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On geometric objects

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The definition of the concept of geometric object was formulated by Wundheiler [2]. More detailed historical information related to this concept may be found in the paper of Kucharzewski and Kuczma [1]. The present paper is devoted to some generalization of Wundheiler's concept. The definition of the geometric object is formulated without the notion of a groupoid or a pseudogroup. In particular, the set of admissible local coordinates is not supposed to satisfy any conditions.

1. Geometric objects in the wide sense. Let Q be an arbitrary set of one-to-one functions f such that the element x belongs to the domain of f, and let F be a function. Any function ω whose domain is the set Q we call a *geometric object, in the wide sense*, at the point x with the transformation function F and the set Q of admissible maps if for every g, $f \in Q$ the triplet $\big(\omega(f), f(x), g\circ f^{-1}\big)$ belongs to the domain of F and

$$\omega(g) = F\big(\omega(f), f(x), g\circ f^{-1}\big). \tag{1}$$

The function ω is called a *geometric object, in the wide sense*, at the point x if there exist a set Q and a function F such that ω is a geometric object, in the wide sense, at the point x with the transformation function F and the set Q of admissible maps. By $\mathrm{Obj}(x, F, Q)$ we shall denote the set of all geometric objects, in the wide sense, at the point x with the transformation function F and the set Q of admissible maps. For any $\omega \in \mathrm{Obj}(x, F, Q)$ we may define the set $[x, \omega]$ of all triplets $\big(\omega(f), f(x), g\circ f^{-1}\big)$, where $g, f \in Q$. From the definition of $\mathrm{Obj}(x, F, Q)$ it follows that the set $[x, \omega]$ is contained in the domain of F. Thus the union $[x, F, Q]$ of all sets $[x, \omega]$, where $\omega \in \mathrm{Obj}(x, F, Q)$ is included in the domain of F. We state that

$$\mathrm{Obj}(x, F, Q) = \mathrm{Obj}(x, F|[x, F, Q], Q). \tag{2}$$

Indeed, the set $\mathrm{Obj}(x, F|[x, F, Q], Q)$ is contained in the set $\mathrm{Obj}(x, F, Q)$, because $[x, F, Q]$ is a subset of the domain of F. To verify the inverse inclusion we take an arbitrary ω of $\mathrm{Obj}(x, F, Q)$. Then

$$\big(\omega(f), f(x), g\circ f^{-1}\big) \in [x, \omega] \subset [x, F, Q]$$

and

$$F|[x, F, Q]\big(\omega(f), f(x), g\circ f^{-1}\big) = F\big(\omega(f), f(x), g\circ f^{-1}\big) = \omega(g)$$

for $g, f\epsilon Q$. Therefore, ω belongs to $\mathrm{Obj}(x, F|[x, F, Q], Q)$.

1.1. *The set $[x, F, Q]$ is the smallest of the sets A contained in the domain of F and such that*

$$\mathrm{Obj}(x, F, Q) = \mathrm{Obj}(x, F|A, Q). \tag{3}$$

Proof. Let A be a subset of the domain of the function F fulfilling (3). If $m\epsilon[x, F, Q]$, then there exists an $\omega\epsilon\mathrm{Obj}(x, F, Q)$ such that $m\epsilon[x, \omega]$. Thus

$$m = \big(\omega(f), f(x), g\circ f^{-1}\big) \quad \text{for some } g, f\epsilon Q.$$

According to the mean of the $\mathrm{Obj}(x, F|A, Q)$, by (3), we get $m\epsilon A$. Therefore, $[x, F, Q]$ is contained in A. This ends the proof.

What is proved ensures a possibility to define the part of the domain of F, where F is the transformation function of a geometric object, that is not interesting from the point of view of the examination of geometric object at x. This part is the complementation of $[x, F, Q]$ to the domain of F. The definition of the set $[x, F, Q]$ has a certain defect. It gives the definition of the smallest set contained in the domain of F for which the set of all objects, in the wide sense, with the transformation function F and the set Q of admissible maps is retained, but this definition uses the notion of the geometric object (in the wide sense). We shall give the definition of the same set without the notion of the geometric object.

Let us consider any function F and a set Q of one-to-one functions, and define the set $\mathfrak{T}(x, F, Q)$ of all sets T included in the domain of F and satisfying the following condition:

if $m\epsilon T$, then there exist $p, q\epsilon Q$ such that for every $g, f\epsilon Q$ the triplets

$$\big(F(m), q(x), f\circ q^{-1}\big) \quad \text{and} \quad \big(F\big(F(m), q(x), f\circ q^{-1}\big), f(x), g\circ f^{-1}\big)$$

belong to T and the equalities

$$F\big(F\big(F(m), q(x), f\circ q^{-1}\big), f(x), g\circ f^{-1}\big) = F\big(F(m), q(x), g\circ q^{-1}\big)$$

$$\big(F\big(F(m), q(x), p\circ q^{-1}\big), p(x), q\circ p^{-1}\big) = m$$

hold.

Now, we shall examine some properties of sets belonging to the set $\mathfrak{T}(x, F, Q)$. By easy verification we state that

1.2. *If $\omega\epsilon\mathrm{Obj}(x, F, Q)$, then $[x, \omega]\epsilon\mathfrak{T}(x, F, Q)$. The union of an arbitrary set of sets belonging to $\mathfrak{T}(x, F, Q)$ belongs to $\mathfrak{T}(x, F, Q)$.*

We shall prove that

1.3. *The set T belongs to $\mathfrak{T}(x, F, Q)$ if and only if T is contained in the domain of F and*

$$[x, F|T, Q] = T. \tag{4}$$

Proof. Let T belong to $\mathfrak{T}(x, F, Q)$. Then T is included in the domain of F. Consider any $m \in T$. From the definition of $\mathfrak{T}(x, F, Q)$ it follows that the function ω_m defined by the formula

$$\omega_m(f) = F\big(F(m), q(x), f \circ q^{-1}\big) \quad \text{for } f \in Q$$

belongs to $\mathrm{Obj}(x, F, Q)$, $\big(\omega_m(f), f(x), g \circ f^{-1}\big) \in T$ for any $g, f \in Q$ and

$$\big(\omega_m(p), p(x), q \circ p^{-1}\big) = m.$$

Hence it follows that

$$\omega_m(g) = F\big(\omega_m(f), f(x), g \circ f^{-1}\big) = (F|T)\big(\omega_m(f), f(x), g \circ f^{-1}\big).$$

Thus, $\omega_m \in \mathrm{Obj}(x, F|T, Q)$ and $m \in [x, \omega_m]$. In other words, m belongs to $[x, F|T, Q]$. Therefore (4) is satisfied. Now, we suppose that the set T contained in the domain of F fulfils equality (4). Let us take $m \in T$. Then there exists an $\omega \in \mathrm{Obj}(x, F|T, Q)$ such that $m \in [x, \omega]$. Thus

$$m = \big(\omega(p), p(x), q \circ p^{-1}\big) \quad \text{for some } p, q \in Q,$$

and, for any $g, f \in Q$, we have $F(m) = \omega(q)$,

$$\big(F(m), q(x), f \circ q^{-1}\big) = \big(\omega(q), q(x), f \circ q^{-1}\big) \in T,$$

$$\Big(F\big(F(m), q(x), f \circ q^{-1}\big), f(x), g \circ f^{-1}\Big) = \big(\omega(f), f(x), g \circ f^{-1}\big) \in T$$

and

$$\begin{aligned} F\Big(F\big(F(m), q(x), f \circ q^{-1}\big), f(x), g \circ f^{-1}\Big) &= \omega(g) \\ &= F\big(\omega(q), q(x), g \circ q^{-1}\big) = F\big(F(m), q(x), g \circ q^{-1}\big). \end{aligned}$$

Therefore, $T \in \mathfrak{T}(x, F, Q)$.

1.4. *Among the sets belonging to $\mathfrak{T}(x, F, Q)$ there exists the greatest. This set is equal to the set $[x, F, Q]$.*

Proof. Denoting by T_0 the union of all sets of $\mathfrak{T}(x, F, Q)$, according to 1.2 we state that T_0 is an element of $\mathfrak{T}(x, F, Q)$. It is easy to verify that $[x, F, Q] \in \mathfrak{T}(x, F, Q)$. Consider any set T belonging to $\mathfrak{T}(x, F, Q)$. By 1.3, the set T is equal to $[x, F | T, Q]$. Then T is contained in $[x, F, Q]$. Hence it follows that T_0 is contained in $[x, F, Q]$. This ends the proof.

2. Geometric objects. In the previous section we suppose no special restrictions concerning the set Q of admissible maps and we adjoin no structures, for example, topological structures to the set Q or to the domains of functions belonging to Q. In interesting studies from the point of view of differential geometry all maps of the set Q are charts of the atlas of a differentiable manifold. Objects considered in differential geometry have the essential property that the values of an object corresponding to the charts f and g are equal if those charts are identical on some neighbourhood of the point x. We now define the notion of a geometric object in such a way as to retain this condition.

Let Q be an arbitrary set of one-to-one functions such that the domains of these functions contain a point x. Consider the weakest topology such that all domains of functions belonging to Q are open. Denote by $\mathrm{N}Q$ the topological space obtained in this way. We introduce the following relation between functions of the set Q:

(e_Q) $f_1 \equiv_Q f_2$ if and only if there exists a set A which is open in the topology of the space $\mathrm{N}Q$ and such that $f_1|A = f_2|A$.

It is evident that this relation is an equivalence relation. An object ω, in the wide sense, at the point x with the transformation function F and the set Q of admissible maps is said to be an *object at the point* x with the transformation function F and the set Q of admissible maps, which we shall denote by $\omega \in \mathrm{obj}(x, F, Q)$, if from $f_1 \equiv_Q f_2$ it follows that $\omega(f_1) = \omega(f_2)$. Consider the set Q_0 of all functions that are obtained from function of the set Q by the restriction of their domains to any open sets of the space $\mathrm{N}Q$ which an included in these domains.

It is easy to verify that the topological space $\mathrm{N}Q$ is identical with $\mathrm{N}Q_0$. Hence it follows that the set Q_0 fulfils the following condition:

(i_{Q_0}) if $f \in Q_0$ and A is open in $\mathrm{N}Q_0$ and included in the domain of f, then $f|A \in Q_0$.

This condition ensures the equivalence of the notion of geometric object in the wide sense to a given set of admissible maps and of the notion of geometric object to the same set of admissible maps. More exactly,

2.1. *If the set Q of one-to-one mappings whose domains contain the point x fulfils the condition*

(i_Q) *if $f \in Q$ and A is open in $\mathrm{N}Q$ and included in the domain of f, then $f|A \in Q$,*

then

(ii_Q) *for every transformation function F the equality*

$$\mathrm{Obj}(x, F, Q) = \mathrm{obj}(x, F, Q)$$

holds.

Proof. Under the assumption of (i_Q) we take any ω of the set $\mathrm{Obj}(x, F, Q)$. Let $f \in Q$ and A be an open set of the topological space $\mathrm{N}Q$ included in the domain of f. Then

$$\begin{aligned}\omega(f) &= F\big(\omega(f|A), (f|A)(x), f\circ(f|A)^{-1}\big) \\ &= F\big(\omega(f|A), (f|A)(x), f|A\circ(f|A)^{-1}\big) = \omega(f|A).\end{aligned}$$

Let $g, f \in Q$ and $g \equiv_Q f$. There exists a set A open in $\mathrm{N}Q$ such that $g|A = f|A$. Hence it follows that

$$\omega(g) = \omega(g|A) = \omega(f|A) = \omega(f).$$

Therefore, ω belongs to the set $\mathrm{obj}(x, F, Q)$ and, $\mathrm{Obj}(x, F, Q)$ is included in the $\mathrm{obj}(x, F, Q)$. The inverse inclusion is obvious.

It is interesting, we believe, to give necessary and sufficient conditions for (ii_Q) without the notion of a transformation function.

The topological space $\mathrm{N}Q$ plays an essential part in condition (e_Q) and, consequently, in the concept of geometric object just defined. Let us note that if M is any differentiable manifold of the class C^k and $Q_x(M)$ is the set of all charts of the maximal atlas of this manifold such that their domains contain the point x, then $\mathrm{N}Q_x(M)$ is identical with the topological space for which the set of all open sets is the set of all unions $\bigcup_{f\in S} D_f$, where S is any subset of $Q_x(M)$. It is easy to state that the topological space of differentiable manifold is identical with the smallest topology such that all sets which are open in some topology $\mathrm{N}Q_x(M)$ are open in it. Denote this topology by X; i.e. the union of all sets of open sets of topological spaces $\mathrm{N}Q_x(M)$, where x is a point of M, is a base of the topological space X. Indeed, if A is open in the topology of M, then, for every point x belonging to A, there exists a chart $f \in Q_x(M)$ such that its domain is included in A. Hence it follows that x belongs to the domain of f and this domain is open in $\mathrm{N}Q_x(M)$. Thus, A is open in X. Similarly, if A is open in X, then, for any point x of A, there exist finite sequences: of points $x_1, \ldots, x_s$ of M and of sets $A_1, \ldots, A_s$ open in the spaces $\mathrm{N}Q_{x_1}(M), \ldots, \mathrm{N}Q_{x_s}(M)$, respectively, and fulfilling the condition

$$x \in A_1 \cap \ldots \cap A_s \subset A.$$

On the other hand, A is open in the topology of M, because every set open in $\mathrm{N}Q_x(M)$ is open in M. Hence A is open in M.

Let now ω be any object at x with the transformation function F and the set Q of admissible maps. From the definition of the concept of an object follows the correctness of the following definition of the function ω_0: for every $f \in Q_0$ we set

$$\omega_0(f) = \omega(g), \quad \text{where } f = g|A,\ g \in Q. \tag{5}$$

The set A is here open in NQ and contained in the domain of g. It is an immediate consequence of (5) (and the fact that Q is included in Q_0) that the function ω_0 is an extension of ω. In general, ω_0 need not be any element of $\mathrm{Obj}(x, F, Q)$.

Let us make the following supposition about the transformation function F:

(i_{QF}) if A, A', B, B' are open sets in the topology of the space NQ contained respectively in the domains of the maps f, f', g, g' of Q such that $g\circ f^{-1}|f[A \cap B] = g'\circ f'^{-1}|f'[A' \cap B']$ and the triplets $(w, u, g\circ f^{-1}), (w, u, g'\circ f'^{-1})$ belong to the domain of F, then

$$F(w, u, g\circ f^{-1}) = F(w, u, g'\circ f'^{-1}). \tag{6}$$

We prove that

2.2. *If the transformation function F satisfies condition* (i_{QF}), *then there exists a transformation function F_0, which is an extension of the function $F|[x, F, Q]$, such that for any object $\omega \in \mathrm{obj}(x, F, Q)$ the function ω_0 defined by formula* (5) *is an element of* $\mathrm{obj}(x, F_0, Q_0)$.

Proof. Let T be the set of all triplets $(w, u, g\circ f^{-1}|f[A \cap B])$, where A and B are open sets in the topology of the space NQ, contained in the domains of f and g respectively, such that $(w, u, g\circ f^{-1})$ belongs to the domain of F. We shall define the function F_0 on the set T and we shall verify that this function fulfils the required condition. For this purpose, let us take an arbitrary triplet (w, u, φ) of the set T. Consider any maps f, f', g, g' of the set Q and the sets A, A', B, B', open in the space NQ, contained in the domains of the maps, f, f', g, g' respectively and such that

$$(w, u, g\circ f^{-1}|f[A \cap B]) = (w, u, \varphi) = (w, u, g'\circ f'^{-1}|f'[A' \cap B']). \tag{7}$$

Condition (i_{QF}) enssures the fulfilling of (6). Then, we may set

$$F_0(w, u, \varphi) = F(w, u, g\circ f^{-1}), \tag{8}$$

where f, g, A and B are as in the first of equalities (7). The function F_0 is, in this manner, well defined on the set T.

Let $f_0, g_0 \in Q_0$. Then there exist open sets A and B and functions $f, g \in Q$ such that $f_0 = f|A$, $g_0 = g|B$. Suppose that ω is a geometric object at the point x with the transformation function F and the set Q of admissible maps. Thus, $\omega_0(f_0) = \omega(f)$ and $\omega_0(g_0) = \omega(g)$. Moreover, the triplet $(\omega(f), f(x), g\circ f^{-1})$ belongs to the domain of F and equality (1) holds. Hence it follows that

$$\big(\omega_0(f_0), f_0(x), g_0\circ f_0^{-1}\big) = \big(\omega(f), f(x), g\circ f^{-1}|f[A \cap B]\big) \in T$$

and, by (8),

$$F_0\big(\omega_0(f_0), f_0(x), g_0 \circ f_0^{-1}\big) = F\big(\omega(f), f(x), g \circ f^{-1}\big) = \omega(g) = \omega_0(g_0).$$

Directly, from the definition of ω_0 it follows that if $f_1 \equiv {}_{Q_0} f_2$, then $\omega_0(f_1) = \omega_0(f_2)$. This ends the proof.

Let Q be a set of one-to-one mappings. The ordered pair (x, f) is said to be *Q-equivalent* to the pair (x', f') if and only if $x = x'$, x belongs to all domains of functions belonging to Q, $f, f' \epsilon Q_0$ and there exists a set A open in NQ such that $f|A = f'|A$. It is easy to state that the Q-equivalence just defined is an equivalence relation, i.e. it is reflexive, symmetric and transitive. Any equivalence class of this relation is called a *Q-local coordinate system.* From 2.2 it follows that, if condition (i_{QF}) is satisfied, we can regard a geometric object with the set Q of admissible maps as a function defined on the set of all Q-local coordinate systems.

References

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