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The global Yang–Mills equations depending on an arbitrary metric

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Abstract. One of us (J. Ławrynowicz, [5]) has recently posed the problem of replacing a given Yang–Mills field, in general with currents, by a Yang–Mills field in a different, curved space-time, by way of including currents in the geometry so that the field becomes Maxwell-like. This is closely related to a rigorous derivation of the global Yang–Mills equations depending on an arbitrary metric (riemannian or pseudo-riemannian), distinguishing their "solenoidal" and "nonsolenoidal" parts, studying their global properties, and motivating a suitable choice of the principal fibre bundle (real or complex, and in particular holomorphic) with the manifold in question as the base space. Our paper aims at discussing these questions.

Introduction. Denote by su(2) the Lie algebra corresponding to SU(2). By a *Yang–Mills field* we mean any vector field $A = (A_k)$, $k = 1, \ldots, n$, in an open set U in $\boldsymbol{R}^n$, with values in su(2) ([1], p. 12–13).

Suppose that T is a representation of SU(2) in a vector space $\boldsymbol{F}$ and Ψ: $U \to \boldsymbol{F}$ is a C^∞-mapping. Then the *covariant derivative* of the vector field Ψ is given by the formulae

$$\nabla_k \Psi = (\partial/\partial x^k)\Psi + t(A_k)\Psi,$$

t being the representation of su(2) corresponding to T and $x = (x^k)$, $k = 1, \ldots, n$, denoting the coordinate system in U. It satisfies the identity

$$(\nabla_j \circ \nabla_k - \nabla_k \circ \nabla_j)\Psi = t(F_{jk})\Psi,$$

where

$$F_{jk} = (\partial/\partial x^j) A_k - (\partial/\partial x^k) A_j + [A_j, A_k]. \tag{1}$$

If SU(2) is replaced by SU(1), then $[A_j, A_k] = 0$ and $[F_{jk}]$ is the *electromagnetic tensor*.

Take now a C^∞-function ϱ: $U \to$ SU(2) and consider the following *gauge transformation* of the pair $(\Psi(x), A_k(x))$, where $x \in U$:

$$\begin{aligned} &\Psi(x) \mapsto \Psi'(x) = T(\varrho(x))\Psi(x), \\ &A_k(x) \mapsto A'_k(x) = \varrho(x) A_k(x) \varrho^{-1}(x) - [(\partial/\partial x^k)\varrho(x)]\varrho^{-1}(x). \end{aligned} \tag{2}$$

With each functional of the form (2) we may associate the functional [4]:

$$L^{\sim}[\Psi, A] = \int L[\Psi(x), (\nabla_j \Psi(x))]\, dx + \mathscr{L}[A], \tag{3}$$

where $\mathscr{L}[A]$ is a volume integral involving the Killing form acting on $[F_{jk}]$ (we apply the Einstein summation convention):

$$\mathscr{L}[A] = -\tfrac{1}{4}\int g^{jm} g^{lk} \operatorname{Tr}(F_{jl} F_{mk})\, dV, \tag{4}$$

$g = [g_{jk}]$ is a (pseudo-)riemannian (which means: pseudo-riemannian, and in particular Riemannian) tensor on U, and dV is the volume element in the manifold (U, g). It can be seen that (3) is invariant with respect to (2). Thus, $\mathscr{L}^{\sim}$ expresses the action of the vector field Ψ on the Yang–Mills field A, which may be interpreted as a compensation field, whereas $\mathscr{L}$ expresses the action of a free compensation field.

In the case of the Minkowski space-time we have $g^{jj} = -1$ for $j = 1, 2, 3$, $g^{00} = 1$, $g^{jk} = 0$ for $j \neq k$, $j, k = 1, 2, 3, 0$, and the extremals for $\mathscr{L}$ satisfy the Yang–Mills equations ([1], p. 12–13):

$$\nabla^k F_{jk} = 0, \quad j = 1, 2, 3, 0, \tag{5}$$

where

$$\nabla^k = g^{kl}\nabla_l, \qquad \nabla_l F_{jk} = (\partial/\partial x^l) F_{jk} + [A_l, F_{jk}]. \tag{6}$$

In the present paper we start with a rigorous local generalization of (5) to the case of a compact orientable (pseudo-)riemannian manifold N (Section 1) which from the physical point of view is the space of a particle (cf. [2]). Then we take an arbitrary G-vector bundle over the base space N, where $G \subset \mathrm{SO(m)}$ or $\mathrm{SU(m)}$, and consider the Lie algebra $\mathscr{G}$ corresponding to G. We define the $\mathscr{G}$-vector field which is a generalization of the Yang–Mills field and we extend (5) for such fields using the *codifferential operator* $\delta = *d*$, where $*$ is the metric-dependent Hodge $*$-operator (Section 2). Physically this means that now we have much more freedom in choosing symmetries of the field and we allow hermitian structures via the constructed complex (in particular, holomorphic) vector bundles. Now we are able to derive the global Yang–Mills equations depending on an arbitrary metric g and an arbitrary non-abelian compact Lie group G (Section 3), to distinguish their "solenoidal" and "nonsolenoidal" parts, and to discuss their global properties. Finally, we give a motivation for a suitable choice of the principal fibre bundle (real or complex, in particular, holomorphic) with N as the base space and discuss the physical consequences of our results.

1. The Yang–Mills equations in the presence of external fields. We are going to derive rigorously the system of local Yang–Mills equations in the case of an arbitrary (pseudo-)riemannian metric or, differently speaking, the Yang–Mills equations in the presence of external fields.

LEMMA 1. *Suppose that N is an n-dimensional compact orientable (pseudo-) riemannian manifold with metric g of index* 0 *or* 1. *Let* su(2) *be the Lie algebra corresponding to* SU(2) *and $A = (A_k)$ a C^∞-Yang–Mills field in a coordinate neighbourhood U of N. Further, in a coordinate system (x^k) in U, let F_{jk} be given by* (1). *Consider functional* (4), *where dV is the volume element in U. If the functional attains its stationary value for some Yang–Mills field A, then this field satisfies the generalized Yang–Mills equations*

$$(\mathrm{Div}\, F)^j + [A_k, F^{jk}] = 0, \quad j = 1, 2, \ldots, n, \tag{7}$$

where

$$\begin{aligned} &(\mathrm{Div}\, F)^j = g^{mk}[(\partial/\partial x^m) F^j_k - \Gamma^l_{mk} F^j_l], \\ &F^{jk} = g^{jl} g^{mk} F_{lm}, \quad F^j_k = g^{jm} F_{mk}, \end{aligned} \tag{8}$$

and Γ^l_{jk} are the usual Christoffel symbols:

$$g_{lm}\Gamma^l_{jk} = \tfrac{1}{2}\left(\frac{\partial}{\partial x^j} g_{mk} + \frac{\partial}{\partial x^k} g_{jm} - \frac{\partial}{\partial x^m} g_{jk}\right).$$

Proof. Let us denote the Killing form $-\mathrm{Tr}(F_{jl}F_{mk})$ in (4) by $\langle F_{jl}, F_{mk}\rangle$. We calculate the first local Gateaux variation $\delta\mathscr{L}[A]$; this means that the supports of δA_j, $j = 1, 2, \ldots, n$, are compact:

$$\begin{aligned} \delta\mathscr{L}[A] &= -\tfrac{1}{4}\int g^{jm} g^{lk}(\langle \delta F_{jl}, F_{mk}\rangle + \langle F_{jl}, \delta F_{mk}\rangle)\, dV \\ &= -\tfrac{1}{2}\int g^{jm} g^{lk}\langle \delta F_{jl}, F_{mk}\rangle\, dV. \end{aligned}$$

On the other hand,

$$\delta F_{jl} = \frac{\partial}{\partial x^j}\delta A_l - \frac{\partial}{\partial x^l}\delta A_j + [\delta A_j, A_l] + [A_j, \delta A_l].$$

Therefore, integrating the integrand by parts under the hypothesis that the supports of δA_j are compact, we obtain

$$\delta\mathscr{L}[A] = \sum_{k=1}^{10} \delta_k, \tag{9}$$

where, with the notation $_{|j} = \partial/\partial x^j$ and $dV_{\mathrm{eucl}} = (\pm\det g)^{-1/2}\, dV$,

$$\begin{aligned} \delta_1 &= \tfrac{1}{2}\int g^{jm} g^{lk}\langle \delta A_l, (\partial/\partial x^j) F_{mk}\rangle (\pm\det g)^{1/2}\, dV_{\mathrm{eucl}}, \\ \delta_2 &= \tfrac{1}{2}\int g^{jm}_{|j} g^{lk}\langle \delta A_l, F_{mk}\rangle (\pm\det g)^{1/2}\, dV_{\mathrm{eucl}}, \\ \delta_3 &= \tfrac{1}{2}\int g^{jm} g^{lk}_{|j}\langle \delta A_l, F_{mk}\rangle (\pm\det g)^{1/2}\, dV_{\mathrm{eucl}}, \\ \delta_4 &= \tfrac{1}{2}\int g^{jm} g^{lk}\langle \delta A_l, F_{mk}\rangle (\partial/\partial x^j)(\pm\det g)^{1/2}\, dV_{\mathrm{eucl}}, \end{aligned}$$

$$\delta_5 = -\tfrac{1}{2}\int g^{jm} g^{lk} \langle \delta A_j, (\partial/\partial x^l) F_{mk}\rangle (\pm \det g)^{1/2}\, dV_{\text{eucl}},$$

$$\delta_6 = -\tfrac{1}{2}\int g^{jm}_{|l} g^{lk} \langle \delta A_j, F_{mk}\rangle (\pm \det g)^{1/2}\, dV_{\text{eucl}},$$

$$\delta_7 = -\tfrac{1}{2}\int g^{jm} g^{lk}_{|l} \langle \delta A_j, F_{mk}\rangle (\pm \det g)^{1/2}\, dV_{\text{eucl}},$$

$$\delta_8 = -\tfrac{1}{2}\int g^{jm} g^{lk} \langle \delta A_j, F_{mk}\rangle (\partial/\partial x^l)(\pm \det g)^{1/2}\, dV_{\text{eucl}},$$

$$\delta_9 = -\tfrac{1}{2}\int g^{jm} g^{lk} \langle [A_j, \delta A_l], F_{mk}\rangle (\pm \det g)^{1/2}\, dV_{\text{eucl}},$$

$$\delta_{10} = -\tfrac{1}{2}\int g^{jm} g^{lk} \langle [\delta A_j, A_l], F_{mk}\rangle (\pm \det g)^{1/2}\, dV_{\text{eucl}}.$$

By the properties of the Killing form, we have

$$\begin{aligned}\delta_9 &= \tfrac{1}{2}\int g^{jm} g^{lk} \langle [\delta A_l, A_j], F_{mk}\rangle (\pm \det g)^{1/2}\, dV_{\text{eucl}}\\ &= -\tfrac{1}{2}\int g^{jm} g^{lk} \langle \delta A_l, [F_{mk}, A_j]\rangle (\pm \det g)^{1/2}\, dV_{\text{eucl}}\\ &= \tfrac{1}{2}\int g^{jm} g^{lk} \langle \delta A_j, [A_l, F_{mk}]\rangle (\pm \det g)^{1/2}\, dV_{\text{eucl}} = \delta_{10}.\end{aligned}$$

Moreover, $\delta_1 = \delta_5$, $\delta_2 = \delta_7$, $\delta_3 = \delta_6$, and $\delta_4 = \delta_8$. On the other hand, with the notation $\Delta^{jm} = g^{jm}\det g$, we get

$$\frac{\partial}{\partial x^l}(\pm \det g)^{1/2} = \tfrac{1}{2}(\pm \det g)^{1/2} g^{jm}\frac{\partial}{\partial x^l} g_{jm} = -\tfrac{1}{2}(\pm \det g)^{1/2} g_{jm}\frac{\partial}{\partial x^l} g^{jm},$$

the latter equality being the consequence of the identity $g^{jm} g_{jm} = 4$. Now we express the derivatives of g^{jm} by g and the related Christoffel symbols:

$$g^{jm}_{|l} = -\Gamma^j_{rl} g^{rm} - \Gamma^m_{rl} g^{jr}; \tag{10}$$

so, finally,

$$(\partial/\partial x^l)(\pm \det g)^{1/2} = (\pm \det g)^{1/2}\, \Gamma^r_{rl}. \tag{11}$$

Thus, the corresponding addends δ_k become:

$$\begin{aligned}\delta_2 &= \delta_7\\ &= -\tfrac{1}{2}\int g^{jm}(-\Gamma^l_{rl} g^{rk} - \Gamma^k_{rl} g^{lr}) \langle \delta A_j, F_{mk}\rangle (\pm \det g)^{1/2}\, dV_{\text{eucl}}\\ &= \tfrac{1}{2}\int (g^{jm} g^{rk}\Gamma^l_{rl} + g^{jm} g^{lr}\Gamma^k_{rl}) \langle \delta A_j, F_{mk}\rangle (\pm \det g)^{1/2}\, dV_{\text{eucl}},\end{aligned}$$

$$\begin{aligned}\delta_3 &= \delta_6\\ &= \tfrac{1}{2}\int (g^{lk} g^{rm}\Gamma^j_{rl} + g^{lk} g^{jr}\Gamma^m_{rl}) \langle \delta A_j, F_{mk}\rangle (\pm \det g)^{1/2}\, dV_{\text{eucl}},\end{aligned}$$

$$\begin{aligned}\delta_4 &= \delta_8\\ &= -\tfrac{1}{2}\int g^{jm} g^{lk}\Gamma^r_{rl} \langle \delta A_j, F_{mk}\rangle (\pm \det g)^{1/2}\, dV_{\text{eucl}}.\end{aligned}$$

By the above calculations we have

$$\delta_1 + \delta_5 + \delta_9 + \delta_{10} = -\int g^{jm} g^{lk} \langle \delta A_j, \nabla_l F_{mk}\rangle (\pm \det g)^{1/2}\, dV_{\text{eucl}},$$

$$\delta_2 + \delta_7 + \delta_3 + \delta_6 + \delta_4 + \delta_8 = \int R^{jmk} \langle \delta A_j, F_{mk}\rangle (\pm \det g)^{1/2}\, dV_{\text{eucl}},$$

where ∇F_{mk} is given in (6) and

$$R^{jmk} = g^{jm} g^{lr} \Gamma^k_{rl} + g^{lk} g^{rm} \Gamma^j_{rl} + g^{lk} g^{jr} \Gamma^m_{rl}.$$

Since δA_j, $j = 1, \ldots, n$, are arbitrary and the Killing form in question is non-degenerate, $\delta \mathscr{L}[A] = 0$ implies by (8)

$$(12) \qquad g^{jm} g^{lk} \nabla_l F_{mk} = R^{jmk} F_{mk}, \quad j = 1, 2, \ldots, n.$$

In order to prove that the systems (12) and (7) are equivalent we have to introduce the notation (9) and to apply again the formulae (10). Namely, from (12) we deduce that

$$\begin{aligned} g^{lk} \{(\partial/x^l) F^j_k + [A_l, F^j_k]\} &= g^{lr} \Gamma^k_{rl} F^j_k + g^{lk} \Gamma^j_{rl} F^r_k - g^{jr} \Gamma^m_{rl} F^l_m + g^{lk} [(\partial/\partial x^l) g^{jm}] F_{mk} \\ &= g^{lr} \Gamma^k_{rl} F^j_k + g^{lk} \Gamma^j_{rl} F^r_k - g^{jr} \Gamma^m_{rl} F^l_m + g^{rm} \Gamma^j_{rl} F^l_m - g^{jr} \Gamma^m_{rl} F^l_m \end{aligned}$$

and distinguish in a natural way five addends $a_1, \ldots, a_5$ in the expression obtained. Clearly, $a_3 = -a_5$. Since $F^{lr} = g^{mr} F^l_m$ equals $-F^{rl} = g^{lk} F^r_k$, we also have $a_2 = -a_4$. Therefore

$$g^{lk} \{\partial/\partial x^l F^j_k + [A_l, F^j_k]\} = g^{lr} \Gamma^k_{rl} F^j_k, \quad j = 1, 2, \ldots, n;$$

and hence

$$(13) \qquad g^{lk} \{(\partial/\partial x^l) F^j_k - \Gamma^r_{lk} F^j_r + g^{lk} [A_l, F^j_k]\} = 0, \quad j = 1, 2, \ldots, n;$$

so we arrive indeed at (7) with the notation (8). It is clear that we can proceed as well in the opposite direction, and so the proof is completed.

2. The case of an arbitrary symmetry within SO(m) **or** SU(m)**.** Now we should like to extend the situation considered to vector fields of a more general symmetry than the Yang–Mills fields, namely of any symmetry within SO(m) or SU(m).

Let $\boldsymbol{E} = (E, \pi, N)$ be a G-vector bundle over the base space N — an n-dimensional compact orientable (pseudo-)riemannian manifold with metric g, where E denotes the boundle space, π: $E \to N$ is the projection, and G is an arbitrary compact subgroup of SO(m) or SU(m). Among other things, we allow complex (in particular, holomorphic) vector bundles. It is obvious that there is a covering $\mathscr{U} = \{U_j: j \in I\}$ of N with local formes over U_j such that the corresponding transition matrices have their values in SU(2). Then a connection ∇ on $\boldsymbol{E}$ is called a *G-connection* if for every local frame in question the connection matrix ω has its values in the Lie algebra $\mathscr{G}$ corresponding to G. It is not difficult to prove (cf. [6]), using the partition of unity, that on a given G-vector bundle there always exists a G-connection.

By a *$\mathscr{G}$-vector field* we mean any vector field A on N with values in $\mathscr{G}$. Thus, a Yang–Mills field is an su(2)-vector field. A G-connection ∇ is called the *connection corresponding to a $\mathscr{G}$-vector field* if in any local frame belonging to the G-structure the curvature form Ω^2 corresponding to A satisfies the differential equation

$$\delta F + 2\mathrm{Tr}_g(A \otimes_{\mathscr{G}} F) = 0, \tag{14}$$

where $\otimes_{\mathscr{G}}$ denotes the G-dependent tensor product operator and, locally,

$$F = F_{jk}\, dx^j \wedge dx^k, \qquad A = A_l\, dx^l, \tag{15}$$

F_{jk} and A_l being given by (1) and $A = (A_l)$, respectively, and $\wedge$ denoting the wedge product operator. This definition is motivated by the following

LEMMA 2. *Let N, g, G, $\mathscr{G}$, and $\mathbf{E}$ be as before. Then the system of differential equations* (7) *with notation* (8) *is well-posed and equivalent to* (14).

Proof. By (13) and the relation

$$g_{jm|l} = \Gamma^r_{lm} g_{jr} + \Gamma^r_{lj} g_{rm}, \tag{16}$$

analogous to (10), equations (7) with notation (8) become

$$g^{lk}[(\partial/\partial x^l) F^j_k - \Gamma^r_{lk} F_{mr}] + g^{lk}[A_l, F_{mk}] = 0, \qquad m = 1, 2, \ldots, n.$$

Hence

$$g^{lk}[\partial/\partial x^l F_{mk} - \Gamma^r_{lk} F_{mr} - \Gamma^r_{lm} g_{jr} F^j_k - \Gamma^r_{lj} g_{rm} F^j_k] + g^{lk}[A_l, F_{mk}] = 0$$

and, consequently,

$$g^{lk}[(\partial/\partial x^l) F_{mk} - \Gamma^r_{lk} F_{mr} - \Gamma^r_{lm} F_{rk} - \Gamma^r_{lj} g_{rm} F^j_k] + g^{lk}[A_l, F_{mk}] = 0.$$

Since, by the antisymmetry of F^{lj},

$$g^{lk} \Gamma^r_{lj} g_{rm} F^j_k = \Gamma^r_{lj} g_{rm} F^{lj} = \Gamma^r_{lj} F^{lj} g_{rm} = 0,$$

we arrive at

$$-g^{lk}(\partial/\partial x^l) F_{mk} - \Gamma^r_{lk} F_{mr} - \Gamma^r_{lm} F_{rk} + g^{lk}[A_l, F_{km}] = 0, \qquad m = 1, 2, \ldots, n.$$

By the definitions of δ and $\otimes_{\mathscr{G}}$, the above system is identical with (14), as desired.

3. The global Yang–Mills equations. We are now going to derive the global Yang–Mills equations depending on an arbitrary (pseudo-)riemannian metric g and an arbitrary non-abelian compact Lie group G.

THEOREM. *Suppose that N is an n-dimensional compact orientable* (*pseudo-*) *riemannian manifold with metric g of index* 0 *or* 1. *Let $\mathbf{E} = (E, \pi, N)$ be a real or complex G-vector boundle over the base space N, where E denotes the boundle space, $\pi\colon E \to N$ is the projection, and G is an arbitrary compact subgroup of* SO(m) *or* SU(m). *Suppose further that, in a local frame,*

$$F = dA + A \wedge A \quad \text{with} \quad A = A_k dx^k, \tag{17}$$

where (x^k) *is a coordinate system in a coordinate neighbourhood of* N, *and* $\wedge$ *denotes the wedge product operator. Consider the functional*

$$\mathcal{L}[A] = -\tfrac{1}{4}\int_N \operatorname{Tr}(F \wedge *_g F), \tag{18}$$

where A *is the* C^r G*-vector field on* N *such that, locally,* $A = (A_k)$ *and* $A = A_k dx^k$, *and* $*_g$ *is the* g*-dependent Hodge* $*$*-operator. Finally, suppose that functional* (18) *attains its stationary value for some* $\mathcal{G}$*-vector field* A. *Let* $P(N, G)$ *be the boundle of orthonormal frames of* E, *equipped with the connection* Γ *induced by a given* G*-connection* ∇ *on* E, *corresponding to* A. *Then the field* A *satisfies on* N *the system of generalized Yang–Mills equations*

$$D(*_{\mathcal{g}}\Omega^2) = 0, \tag{19}$$

where D *is the covariant derivative operator related to* Γ,

$$*_{\mathcal{g}}\colon A^p P \to A^{n-p} P, \tag{20}$$

where $A^r P$ *is the modulus of horizontal* r*-forms on* P, *of the type* ad G, *and* Ω^2 *is the curvature form corresponding to* A.

Proof. By Lemmas 1 and 2, the $\mathcal{G}$-vector field A satisfies the differential equation (13). Let ω denote the connection matrix corresponding to A. This means that, for any local cross-section s of $P(N, G)$, we have

$$s^*\Omega^2 = F, \quad s^*\omega = A, \tag{21}$$

where F and A are given locally by (15), and also

$$D(*_{\mathcal{g}}\Omega^2) = d(*_{\mathcal{g}}\Omega^2) + \omega \wedge *_{\mathcal{g}}\Omega^2. \tag{22}$$

Consequently,

$$s^* D(*_{\mathcal{g}}\Omega^2) = s^* d(*_{\mathcal{g}}\Omega^2) + (s^*\omega) \wedge (s^* *_{\mathcal{g}}\Omega^2) = ds^*(*_{\mathcal{g}}\Omega^2) + (s^*\omega) \wedge (s^* *_{\mathcal{g}}\Omega^2).$$

Since, as it can easily be verified, $s^* *_{\mathcal{g}} = *_g s^*$, relation (22) yields

$$*_g s^* D(*_{\mathcal{g}}\Omega^2) = (*_g d *_g) s^*\Omega^2 + *_g[(\delta^*\omega) \wedge *_g(\delta^*\Omega^2)]. \tag{23}$$

Therefore, by the definition of the codifferential operator: $\delta_g = *_g d *_g$ and equalities (21), relation (23) becomes

$$*_g s^* D(*_{\mathcal{g}}\Omega^2) = \delta_g F + *_g(A \wedge *_g F). \tag{24}$$

If we succeed to prove that

$$*_g(A \wedge *_g F) = 2\operatorname{Tr}_g(A \otimes_{\mathcal{g}} F), \tag{25}$$

then the global differential equation (14) is shown to be equivalent to

$$*_g s^* D *_{\mathcal{g}} \Omega^2 = 0. \tag{26}$$

In order to demonstrate the statement formulated above, let us consider in any coordinate neighbourhood u of N, mentioned in Lemma 1, an orthonormal system (e_r), $r = 1, \ldots, m$, of vector fields. Consider further the system (e_j^*), $j = 1, \ldots, n$, of one-forms on U such that $e_j^*[e_k] = \delta_{jk}$ for each (j, k). Then (15) becomes

$$F = \tfrac{1}{2} f^{jk} e_j^* \wedge e_k^*, \qquad A = a^l e_l^*, \tag{27}$$

and we calculate, subsequently,

$$*_g F = \tfrac{1}{2} f^{jk} *_g (e_j^* \wedge e_k^*) = \tfrac{1}{4} f^{jk} \varepsilon_{jk}^{lm} e_l^* \wedge e_m^*,$$

where ε_{jk}^{lm} denotes the totally antisymmetric Levi-Civita tensor, and

$$A \wedge *_g F = \tfrac{1}{4} [a^r, f^{jk}] \varepsilon_{jk}^{lm} e_l^* \wedge e_m^* \wedge e_r^*, \qquad *_g (*_g F \wedge A) = \tfrac{1}{4} [a^r, f^{jk}] \varepsilon_{jk}^{lm} \varepsilon_{lm}^{rs} e_s^*,$$

where $\varepsilon_{jk}^{lm} \varepsilon_{lm}^{rs} = 2(\delta_j^r \delta_k^s - \delta_j^s \delta_k^r)$ and δ_i^k denotes the Kronecker symbol. Thus,

$$*_g (A \wedge *_g F) = \tfrac{1}{4} \cdot 2 \cdot 2 [a^l, f^{rs}] \delta_{lr} e_s^* = 2\mathrm{Tr}_g (A \otimes_{\mathscr{G}} F),$$

where δ_{jk} is the Kronecker symbol, and the global differential equation (14) is indeed equivalent to (26).

The system of differential equations (26) holds in particular for every local cross-section s on $P(N, G)$. On the other hand, the Hodge operator $*_g$ is an isomorphism, and so we conclude that the $\mathscr{G}$-vector field A satisfies on N the system of differential equations (19), as desired.

After Lemma 1, following the classical analogies, it would be natural to distinguish the "solenoidal" and "nonsolenoidal" parts of the generalized Yang–Mills equations as $(\mathrm{Div}\, F)^j$ and $[A_k, F^{jk}]$, respectively $(n = 1, 2, \ldots, n)$. After Lemma 2 the natural candidates would be δF and $2\mathrm{Tr}_g(A \otimes_{\mathscr{G}} F)$, respectively. However, if we look for a suitable global decomposition of the final equation (19), the proper choice is $d(*_{\mathscr{G}} \Omega^2)$ for the "solenoidal" part and $*_{\mathscr{G}} \Omega^2 \wedge_{\mathscr{G}} \omega$ for the "nonsolenoidal" part, where $\wedge_{\mathscr{G}}$ denotes the $\mathscr{G}$-dependent wedge product operator. Namely, the following result holds true:

COROLLARY. *Under the hypotheses of the theorem, without assuming that A corresponds to a stationary value of* (18), *we have*

$$D(*_{\mathscr{G}} \Omega^2) = d(*_{\mathscr{G}} \Omega^2) + *_{\mathscr{G}} \Omega^2 \wedge_{\mathscr{G}} \omega, \tag{28}$$

where both addends are well-defined global tensorial forms. An analogous statement for the decomposition (24) *is, in general, false.*

Proof. Consider the global tensorial form $D(*_{\mathscr{G}} \Omega^2)$ which, by the theorem, or – more exactly – by (24) and (25) without assuming that A corresponds to a stationary value of (18), is equivalent to the left-hand side of (14), where F and A can locally be expressed by (27), where (e_r) and (e_j^*) are as in the proof of the theorem. The forms F and A themselves have their values in the Lie algebra $\mathscr{G}$, defined over a coordinate neighbourhood U of N such that $(\pi^{-1}(U), \pi, U)$ is a trivial bundle.

The first statement of the corollary is obvious. In order to prove the second statement, in view of (25) we have to check whether vanishing of the one-form $\mathrm{Tr}_g(A\otimes_{\mathscr{G}}F)$ depends on the choice of a local trivialization. Thus, let us take into account another local trivialization over U. The forms F and A are then transformed according to the formulae

$$\tilde{F}=\hat{g}^{-1}F\hat{g} \quad \text{and} \quad \tilde{A}=\hat{g}^{-1}A\hat{g}+\hat{g}^{-1}d\hat{g},$$

respectively, where $\hat{g}\colon U\to \mathrm{GL}(\mathrm{m}, \boldsymbol{R})$ or $\mathrm{GL}(\mathrm{m}, \boldsymbol{C})$. Consequently,

$$\begin{aligned}\mathrm{Tr}_g(\tilde{A}\otimes_{\mathscr{G}}\tilde{F}) &= \mathrm{Tr}_g[(\hat{g}^{-1}A\hat{g}+\hat{g}^{-1}d\hat{g})\otimes_{\mathscr{G}}(\hat{g}^{-1}F\hat{g})]\\ &= \mathrm{Tr}_g(\hat{g}^{-1}A\hat{g}\otimes_{\mathscr{G}}\hat{g}^{-1}F\hat{g})+\mathrm{Tr}_g(\hat{g}^{-1}d\hat{g}\otimes_{\mathscr{G}}\hat{g}^{-1}F\hat{g})\\ &= \mathrm{Tr}_g[\hat{g}^{-1}(A\otimes_{\mathscr{G}}F)\hat{g}]+\mathrm{Tr}_g\{\hat{g}^{-1}(d\hat{g})\hat{g}^{-1}\otimes_{\mathscr{G}}F]\hat{g}\}\\ &= \hat{g}^{-1}[\mathrm{Tr}_g(A\otimes_{\mathscr{G}}F)]\hat{g}-\hat{g}^{-1}[\mathrm{Tr}_g(\hat{g}d\hat{g}^{-1}\otimes_{\mathscr{G}}F)]\hat{g}.\end{aligned}$$

Hence the equation $\mathrm{Tr}_g(A\otimes_{\mathscr{G}}F)=0$ is not invariant with respect to local trivializations, and this completes the proof.

4. The canonical principal fibre boundle and physical consequences. By the theorem proved in Section 3, the global formulation of the generalized Yang–Mills problem, depending on an arbitrary metric g and an arbitrary non-abelian Lie group $\mathscr{G}$, involves in a natural way the bundle $P(N, G)$ of orthonormal frames of E, which is equipped with the connection Γ induced by a given G-connection ∇ on E, corresponding to the $\mathscr{G}$-vector field A. We can go still further, considering a more general situation, namely that $P(N, G)$ is an arbitrary principal fibre bundle of E – a real or complex G-vector bundle over N, where G is an arbitrary compact subgroup of some SO(m) or SU(m). Then we can construct on E the canonical riemannian (hermitian) metric h (cf. e.g. [6], p. 69) and consider the h-depending Hodge $*$-operator $*_{\mathscr{G}}$. The distinction between the "solenoidal" and "nonsolenoidal" parts of the generalized Yang–Mills equations now motivates, even purely mathematically, the following definition: a connection ∇ corresponding to a $\mathscr{G}$-vector field is called *solenoidal* if the corresponding connection matrix ω satisfies the condition

$$(29) \qquad *_{\mathscr{G}}\Omega^2\wedge_{\mathscr{G}}\omega=0.$$

In [3] we have given another motivation for the above definition, proving that, if φ is a horizontal tensorial two-form on the principal fibre bundle P in question, then

$$(30) \qquad D\varphi(X, Y, Z)=d\varphi(X, Y, Z)-\tfrac{1}{3}\{[\varphi(X, Y), \omega(Z)]+ +[\varphi(Y, Z), \omega(X)]+[\varphi(Z, X), \omega(Y)]\}$$

for arbitrary vector fields X, Y, and Z on P. From this fact we have deduced that a connection ∇ corresponding to a $\mathscr{G}$-vector field, where $\mathscr{G}$ is the Lie algebra of a semi-simple Lie group G, is solenoidal if and only if

$$\Omega^2 \equiv D\omega = 0. \tag{31}$$

If, in particular, the principal fibre bundle $P(N, G)$ is not trivial and admits a solenoidal connection, then N is multiply connected.

The physical consequences of the above results, obtained in our paper [3], have already been indicated in the paper. Here we indicate only the additional physical consequences resulting from the present paper. Namely, the possibility of obtaining the Yang–Mills equations, reduced to a form equivalent to the Maxwell equations appearing in some class of metrics, shows that the variety of physical fields can be treated as a result of geometry while their sources are of the same nature by physical principles. This observation will be treated by us rigorously in a subsequent paper. The whole research elucidates the problem, posed recently by one of us in [5], of replacing a given Yang–Mills field, in general with currents, by a $\mathscr{G}$-vector field in a different, curved (pseudo-)riemannian manifold, by way of including currents in the geometry, so that the field becomes Maxwell-like.

References

[1] M. F. Atiyah, *Geometry of Yang–Mills fields*, Lezioni Fermiane, Academia Nazionale dei Lincei–Scuola Normale Superiore, Pisa 1979.

[2] –, N. J. Hitchin and I. M. Singer, *Self-duality in four-dimensional Riemannian geometry*, Proc. Royal Soc. London Ser. A 362 (1978), 425–461.

[3] B. Gaveau, J. Kalina, J. Ławrynowicz, P. Walczak and L. Wojtczak, *On solenoidal Yang–Mills fields*, Proc. of the 2nd Internat. Conf. on Complex Analysis and its Applications, Varna 1983, to appear.

[4] S. G. Gindikin i G. M. Henkin, *Preobrazovanie Penrouza i kompleksnaya integralnaya geometriya*, in: *Itogi nauki i tehniki*, Seriya: Sovremennye problemy matematiki 17, Moskva 1981, 57–111.

[5] J. Ławrynowicz, *Connections corresponding to hermitian structures* [*Problems in the analysis on complex manifolds*], in: *Analytic functions*, Błażejewko 1982, Proceedings. Ed. by J. Ławrynowicz (Lecture Notes in Math. 1039), Springer-Verlag, Berlin–Heidelberg–New York–Tokyo 1983, 488–490.

[6] R. O. Wells, Jr., *Differential analysis on complex manifolds*, Prentice-Hall, Englewood Cliffs, NJ., 1973.

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