a meromorphic solution of equation (1) that is analytic at $z = a$. Putting in (1) $g(z) = g_1(z)/(z-a)^p$ and $h(z) = (z-a)^q h_1(z)$, where $g_1(z)$ and $h_1(z)$ are analytic at $z = a$, $g_1(a) \neq 0$, $h_1(a) \neq 0$, we see that $Z(\varphi) = p + q$. Hence $\varphi(z)$ must have the form $\varphi(z) = (z-a)^{p+q} \varphi_1(z)$, $\varphi_1(a) \neq 0$. For $\varphi_1(z)$ we get the equation

$$
\varphi_1(z) = \frac{(z-a)^p [f_1(z)]^{p+q} \varphi_1[f(z)] - h_1(z)}{g_1(z)}.
$$

(14)

From Theorem I and Lemma 2 it follows that there exists exactly one meromorphic solution in $U$ of equation (14). Let $\varphi_1(z)$ be this solution. Then $\varphi(z) = (z-a)^{p+q} \varphi_1(z)$ is a meromorphic solution of equation (1). Suppose that there exists another meromorphic solution of equation (1). Since we have already found all solutions that are analytic at $z = a$, it must have form (6), where $\varphi_1(a) \neq 0$ and $r \in N$. Thus we get for $\varphi_1(z)$ the equation

$$
\frac{\varphi_1[f(z)]}{(z-a)^r [f_1(z)]^r} - \frac{g_1(z)}{(z-a)^p} - \frac{\varphi_1(z)}{(z-a)^q} = h(z).
$$

On the left-hand side there is a pole of order $p+q$ and on the right-hand side there is an analytic function at $z = a$. This is impossible, which shows that there are no other meromorphic solutions and completes the proof of the theorem.

Similarly we can prove the following

**Theorem 4.** Let (I)-(III) be fulfilled. Moreover, suppose that $g(z) = g_1(z)/(z-a)^p$, $h(z) = h_1(z)/(z-a)^q$, where $g_1(a) \neq 0$, $h_1(a) \neq 0$, and $p, q \in N$.

Then there exists exactly one meromorphic solution $\varphi(z)$ of equation (1), and

(a) in the case of $p = q$ it is analytic at the point $z = a$ and $\varphi(a) = -h_1(a)/g_1(a)$;

(b) in the case of $p < q$ it has a pole at $z = a$ and $P(\varphi) = q - p$;

(c) in the case of $p > q$ it is analytic at the point $z = a$ and $Z(\varphi) = p - q$. 

References


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