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# SCIENTIFIC WORK OF EDWARD MARCZEWSKI

BY

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(WROCŁAW)

Edward Marczewski was a disciple of the Warsaw school of mathematics in the interwar period, known for its contribution to the development of set theory, topology and the theory of real functions. His teachers and later colleagues were S. Mazurkiewicz, W. Sierpiński, S. Saks, K. Kuratowski, and B. Knaster.

Among subjects of interest to the Warsaw school the theory of analytic sets figured largely. Its discovery by Suslin, as a consequence of Lebesgue's "creative" error (the latter did not realize that the intersection of projections can be larger than the projection of the intersections), was still, at that time, engaging attention. Mathematicians were fascinated by the various ways of characterizing analytic sets and constructing analytic non-Borel sets. The Tarski-Kuratowski method of estimating the Borel or projective class of a set by means of quantifiers occurring in its definition both helped to solve many problems and suggested new ones. Certain problems were soon considered to be hopeless as, for instance, the question of what cardinality the complement of an analytic set can be ("nous ne saurons jamais" said Lusin, and his words made a great impression on Sierpiński) or whether the projection of such a complement is always measurable. Some mathematicians considered such problems to be very special, of little use and beyond the region of "true mathematics". Their role in topology soon diminished but they continued to be of interest and a challenge for mathematicians working on the foundations of mathematics and the logical discoveries of the sixties (J. Addison, P. Cohen) proved their real significance in mathematics. Marczewski was profoundly interested in the theory of analytic sets. Due to his inborn tendency to pursue essential features of mathematical arguments and thus to arrive at generalizations, he found a purely set theoretical approach to theorems which say that every analytic set in $R^n$ is measurable in the sense of Lebesgue (Lusin) and satisfies Baire's condition, i.e. it differs from an open set for a set of the first Baire category (Sierpiński). His result [9] (¹) is as follows:

*A σ-algebra $\boldsymbol{K}$ is invariant under the (Suslin) A-operation if it has the following property: every set $E$ is contained in a set $K \in \boldsymbol{K}$ such that if $Z \supset E$, $Z \in \boldsymbol{K}$, then every subset of $K \setminus Z$ is in $\boldsymbol{K}$.*

Now if $\boldsymbol{K}$ is the algebra of measurable sets, it is enough to choose $K$ to be a $G_\delta$-set containing $E$ and such that the inner measure of $K \setminus E$ is 0. If $\boldsymbol{K}$ is the algebra of sets satisfying Baire's condition, one may choose $K \supset E$, open and such that $K \setminus E$ does not contain any set of second category.

An interesting application of the sieve method can be found in a joint paper of Marczewski and Kuratowski [7]. If $Y$ is a metric compact uncountable space, then the space $2^Y$ of closed subsets of $Y$ with the Hausdorff distance is a continuous image of the Cantor set $C$: $x \mapsto H(x) \in 2^Y$. The set $U = \{(x, y): y \in H(x)\}$ is a universal closed set in $C \times Y$. Let $\boldsymbol{K}$ be a family of closed non-empty sets in $Y$. We say that the set $\Gamma(U, \boldsymbol{K}) = \{x: H(x) \in \boldsymbol{K}\} \subset C$ is obtained by sifting $C$ through the sieve $U$ by means of $\boldsymbol{K}$. In [7] the authors prove that if $\Gamma(U, \boldsymbol{K})$ is Borel (analytic) in $C$, then $\boldsymbol{K}$ is Borel (analytic) in $2^Y$. We may replace $C$ by an arbitrary uncountable Polish space $X$, and $U$ by an arbitrary closed set $F$ in $X \times Y$. Then the sifted set, i.e. $\Gamma(F, \boldsymbol{K}) = \{x: (\{x\} \times Y) \cap F \in \boldsymbol{K}\}$, is Borel (analytic) in $X$ if $\boldsymbol{K}$ is Borel (analytic) in $2^Y$. Let $F$ run over the whole class of closed sets in $X \times Y$, and $\Phi(\boldsymbol{K})$ be the family of all $\Gamma(F, \boldsymbol{K})$. It is proved that the family $\Phi(\boldsymbol{K})$ consists of Borel (analytic) sets if and only if $\boldsymbol{K}$ is Borel (analytic) in $2^Y$. If $Y = [0, 1] = I$ and $\boldsymbol{K}$ is the family of closed sets in $Y$ containing at least one irrational, then $\Phi(\boldsymbol{K})$ is the family of all analytic sets in $X$ and thus $\boldsymbol{K}$ is analytic non-Borel in $2^I$.

Earlier results ensure that $\Phi(\boldsymbol{K})$ consists of all analytic sets also in some other simple cases; e.g. if $\boldsymbol{K}$ is the family of all uncountable closed sets in $Y$. Consequently, also this class is an analytic non-Borel set in $2^Y$.

The papers [17], [24], [26], and [30] are devoted to the equivalence of classes of sets and to related notions. Two sequences or two indexed families $(A_\xi)$ and $(B_\xi)$ of sets contained in spaces $X$ and $Y$, respectively, are called *equivalent* if there exists a one-to-one onto map $\varphi: X \to Y$ such that $\varphi(A_\xi) = B_\xi$ [17]. In general, the equivalence of classes $\boldsymbol{A}$ and $\boldsymbol{B}$ means the existence of a one-to-one onto map such that $\varphi(A) \in \boldsymbol{B}$ for each $A \in \boldsymbol{A}$ and $\varphi^{-1}(B) \in \boldsymbol{A}$ for each $B \in \boldsymbol{B}$. The problem of equivalence of set classes seems to originate from Ulam who asked whether for every sequence $\boldsymbol{A}$ of sets in a space of cardinality $\mathfrak{c}$ there exists a sequence not equivalent to any subsequence of $\boldsymbol{A}$. Marczewski gave a positive answer to this in [17], simultaneously showing that in $R^n$ there exists a sequence

(¹) The numbers in brackets refer to the list of papers on p. 13-17.

of non-dense sets of measure 0 not equivalent to any sequence of projective sets. This follows from the fact that there exist $2^{\mathfrak{c}}$ sequences of non-dense pairwise non-equivalent zero-sets which in turn can be deduced from the following purely set-theoretical proposition:

If $\omega \leqslant |X| \leqslant \mathfrak{c}$, then there exist $2^{|X|}$ non-equivalent sequences of sets in $X$.

Such and similar problems have been solved by Marczewski by means of characteristic functions of sequences. The *characteristic function* (often called the *Marczewski function*) of a sequence $(E_n)$ is defined as the map $X \to C$ ($C$ = ternary Cantor set):

$$x \mapsto 2\sum_{n=1}^{\infty} c_n(x)/3^n,$$

where $c_n$ is the characteristic function of $E_n$.

If we assume CH, then there exists a sequence of sets no subsequence of which is equivalent to any sequence of measurable sets [17]. Given two equivalent set families in topological spaces $X$ and $Y$, it may happen that $\varphi$ can be chosen to be a generalized homeomorphism in the sense of Kuratowski. Then we call these families *B-equivalent*. For example, let $X$ and $Y$ be Polish spaces without isolated points and of the second category and let $\mu_1, \mu_2$ be continuous Borel measures. If $\mu_1(X) = \mu_2(Y)$, then there exists a generalized homeomorphism $h\colon X \to Y$ preserving Baire category and such that $\mu_1(A) = \mu_2(h(A))$ for every Borel set $A \subset X$ [24]. This theorem is derived from the following decomposition theorem proved by Sierpiński: a separable metric space equipped with a continuous Borel measure can be decomposed into an $F_\sigma$-set of the first category and a $G_\delta$-set of measure 0. The problem of decomposing a space into a "big" set of measure 0 and a "small" set of full measure was further examined by Marczewski in the post-war period in collaboration with Sikorski. They proved [44] that for a metric space $X$ the following conditions are equivalent: (1) *for every finite or $\sigma$-finite Borel measure on $X$ there exists a decomposition $X = X_1 \cup X_2$ with $X_1$ of measure 0 and $X_2$ separable,* (2) *the cardinality of $X$ is not real measurable.*

Sierpiński's decomposition theorem has stimulated some further results on two classes of sets which were investigated by the Warsaw school with particular interest: the class of Lebesgue-measurable sets and that of sets satisfying the Baire condition. Marczewski and Sierpiński were fascinated by the perceptible analogy between these classes and even more intrigued by the difference between them. The two classes have turned out to be non-equivalent [13]. Assuming CH, the ideal of sets of measure 0 is equivalent to that of sets of first category (Sierpiński) but they are not $B$-equivalent [24].

Apart from equivalence, Marczewski introduced and examined some weaker relations between families of sets, e.g. the "weak isomorphism", that is, the existence of a map $X \to Y$ preserving inclusion between sets belonging to these families. In [30] he proved that if each of them contains all singletons and if they are weakly isomorphic, then they are equivalent. An application of this theorem to the classes of closed sets in two topological spaces shows that any $T_1$-topology $\tau$ is determined up to equivalence by the lattice structure of the class of $\tau$-closed sets.

Edward Marczewski always favoured measure theory. Even though he abandoned it occasionally, especially in the sixties when he was engaged in universal algebra, he never really lost interest in it and it was the centre of his didactic activities in the last years of his life. We have already mentioned his important result on equivalence of measures [24]. Before characterizing his later achievements in measure theory, we would like to mention one striking result belonging both to measure theory and topology: every $n$-dimensional set in a euclidean space is homeomorphic to some set in $R^{2n+1}$, the $(n+1)$-dimensional measure of which vanishes [21].

Invariant extensions of Lebesgue measure are treated in [15]. Marczewski, referring to the unpublished and, consequently, probably forgotten result of Jankowska-Wiatr ([2]), gives a construction of a large class of extensions of Lebesgue measure by means of a $\sigma$-ideal $I$ of sets of inner Lebesgue measure zero, invariant under translations (or generally — isometries). The measure algebras of these extensions do not differ essentially from the Boolean algebra of Lebesgue measurable sets. In this way Marczewski obtained an invariant extension in which the sets of (new) measure zero have no basis of power $\mathfrak{c}$. (For sets of Lebesgue measure zero such a basis consists of $G_\delta$-sets.) In the same paper another significant method of constructing extensions with essentially new sets of positive measure was given. This method makes use of Sierpiński's set which together with its complement is totally imperfect (i.e., every perfect subset is countable).

The problem of invariant extensions of Lebesgue measure was takeg up by various authors. In 1950 two papers appeared, one by Kakutani and Oxtoby and the other by Kodaira and Kakutani. The first uses the same idea of constructing invariant extensions of Lebesgue measure (on the one-dimensional torus $T$) as was presented by Marczewski in [15], whereas in the second paper a different method, consisting in adding new non-measurable characters of $T$, is used.

Further work on invariant measures was done by Hulanicki for compact abelian groups and by Hulanicki and Ryll-Nardzewski for com-

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([2]) The only available source of information on her result is in [15].

pact groups which are not necessarily abelian (this volume, p. 223-227). In addition, since the fifties, investigations of invariant measures have been systematically carried out by a group of mathematicians from Tbilisi, lead by Pkhakadze.

A result of a different type on the existence of extensions of finitely additive measures for arbitrary algebras of sets was obtained by Marczewski and Łoś [53].

In the general measure theory of fundamental importance are theorems on classification of measure spaces. We have in mind Maharam's theorem on the structure of measure algebras (Boolean isomorphisms) and papers of Rokhlin on pointwise isomorphisms of abstract Lebesgue spaces. It is little known that Marczewski was a pioneer in this field and the first to prove [28] that a normalized measure is isomorphic to the Lebesgue measure on the unit interval if and only if it is atomless and its Fréchet-Nikodym space is separable. Later Marczewski and Sikorski [58] obtained Cantor-Bernstein type theorems for arbitrary measure algebras.

In 1919 Borel introduced the following property of sets on the line:

A set has property (c) if it can be covered by a sequence of intervals of arbitrarily small length.

Sierpiński studied this condition very extensively and proved (assuming CH) that, e.g., there exist uncountable sets with property (c). In [1] Marczewski showed that uncountable analytic sets do not have this property and, in general, that sets with property (c) are totally imperfect. Furthermore, he proved that (c) is invariant under continuous mappings and, consequently, that the sets satisfying (c) are of absolute measure zero (i.e., are of measure zero for every continuous Borel measure).

Non-separable measures (i.e., those with the non-separable Fréchet-Nikodym space) are today a subject of research of specialists and sometimes tools for mathematicians in various fields. This has not always been the case. Even the existence of such measures was once an open problem, raised by Nikodym. In [25] Marczewski proved that there exists a family of power $\mathfrak{c}$ of set-theoretically $\sigma$-independent $F_\sigma$-subsets of $[0, 1]$ and he constructed a non-separable measure on a $\sigma$-algebra of Borel subsets of $[0, 1]$. This provides an important example of a finite measure which cannot be extended to a measure on all Borel subsets of $[0, 1]$.

It was the intention to dispense with the topological concepts in measure theory that led Marczewski to the notion of a compact measure [63].

We say that a finitely additive measure $m$ on a finitely additive algebra of sets is *compact* if there exists a compact family $\boldsymbol{F}$ of sets (not necessarily measurable) approximating $m$, i.e., for every measurable set $A$ and every $\varepsilon > 0$ there is a measurable set $B$ and a set $F \in \boldsymbol{F}$ such that $B \subset F \subset A$ and $m(A \setminus B) < \varepsilon$ ($\boldsymbol{F}$ is compact if every countable subfamily of $\boldsymbol{F}$ with the finite intersection property has non-empty intersection).

It turned out that every compact measure $m$ can be extended to a compact (countably additive) measure on the $\sigma$-algebra generated by the algebra of $m$-measurable sets and that for compact measures the theorem on the existence of non-direct products of measures is true [65].

The notion of a compact measure has turned out to be prolific both in probability theory and in measure theory (applications to the theory of projective limits of measures, theory of conditional probabilities, etc.) and is frequently applied today in research.

In [35] Marczewski presented the results he had obtained during the war in Lvov. The reader today may ask why various properties of topological spaces, which in the metric case are equivalent and signify separability, had had to wait so long for systematic studies. One possible reason is that in the first half of the century the role of non-metric spaces in analysis was of less importance than it is today. The existence of a countable basis in a Hausdorff space viz. the second countability axiom (in [35] called property (B)) turns out to be the strongest separability-like condition. Essentially weaker is the existence of a dense countable subset (property (D)) and weaker again is the condition (s) of Suslin that reads: every family of disjoint open sets is countable. Between (D) and (s) there is the following property (k) of Knaster: every uncountable family of open sets contains an uncountable subfamily consisting of sets with pairwise non-void intersection. In [35] it is proved that the cartesian power $X^T$ $(\operatorname{card} X > 1)$ has (i) property (B) iff $X$ has it and $T$ is countable, (ii) property (D) iff $X$ has it and $\operatorname{card} T \leqslant \mathfrak{c}$, and (iii) property (k) if $X$ has it (whatever $T$ is). The proof of (iii) is the most difficult. The author writes that he does not know whether the property (s) is preserved even in the case $\operatorname{card} T = 2$. This question remained unanswered for a long time. In 1971 Arhangel'skiĭ and Juhász proved that the finite multiplicativity of (s) is independent of ZFC. In a paper in Fundamenta Mathematicae 103 (1979) J. Roitman quotes the result of Galvin and Laver stating that the finite multiplicativity of (s) does not hold under the assumption of CH, as well as the result of Kunen asserting that it does hold under assumption of Martin's axiom together with $\neg$CH. In [35] Marczewski states that if $\operatorname{card} T > \omega$, then the space $X^T$ contains an uncountable isolated set and by putting $X = [0, 1]$ he deduces therefrom that $[0, 1]^T$ $(\operatorname{card} T > \omega)$ contains a closed set which is not a continuous image of the whole space, a phenomenon which does not occur if $T = \omega$.

Two papers concerning analysis, [11] and [12], stand separate in Marczewski's work. Strong results are obtained there by set-theoretical methods, in particular by the Baire category method. The first of these papers, joint with S. Kierst, contains striking existence theorems considerably ameliorating some earlier results on the set of values of an entire function or a function holomorphic in the disc or meromorphic in the plane.

Let us say that a set $A \subset C$ belongs to the class $S$ if $A$ contains a connected, unbounded subset having positive distance from the complement of $A$. Using Runge's theorem the authors prove that the set of entire functions taking every complex value on every $A \in S$ is the complement of a set of the first category in the space of entire functions equipped with a metric derived from natural seminorms. Previously, one had only an example of Zygmund of an entire function assuming every complex value in every sector of the plane. Even a stronger theorem holds: except for a set of first category, every entire function satisfies the equation $f(z) = \varphi(z)$ for each rational function $\varphi$, each $A \in S$, and infinitely many $z \in A$. Furthermore, except for a set of first category, every entire function maps any connected unbounded set disjoint from some ray onto a set dense in $C$. Analogous theorems hold for functions holomorphic in the disc.

If in the space $M$ of functions meromorphic in $C$ a distance is introduced by means of the stereographic projection, then it can be shown that every function in $M$, except for a set of first category, maps any set from $S$ onto the whole plane. And again, except for a set of the first category in $M$, every function maps any connected unbounded set onto a set dense in $C$. (An example of such a function had been given by Gross.)

In [12] Marczewski introduced the notion of a function almost subharmonic in a region $G \subset C$, which is what he calls a function if for almost every point $(x, y) \in G$ and every closed disc $K \subset G$ with centre in $(x, y)$ and radius $r$ one has

$$f(x, y) \leqslant \frac{1}{\pi r^2} \int_K f(t, u)\,dt\,du.$$

The author proves that every almost subharmonic function is almost everywhere equal to some subharmonic function. The problem was at the time rather unusual as it was not till much later that the theory of functional equations was developed and problems of the following kind arose: assuming that the function $f$ fulfills some functional equation or inequality almost everywhere, decide whether $f$ is almost everywhere equal to an (exact) solution of this equation or inequality (such was, for example, the problem of Erdös concerning Cauchy's equation, solved by Jurkat and by de Bruijn).

The work of Marczewski in algebra covers the period 1958-1970 and is concerned mostly with the general notion of independence. In a short note [75] published in 1958 he found a general notion of independence which embraced many kinds of independence considered in mathematics such as linear independence in vector spaces, algebraic independence of elements of a field relative to a subfield, independence of sets, etc. The idea is as follows: take a universal algebra and consider the set of all compositions of fundamental operations (projections being considered as fundamental). The operations obtained are called *algebraic*. Now, a subset of the algebra carrier is called *independent* if any two algebraic operations coinciding on it necessarily coincide everywhere. It can easily be seen that this definition — without any reference to algebraic operations — can also be stated as follows: a subset $I$ of an algebra $A$ is independent if and only if every mapping of $I$ into $A$ can be extended to a homomorphism of the algebra generated by $I$ into $A$. This makes this notion close to that of a free algebra.

The new notion of independence gave rise to several problems. One class of questions which arise in this area is concerned with algebras in which independence has certain properties of linear independence like Steinitz's exchange property, the invariance of the cardinality of a maximal independent set, etc. Marczewski himself [77] introduced a class of algebras which have the main properties of linear spaces with respect to independence. Later Urbanik found a characterization theorem for this class.

If we define a basis of an algebra as an independent set of generators, then it is not difficult to construct examples of algebras with bases of different cardinalities. However, it turned out that the set of all possible finite cardinalities of bases in a given algebra always forms an arithmetic progression. This result was proved by Marczewski and Ryll-Nardzewski in 1961 and later Świerczkowski showed that every arithmetical progression can be the set of cardinalities of all bases in some algebra.

Thanks to the interesting problems in universal algebra, inspired by the work of Marczewski, this domain has become attractive for many mathematicians in the world.

Before concluding we would like to pay full credit to Edward Marczewski for the inspiration of research also in other directions. The very lively seminars, he led for many years together with Steinhaus, determined various domains of work which have been systematically cultivated up to the present time by many mathematicians in Wrocław; as, for instance, the theory of stochastic processes, ergodic theory, measure theory and harmonic analysis.

Edward Marczewski was a familiar and beloved figure to the Polish mathematicians, known for his enthusiasm for science and kindhearted approach to people. Even when defeated by a long illness he was still a part of the Polish mathematical life.

# List of mathematical papers of Edward Marczewski *

**Abbreviations**

ASPM — Annales de la Société Polonaise de Mathématique
BAPS — Bulletin de l'Académie Polonaise des Sciences, Série des sciences mathématiques, astronomiques et physiques
CM — Colloquium Mathematicum
CRV — Comptes Rendus des Séances de la Société des Sciences et des Lettres de Varsovie, Classe III
FM — Fundamenta Mathematicae

Other abbreviations follow those of Mathematical Reviews.


### 1930

[1] *Sur une hypothèse de M. Borel*, FM 15, p. 126-127.

[2] *Sur une classe d'ensembles linéaires*, CRV 22, p. 179-184.

[3] *Sur un problème de M. Banach*, FM 15, p. 212-214.

[4] *O mierzalności i warunku Baire'a* (*On measurability and Baire's condition*), CR du I Congrès des Mathématiciens des Pays Slaves, Warszawa 1929, p. 297-303 [in Polish].

[5] *Sur l'extension de l'ordre partiel*, FM 16, p. 386-389.

[6] *Un théorème sur les opérations de M. Hausdorff*, CRV 23, p. 13-15.

### 1931

[7] (and K. Kuratowski) *Sur les cribles fermés et leurs applications*, FM 18, p. 160-170.

[8] *Sur un ensemble non mesurable de M. Sierpiński*, CRV 24, p. 78-85.

### 1933

[9] *Sur certains invariants de l'opération* (A), FM 21, p. 229-235.

[10] *Remarque sur la dérivée symétrique*, FM 21, p. 226-228; Reconnaissance du droit d'auteur, FM 22 (1934), p. 319.

[11] (and S. Kierst) *Sur certaines singularités des fonctions analytiques uniformes*, CR Acad. Sci. Paris 196, p. 1453-1455, and FM 21, p. 276-294.

[12] *Remarques sur les fonctions sousharmoniques*, Ann. of Math. (2) 34, p. 588-594.

### 1934

[13] *Remarques sur les fonctions complètement additives d'ensembles et sur les ensembles jouissant de la propriété de Baire*, FM 22, p. 303-311.


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* The papers [1]-[33] and [36] have been published under the name of Edward Szpilrajn. Bibliography of papers of Edward Marczewski, including non-mathematical ones, can be found in Wiadomości Matematyczne 23 (1979).

**1935**

[14] *Sur une classe de fonctions de M. Sierpiński et la classe correspondante d'ensembles*, FM 24, p. 17-34.

[15] *Sur l'extension de la mesure lebesguienne*, FM 25, p. 551-558.

**1936**

[16] (and W. Sierpiński) *Remarque sur le problème de la mesure*, FM 26, p. 256-261.

[17] *Sur l'équivalence des suites d'ensembles et l'équivalence des fonctions*, FM 26, p. 302-326; Correction, FM 27, p. 294.

[18] (and W. Sierpiński) *Sur les transformations continues biunivoques*, FM 27, p. 289-292.

[19] (and W. Sierpiński) *Sur un ensemble toujours de I^re catégorie de dimension positive*, Publ. Math. Univ. Belgrade 5, p. 117-123.

**1937**

[20] *O zbiorach i funkcjach bezwzględnie mierzalnych* (*On absolutely measurable sets and functions*), CRV 30, p. 39-68 [in Polish, with a summary in French].

[21] *La dimension et la mesure*, FM 28, p. 81-89.

[22] (and S. Mazurkiewicz) *Sur la dimension de certains ensembles singuliers*), FM 28, p. 305-308.

[23] *Remarques sur les ensembles plans fermés*, FM 29, p. 304-306.

**1938**

[24] *On the equivalence of some classes of sets*, FM 30, p. 235-241.

[25] *Ensembles indépendants et mesures non séparables*, CR Acad. Sci. Paris 207, p. 768-770.

[26] *The characteristic function of a sequence of sets and some of its applications*, FM 31, p. 207-223.

[27] *Concerning convergent sequences of sets*, ASPM 17, p. 115.

[28] *On the space of measurable sets*, ASPM 17, p. 120-121.

[29] *Operations upon sequences of sets*, ASPM 17, p. 123-124.

**1939**

[30] *On the isomorphism and the equivalence of classes and sequences of sets*, FM 32, p. 133-148.

**1940**

[31] *Remarques sur l'ensemble de Lusin*, Mathematica (Cluj) 16, p. 50-52.

**1941**

[32] *Remarques sur les produits cartésiens d'espaces topologiques*, CR (Doklady) Acad. Sci. URSS (NS) 31, p. 525-527.

[33] *Sur les mesures dans les produits cartésiens*, Recueil de Travaux, Inst. Math. Kiev.

**1945**

[34] *Sur deux propriétés des classes d'ensembles*, FM 33, p. 303-307.

**1946**

[35] *Séparabilité et multiplication cartesiènne des espaces topologiques*, FM 34, p. 127-143.

[36] *К проблематике теории меры*, Успехи мат. наук 1 (2), p. 179-188.

**1947**

[37a] *Sur les mesures à deux valeurs et les idéaux premiers dans les corps d'ensembles*, ASPM 19, p. 232-233.

[37b] *Two-valued measures and prime ideals in fields of sets*, CRV 40, p. 11-17.

[38] *Remarques sur l'équivalence des classes d'ensembles*, ASPM 19, p. 228.

[39] *Mesures dans les corps de Boole*, ASPM 19, p. 243-244.

[40] (and M. Nosarzewska) *Sur la convergence uniforme et la mesurabilité relative*, CM 1, p. 15-18.

[41] *Sur l'isomorphie des mesures séparables*, CM 1, p. 39-40.

[42] *Remarque sur la mesurabilité absolue*, CM 1, p. 42-43.

**1948**

[43] *Indépendance d'ensembles et prolongement de mesures (Résultats et problèmes)*, CM 1, p. 122-132.

[44] (and R. Sikorski) *Measures in non-separable metric spaces*, CM 1, p. 133-139.

[45] *Ensembles indépendants et leurs applications à la théorie de la mesure*, FM 35, p. 13-28.

[46] *Une généralisation de la mesure relative*, CM 1, p. 171-172.

[47] *Un théorème d'Ulam et les espaces absolument mesurables*, CM 1, p. 186-187.

[48] *Concerning the symmetric difference in the theory of sets and in Boolean algebras*, CM 1, p. 199-202.

[49] *Généralisations du théorème de Steinhaus sur l'ensemble des distances*, CM 1, p. 248-249.

[50] *Remarques sur les fonctions de Hamel*, CM 1, p. 249-250.

[51] *Sur un théorème de Banach et ses conséquences*, CM 1, p. 253-254.

[52] *Sur l'isomorphie des relations et l'homéomorphie des espaces*, ASPM 21, p. 336-342.

**1949**

[53] (and J. Łoś) *Extensions of measure*, FM 36, p. 267-276.

[54] (and R. Sikorski) *Remarks on measure and category*, CM 2, p. 13-19.

**1950**

[55] (and S. Hartman) *On the convergence in measure*, Acta Sci. Math. (Szeged) 12A, p. 125-131.

[56] (and S. Hartman, C. Ryll-Nardzewski) *Théorèmes ergodiques et leurs applications*, CM 2, p. 109-123.

**1951**

[57] *Sur les congruences et les propriétés positives d'algèbres abstraites*, CM 2, p. 220-228.

[58] (and R. Sikorski) *On isomorphism types of measure algebras*, FM 38, p. 92-98.

[59] *Measures in almost independent fields*, FM 38, p. 217-229.

**1952**

[60] (and C. Ryll-Nardzewski) *Sur la mesurabilité des fonctions de plusieurs variables*, ASPM 25, p. 145-154.

**1953**

[61] (and K. Florek, C. Ryll-Nardzewski) *Remarks on the Poisson stochastic processes I*, Studia Math. 13, p. 122-129.

[62] *Remarks on the Poisson stochastic processes II*, ibidem 13, p. 130-136.

[63] *On compact measures*, FM 40, p. 113-124.

[64] (and C. Ryll-Nardzewski) *Projections in abstract sets*, FM 40, p. 160-164.

[65] (and C. Ryll-Nardzewski) *Remarks on the compactness and non-direct products of measures*, FM 40, p. 165-170.

**1954**

[66] (and A. Goetz) *On the frequency of numbers in certain expansions*, CM 3, p. 86.

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