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An example of a function which is locally constant in an open dense set, everywhere differentiable but not constant

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Abstract. The aim of this paper is to give an example of a function which is locally constant in an open dense set, everywhere differentiable, but not constant. We construct the derivative of the function we look for i.e. we construct $\varphi\colon [0,1]\to[0,1]$ such that $f(x) = \int_0^x \varphi(t)\,dt$ has all required properties. We obtain φ as a limit of a sequence of continuous functions. The methods used in the proof are simple and elementary.

A construction of such a function has been given in [1]–[9]. In this note we give another one, which is—together with the proofs of the required properties—simple and elementary. The idea has been suggested to me by Professor A. Pliś. Clearly, it is sufficient to construct a $\varphi\colon [0,1]\to[0,1]$ which satisfies the conditions:

(a) φ is summable in $[0,1]$,

(b) $\varphi = 0$ in an open dense subset of $[0,1]$,

(c) $\lim\limits_{h\to 0}\frac{1}{h}\int_x^{x+h}\varphi(t)\,dt = \varphi(x)$ for $x\in[0,1]$,

(d) $\int_0^1 \varphi(t)\,dt \neq 0$,

since then $f(x) = \int_0^x \varphi(t)\,dt$ has all the required properties.

LEMMA 1. *There exists a function $\psi\colon (0,1]\to\mathbf{R}$ which is continuous, strictly increasing and satisfies the conditions:*

$$\lim_{x\to 0}\psi(x) = -\infty, \tag{1}$$

$$\lim_{x\to 0}\left(\frac{1}{x}\int_0^x \psi(t)\,dt-\psi(x)\right) = 0. \tag{2}$$

One can take $\psi = -h^{-1}$, where $h(t) = te^{1-t^2}$ for $1\leqslant t<\infty$, because

$$\frac{1}{h(x)}\int_x^\infty h(t)\,dt\to 0 \quad \text{when } x\to\infty.$$

LEMMA 2. *Take ψ as in Lemma 1, put $\psi(0) = -\infty$, and for any $c \in (0,1)$ and $d > 0$ define*

$$\psi_{c,d}(x) = \max\big(0, d+\psi(|x-c|)\big) \quad \textit{for } x \in [0,1].$$

Then $\psi_{c,d}$ is continuous in $[0,1]$, vanishing in a neighbourhood of c, and such that if $g\colon [a,b] \to [0,1]$, with $[a,b] \subset [0,1]$, is continuous and satisfies

$$g(t) \leqslant \max\big(\psi_{c,d}(a), \psi_{c,d}(b)\big) \quad \textit{in } [a,b], \tag{3}$$

then

$$\frac{1}{b-a}\int_a^b \big(g-\min(g,\psi_{c,d})\big)\,dt \leqslant \varepsilon(d), \tag{4}$$

where $\varepsilon(d)$ does not depend on g, a, b, c and $\varepsilon(d) \to 0$ for $d \to \infty$.

Proof. Suppose that $\psi_{c,d}(b) \geqslant \psi_{c,d}(a)$ and $\alpha = \psi_{c,d}(b) > 0$ (the case of $\alpha = 0$ being trivial); then $c < b$ and therefore $b-c = \psi^{-1}(\alpha-d)$; next, since the function

$$x \to \frac{1}{b-x}\int_x^b \big(\alpha-\psi_{c,d}(t)\big)\,dt$$

is decreasing in $[c,b)$ [1] and $a \geqslant c-(b-c)$, we get

$$\frac{1}{b-a}\int_a^b \big(g-\min(g,\psi_{c,d})\big)\,dt \leqslant \frac{1}{b-a}\int_a^b \big(\alpha-\psi_{c,d}(t)\big)\,dt$$

$$\leqslant \frac{2}{b-c}\int_c^b \big(\alpha-\psi_{c,d}(t)\big)\,dt \leqslant \varepsilon(d) = \sup_{\alpha\in[0,1]} 2\gamma\big(\psi^{-1}(\alpha-d)\big),$$

where $\gamma(x) = -\left\{\frac{1}{x}\int_0^x \psi(t)\,dt - \psi(x)\right\}$. By (2) we have $\varepsilon(d) \to 0$ for $d \to \infty$.

We pass now to the construction of φ. Take a sequence $\{c_n\}$ dense in $(0,1)$; take $d_n > 0$ such that

$$\varepsilon(d_n) < \frac{1}{2^{n+1}}, \quad \psi_{c_n,d_n}(0) \geqslant 1, \quad n = 1, 2, \ldots \tag{5}$$

and define

$$\varphi = \inf(1, \psi_{c_1,d_1}, \ldots) = \lim \varphi_n,$$

where

$$\varphi_n = \min(1, \psi_{c_1,d_1}, \ldots, \psi_{c_n,d_n}), \quad \text{in } [0,1].$$

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[1] To see it observe that if $h\colon [0,s] \to [0,\infty)$ is continuous and increasing, then $x \to \frac{1}{x}\int_0^x h(t)\,dt$ is also increasing.

Then φ_n are continuous, $0 \leqslant \varphi_n \leqslant 1$, $0 \leqslant \varphi \leqslant 1$ and φ is summable; moreover, $\varphi = 0$ in a neighbourhood of any c_n. Thus φ satisfies conditions (a) and (b).

Let us consider condition (c). The case of $\varphi(x) = 0$ being trivial (since the function φ is then continuous at x because of its uppersemicontinuity), assume that $\varphi(x) > 0$. Let $\varepsilon > 0$ be given. Choose N so as to have

$$|\varphi_N(x) - \varphi(x)| < \frac{\varepsilon}{8}, \tag{6}$$

$$\sum_N^\infty \frac{1}{2^{n+1}} < \frac{\varepsilon}{4}. \tag{7}$$

Since

$$\left|\frac{1}{h}\int_x^{x+h} \varphi(t)\,dt - \varphi(x)\right| \leqslant \left|\frac{1}{h}\int_x^{x+h} \varphi(t)\,dt - \frac{1}{h}\int_x^{x+h} \varphi_N(t)\,dt\right| +$$

$$+\left|\frac{1}{h}\int_x^{x+h} \varphi_N(t)\,dt - \varphi_N(x)\right| + |\varphi_N(x) - \varphi(x)|$$

and, by (6) and the continuity of φ_N, the sum of the last two terms is $< \frac{\varepsilon}{2}$ provided that h is sufficiently small, it is sufficient to have

$$\left|\frac{1}{h}\int_x^{x+h} \varphi(t)\,dt - \frac{1}{h}\int_x^{x+h} \varphi_N(t)\,dt\right| < \frac{\varepsilon}{2} \tag{8}$$

for h small enough.

Put $a = \varphi(x)$. Using the identity

$$u = \max(u, a) + \min(u, a) - a,$$

we have

$$\left|\frac{1}{h}\int_x^{x+h} \varphi_N(t)\,dt - \frac{1}{h}\int_x^{x+h} \varphi(t)\,dt\right| \leqslant \frac{1}{h}\int_x^{x+h} \min(\varphi_N, a)\,dt - \tag{9}$$

$$-\frac{1}{h}\int_x^{x+h} \min(\varphi, a)\,dt + \frac{1}{h}\int_x^{x+h} \left(\max(\varphi_N, a) - a\right)dt.$$

But

$$\frac{1}{h}\int_{x}^{x+h}\big(\max(\varphi_N, a)-a\big)dt \leqslant \max_{t\in[x,x+h]}\varphi_N(t)-\varphi(x) < \frac{\varepsilon}{4} \tag{10}$$

for h small enough, by (6) and the continuity of φ_N.

Next we have

$$\frac{1}{h}\Big(\int_{x}^{x+h}\min(\varphi_N, a)\,dt - \int_{x}^{x+h}\min(\varphi, a)\,dt\Big)$$

$$= \sum_{N}^{\infty}\frac{1}{h}\Big(\int_{x}^{x+h}\min(\varphi_N, a)\Big) - \min(\varphi_{n+1}, a)\,dt,$$

so that applying Lemma 2 with $g = \min(\varphi_n, a)$, $c = c_{n+1}$ and $d = d_{n+1}$ ((3) holds since $g(t) \leqslant a = \varphi(x) \leqslant \psi_{c_{n+1}, d_{n+1}}(x)$) we get by (5) and (7)

$$\frac{1}{h}\Big(\int_{x}^{x+h}\min(\varphi_N, a)\,dt - \int_{x}^{x+h}\min(\varphi, a)\Big) < \frac{\varepsilon}{4},$$

which, together with (9) and (10), gives (8).

Condition (d) is satisfied, because we have

$$1-\int_{0}^{1}\varphi(t)\,dt = \sum_{n=0}^{\infty}\Big(\int_{0}^{1}\varphi_n(t)\,dt - \int_{0}^{1}\varphi_{n+1}(t)\,dt\Big)$$

$$= \sum_{n=0}^{\infty}\int_{0}^{1}\Big(\varphi_n(t)-\min\big(\varphi_n(t), \psi_{c_{n+1}, d_{n+1}}(t)\big)\Big)dt \leqslant \sum_{n=1}^{\infty}\frac{1}{2^{n+1}} < 1$$

using Lemma 2 ((3) holds by (5)).

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