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A CHARACTERIZATION OF JORDAN AND LEBESGUE MEASURES

BY

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1. Introduction. In this paper, R^n will denote the n-dimensional Euclidean space, and J_n the ring of all Jordan measurable subsets of R^n. Moreover, B_n and L_n will denote the σ-fields of all Borel measurable and Lebesgue measurable subsets of R^n, respectively. The n-dimensional Jordan and Lebesgue measures will be denoted by ι_n and λ_n.

The following proposition is well known:

Assume that a set function m defined on J_n (on L_n) has the following properties:

(i) *m is non-negative;*

(ii) *m is σ-additive;*

(iii) $m\big(\bigtimes_{i=1}^{n}[0,1)\big) = 1$;

(iv) *if $A \in J_n$ ($A \in L_n$) and a is an arbitrary rigid motion of R^n (that is, a is an isometry preserving orientation), then $m\big(a(A)\big) = m(A)$.*

Then $m = \iota_n$ ($m = \lambda_n$).

That is, properties (i)-(iv) characterize the measures ι_n and λ_n.

It is easy to see that property (iv) cannot be omitted; there is a set function m defined on L_n with properties (i), (ii) and (iii) for which $m \neq \lambda_n$ (see 4.2).

Consider the property

(iii)* *If a is an arbitrary rigid motion of R^n, then*

$$m\big[a\big(\bigtimes_{i=1}^{n}[0,1)\big)\big] = 1.$$

This property is obviously stronger than (iii) and is weaker than the conjunction of (iii) and (iv).

In this paper we show that for $n \geqslant 2$ conditions (i), (ii) and (iii)* are strong enough to characterize the set functions ι_n and λ_n. The following theorems will be proved.

THEOREM 1. *Let m be a non-negative and (finitely) additive set function defined on J_n ($n \geqslant 2$) for which* (iii)* *holds. Then $m = \iota_n$.*

THEOREM 2. *Let m be a σ-additive set function defined on B_n ($n \geqslant 2$) for which* (iii)* *holds. Then $m(X) = \lambda_n(X)$ for every $X \in B_n$.*

Since λ_n is the completion of its restriction to B_n, Theorem 2 together with Proposition 4.4.25 of [3] yield

THEOREM 3. *Let m be a non-negative and σ-additive set function defined on L_n ($n \geqslant 2$) for which* (iii)* *holds. Then $m = \lambda_n$.*

Sets of the form $[a, b) \times [c, d) \subset R^2$ will be called *intervals*. A set $H \subset R^2$ is a *rectangle* if $H = \alpha(I)$, where I is an interval and α is a rigid motion of R^2. If $I = [0, 1) \times [0, 1)$, then $Q = \alpha(I)$ is a *unit square*. The ring generated by the rectangles will be denoted by P. Obviously, $P \subset J_2 \cap B_2$.

Our results are based upon the following

LEMMA. *Let m be an additive set function defined on P. Suppose that m is bounded from below and that $m(Q) = 1$ for every unit square Q. Then there are countable sets X, $Y \subset R^1$ such that, for every $a \in R^1 \setminus X$, $c \in R^1 \setminus Y$ and positive rationals p, q, we have*

$$m\big([a, a+p) \times [c, c+q)\big) = pq. \tag{1}$$

2. Proof of the Lemma. Let m be a set function satisfying the conditions of the Lemma and suppose that $m(X) \geqslant K$ for every $X \in P$ ($K < 0$). Our proof (using an idea of Christov [2]) is carried out in several steps. First we give some additional notation. The interval $I = [a, b) \times [c, d)$ is said to be an *interval of continuity* (with respect to m) if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|m(X) - m(I)| < \varepsilon$$

whenever $X \in P$ and

$$[a+\delta, b-\delta) \times [c+\delta, d-\delta) \subset X \subset [a-\delta, b+\delta) \times [c-\delta, d+\delta).$$

In the course of the proof the points of R^2 will be considered as vectors. We write

$$A + \mathbf{a} = \{\mathbf{x} + \mathbf{a} \colon \mathbf{x} \in A\} \quad \text{and} \quad b \cdot A = \{b \cdot \mathbf{x} \colon \mathbf{x} \in A\}$$

for every $A \subset R^2$, $\mathbf{a} \in R^2$, and $b \in R^1$.

Let $\mathbf{a}$ and $\mathbf{b}$ be perpendicular vectors. The set of points $k \cdot \mathbf{a} + l \cdot \mathbf{b}$ (k and l are arbitrary integers) is said to be a *lattice* and is denoted by $\Lambda(\mathbf{a}, \mathbf{b})$. The lattice $\Lambda(\mathbf{a}, \mathbf{b})$ is a *unit lattice* if $|\mathbf{a}| = |\mathbf{b}| = 1$. The rectangles

$$\{x \cdot \mathbf{a} + y \cdot \mathbf{b} \colon k \leqslant x < k+1,\ n \leqslant y < n+1\}$$

are the *fundamental squares* of $\Lambda(\mathbf{a}, \mathbf{b})$. The rectangle H is a *lattice* *rectangle* if

$$H = \{x\cdot \boldsymbol{a}+y\cdot \boldsymbol{b}:\ k\leqslant x<n,\ r\leqslant y<s\},$$

where k, n, r, and s are integers. Every lattice rectangle is the union of some fundamental squares.

2.1. If $A\in P$, then A is bounded, and thus A can be covered by a finite number of disjoint unit squares $Q_1, Q_2, \ldots, Q_n$. Hence, for every $X\subset A$, $X\in P$, we have

$$K\leqslant m(X) = m\Big(\bigcup_{i=1}^{n} Q_i\Big)-m\Big(\bigcup_{i=1}^{n} Q_i\setminus X\Big)\leqslant n-K,$$

that is, m is of bounded variation on every $A\in P$.

Let π and ν denote the positive and negative variations of m, respectively. Then

$$m(A) = \pi(A)-\nu(A)\quad \text{for every } A\in P$$

by the Jordan decomposition theorem.

π and ν restricted to the semiring of intervals are non-negative and additive interval functions. Hence there are countable sets X_0, $Y_0\subset R^1$ such that for every $a, b\in R^1\setminus X_0$ and $c, d\in R^1\setminus Y_0$ the interval $I = [a, b)\times[c, d)$ is an interval of continuity with respect to π and ν. Thus for every $\varepsilon>0$ there exists $\delta>0$ such that $X\in P$ and

$$(2)\qquad [a+\delta, b-\delta)\times[c+\delta, d-\delta)\subset X\subset[a-\delta, b+\delta)\times[c-\delta, d+\delta)$$

imply

$$|\pi(X)-\pi(I)|<\varepsilon \quad\text{and}\quad |\nu(X)-\nu(I)|<\varepsilon.$$

Hence $|m(X)-m(I)|<2\varepsilon$, that is, I is an interval of continuity with respect to m.

2.2. Let Q be a unit square, and H a rectangle,

$$Q = \alpha\big([0,1)\times[0,1)\big)\quad\text{and}\quad H = \alpha\big([a, b)\times[c, d)\big),$$

where $b-a<1$, $d-c<1$, and α is a rigid motion of R^2. Let $\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r}$, and $\boldsymbol{s}$ denote the vertices of Q listed going around counter-clockwise. We show that

$$(3)\qquad m(H+\boldsymbol{p})-m(H+\boldsymbol{q})+m(H+\boldsymbol{r})-m(H+\boldsymbol{s}) = 0.$$

Since the sets

$$A = \alpha\big([a, a+1)\times[c, c+1)\big),\quad B = \alpha\big([b, b+1)\times[c, c+1)\big),$$

$$C = \alpha\big([b, b+1)\times[d, d+1)\big),\quad D = \alpha\big([a, a+1)\times[d, d+1)\big)$$

are unit squares, we have

$$m(A) = m(B) = m(C) = m(D) = 1.$$

Thus

$$m(A\setminus B) = m(B\setminus A) \quad \text{and} \quad m(D\setminus C) = m(C\setminus D).$$

Subtracting we have

$$m\big((A\setminus B)\setminus(D\setminus C)\big) - m\big((D\setminus C)\setminus(A\setminus B)\big)$$
$$= m\big((B\setminus A)\setminus(C\setminus D)\big) - m\big((C\setminus D)\setminus(B\setminus A)\big),$$

that is,

$$m(H+p) - m(H+s) = m(H+q) - m(H+r),$$

which proves (3).

2.3. Let Q be an arbitrary unit square and let p, q, r, and s denote its vertices (listed going around counter-clockwise). Suppose, for the interval $H = [a, b)\times[c, d)$, that $H+p$, $H+q$, $H+r$, and $H+s$ are intervals of continuity with respect to m. Then (3) holds.

In fact, let $\varepsilon > 0$ be arbitrary. Since $H+p$, $H+q$, $H+r$, and $H+s$ are intervals of continuity, there exists δ, $0 < \delta < 1$, such that

$$(4) \qquad \begin{aligned} &|m(X+p)-m(H+p)| < \varepsilon, \quad &|m(X+q)-m(H+q)| < \varepsilon,\\ &|m(X+r)-m(H+r)| < \varepsilon, \quad &|m(X+s)-m(H+s)| < \varepsilon \end{aligned}$$

whenever $X \in P$ and X satisfies (2).

Consider the lattice

$$\Lambda = \Lambda\left(\frac{\delta}{2}(q-p),\ \frac{\delta}{2}(s-p)\right)$$

and put

$$N = \left\{\frac{\delta}{2}Q + a:\ a \in \Lambda\right\}.$$

Let $N_1, N_2, \ldots, N_k$ be the elements of N having at least one common point with H and let

$$X = \bigcup_{i=1}^{k} N_i.$$

Then (2) holds because of the choice of δ and X. Applying 2.2 for the squares N_i we have

$$m(N_i+p) - m(N_i+q) + m(N_i+r) - m(N_i+s) = 0 \qquad (i = 1, 2, \ldots, k).$$

Adding these equalities we have

$$m(X+p) - m(X+q) + m(X+r) - m(X+s) = 0.$$

Hence, by (4),

$$|m(H+p) - m(H+q) + m(H+r) - m(H+s)| < 4\varepsilon.$$

Since ε is arbitrary, this gives (3).

2.4. Let Λ be a unit lattice and let H be an interval. Suppose that $H+\boldsymbol{x}$ is an interval of continuity for every $\boldsymbol{x} \in \Lambda$. We show that (3) holds if $\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r}$, and $\boldsymbol{s}$ are the vertices of any lattice rectangle of Λ (listed going around counter-clockwise).

In fact, let $\Lambda = \Lambda(\boldsymbol{a}, \boldsymbol{b})$ and let

$$\boldsymbol{p} = k\boldsymbol{a}+l\boldsymbol{b}, \quad \boldsymbol{q} = n\boldsymbol{a}+l\boldsymbol{b}, \quad \boldsymbol{r} = n\boldsymbol{a}+t\boldsymbol{b}, \quad \boldsymbol{s} = k\boldsymbol{a}+t\boldsymbol{b}.$$

Then for the points $\boldsymbol{p}_{i,j} = i\boldsymbol{a}+j\boldsymbol{b}$ $(i, j = 0, \pm 1, \pm 2, \ldots)$ we have

$$m(H+\boldsymbol{p}_{i,j}) - m(H+\boldsymbol{p}_{i+1,j}) + m(H+\boldsymbol{p}_{i+1,j+1}) - m(H+\boldsymbol{p}_{i,j+1}) = 0$$

by 2.3. Adding these equalities for $k \leqslant i < n$ and $l \leqslant j < t$ we get (3).

2.5. Let $\boldsymbol{u}$ be an arbitrary vector of length $12\sqrt{2}$. We prove that there exist countable sets $X_{\boldsymbol{u}}, Y_{\boldsymbol{u}} \subset R^1$ such that if $a, b \notin X_{\boldsymbol{u}}$ and $c, d \notin Y_{\boldsymbol{u}}$, then, for the interval $H = [a, b) \times [c, d)$,

$$m(H) = m(H+\boldsymbol{u}). \tag{5}$$

Let $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ be perpendicular vectors for which $|\boldsymbol{\xi}| = |\boldsymbol{\eta}| = 1$ and $12\boldsymbol{\xi}+12\boldsymbol{\eta} = \boldsymbol{u}$. The number of unit lattices having at least two common points with $\Lambda = \Lambda(\boldsymbol{\xi}, \boldsymbol{\eta})$ is countable. Let $\{\Lambda_i\}_{i=1}^{\infty}$ denote a sequence of these lattices. Then the set $\bigcup_{i=1}^{\infty} \Lambda_i$ is also countable. Let

$$\bigcup_{i=1}^{\infty} \Lambda_i = \{\boldsymbol{p}_n(x_n, y_n)\}_{n=1}^{\infty}.$$

We are going to show that the sets

$$X_{\boldsymbol{u}} = \bigcup_{n=1}^{\infty} (X_0 - x_n) \quad \text{and} \quad Y_{\boldsymbol{u}} = \bigcup_{n=1}^{\infty} (Y_0 - y_n)$$

satisfy the requirements (where X_0 and Y_0 have the same meaning as in 2.1).

Suppose that $a, b \notin X_{\boldsymbol{u}}$, $c, d \notin Y_{\boldsymbol{u}}$ and put $H = [a, b) \times [c, d)$. Since $a+x_n, b+x_n \notin X_0$ and $c+y_n, d+y_n \notin Y_0$, $H+\boldsymbol{p}_n$ is an interval of continuity for every n.

This implies that if $\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r}$, and $\boldsymbol{s}$ are the vertices of a rectangle T (listed counter-clockwise), $\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r}, \boldsymbol{s} \in \Lambda$, and the lengths of the sides of T are integers, then (3) holds. In fact, T is a lattice rectangle of Λ_n for a suitable n, so we can apply 2.4 for Λ_n and H.

Now let $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d} \in \Lambda$ be the vertices of a rectangle (listed counter-clockwise) for which $\boldsymbol{b}-\boldsymbol{a} = \frac{1}{4}\boldsymbol{u}$ and $|\boldsymbol{d}-\boldsymbol{a}| = 4\sqrt{2}$. We prove

$$m(H+\boldsymbol{a}) - m(H+\boldsymbol{b}) + m(H+\boldsymbol{c}) - m(H+\boldsymbol{d}) = 0. \tag{6}$$

Consider the lattice points $\boldsymbol{e}, \boldsymbol{f} \in \Lambda$ as marked in Fig. 1. Then we obtain

$$m(H+\boldsymbol{a}) - m(H+\boldsymbol{f}) + m(H+\boldsymbol{c}) - m(H+\boldsymbol{e}) = 0, \tag{7}$$

(8) $$m(H+f)-m(H+d)+m(H+e)-m(H+b)=0,$$

since for the rectangles a, f, c, e and f, d, e, b we have

$$|e-a|=1, \quad |f-a|=7, \quad \text{and} \quad |d-e|=|b-e|=5.$$

Adding (7) and (8), we get (6).

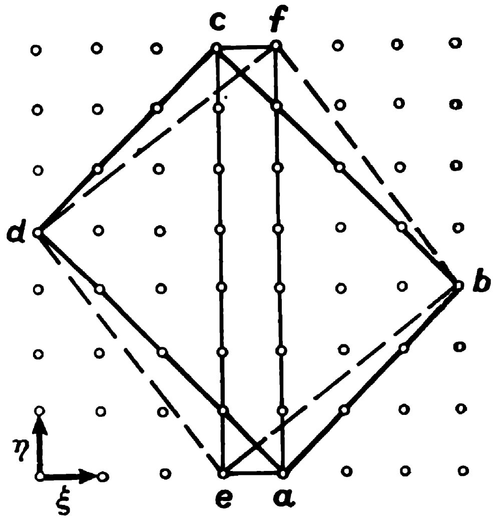

Fig. 1

Suppose now that $p, q, r, s \in \Lambda$ are the vertices of a rectangle T (listed counter-clockwise) for which $q-p=ku$ and $|s-p|=12n\sqrt{2}$. Then (3) holds. In fact, let

$$v=\frac{1}{3n}(s-p) \quad \text{and} \quad q_{i,j}=p+i\frac{1}{4}u+jv.$$

Then, by (6), we have

$$m(H+q_{i,j})-m(H+q_{i+1,j})+m(H+q_{i+1,j+1})-m(H+q_{i,j+1})=0 \quad \text{for every } i, j.$$

Adding these equalities for $0 \leqslant i < 4k$ and $0 \leqslant j < 3n$ we get (3), since $q_{0,0}=p$, $q_{4k,0}=q$, $q_{0,3n}=s$, and $q_{4k,3n}=r$.

Finally, we prove that if a_1 and b_1 are lattice points of Λ for which $b_1-a_1=ku$, where k is an integer, then $m(H+a_1)=m(H+b_1)$.

Let the points c_1, c_2, c_3, a_2, b_2 be chosen as in Fig. 2, that is, let the points a_1, c_1, c_2, b_1 form the four vertices of a square Q_1 listed going around counter-clockwise, let c_3 be the centre of Q_1, and let $a_2=c_1+(a_1-c_3)$ and $b_2=b_1+(c_2-c_3)$. Then c_1, c_2, c_3, a_2, and b_2 are lattice points in Λ and the length of the sides of Q_1 is $12k\sqrt{2}$. Hence

(9) $$m(H+a_1)-m(H+c_1)+m(H+c_2)-m(H+b_1)=0.$$

Since $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{c}_1, \boldsymbol{c}_3$ are the vertices of a square with sides of length $12k$, we have

$$m(H+\boldsymbol{a}_1)-m(H+\boldsymbol{a}_2)+m(H+\boldsymbol{c}_1)-m(H+\boldsymbol{c}_3) = 0. \tag{10}$$

Similarly,

$$m(H+\boldsymbol{c}_3)-m(H+\boldsymbol{c}_2)+m(H+\boldsymbol{b}_2)-m(H+\boldsymbol{b}_1) = 0. \tag{11}$$

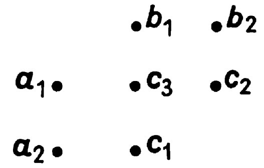

Fig. 2

Adding (9), (10), and (11), we have

$$2[m(H+\boldsymbol{a}_1)-m(H+\boldsymbol{b}_1)]-m(H+\boldsymbol{a}_2)+m(H+\boldsymbol{b}_2) = 0,$$

that is,

$$2[m(H+\boldsymbol{a}_1)-m(H+\boldsymbol{b}_1)] = m(H+\boldsymbol{a}_2)-m(H+\boldsymbol{b}_2). \tag{12}$$

Since $\boldsymbol{b}_2-\boldsymbol{a}_2 = 2k\boldsymbol{u}$, we can repeat the calculations above for $\boldsymbol{a}_2$ and $\boldsymbol{b}_2$, and we get the lattice points $\boldsymbol{a}_3$ and $\boldsymbol{b}_3$ for which

$$\begin{aligned} 4[m(H+\boldsymbol{a}_1)-m(H+\boldsymbol{b}_1)] &= 2[m(H+\boldsymbol{a}_2)-m(H+\boldsymbol{b}_2)] \\ &= m(H+\boldsymbol{a}_3)-m(H+\boldsymbol{b}_3). \end{aligned} \tag{13}$$

Continuing this process, we get $\boldsymbol{a}_n, \boldsymbol{b}_n \in \Lambda$ such that for every n

$$2^{n-1}[m(H+\boldsymbol{a}_1)-m(H+\boldsymbol{b}_1)] = m(H+\boldsymbol{a}_n)-m(H+\boldsymbol{b}_n). \tag{14}$$

The right-hand side of (14) is bounded. In fact, H can be covered by a finite number of disjoint unit squares,

$$H \subset \bigcup_{i=1}^{N} Q'_i,$$

whence

$$H+\boldsymbol{x} \subset \bigcup_{i=1}^{N} (Q'_i+\boldsymbol{x}) \quad \text{for every } \boldsymbol{x}.$$

Repeating the first argument of 2.1 we have

$$K \leqslant m(H+\boldsymbol{x}) \leqslant N-K \quad \text{for every } \boldsymbol{x},$$

thus

$$|m(H+\boldsymbol{a}_n)-m(H+\boldsymbol{b}_n)| \leqslant 2(N-K) \quad \text{for every } n.$$

However, the left-hand side of (14) can be bounded only in the case where $m(H+\boldsymbol{a}_1) = m(H+\boldsymbol{b}_1)$.

Finally, since $\boldsymbol{0}$ and $\boldsymbol{u}$ are lattice points of Λ and their difference is $\boldsymbol{u}$, we get $m(H) = m(H+\boldsymbol{u})$, which was to be proved.

2.6. We show that for every $\mathbf{a} \in R^2$ there exist countable sets $X_{\mathbf{a}}$ and $Y_{\mathbf{a}}$ such that $m(H) = m(H+\mathbf{a})$ whenever $H = [a, b) \times [c, d)$ and $a, b \notin X_{\mathbf{a}}$, $c, d \notin Y_{\mathbf{a}}$.

Let A denote the set of those vectors $\mathbf{a}$ for which $X_{\mathbf{a}}$ and $Y_{\mathbf{a}}$ exist with the property described above. Since every vector can be represented as a sum of vectors of length $12\sqrt{2}$, it is enough to prove that $\mathbf{a} \in A$ and $|\mathbf{u}| = 12\sqrt{2}$ imply $\mathbf{b} = \mathbf{a}+\mathbf{u} \in A$.

Let

$$\mathbf{a} = (x, y), \quad X_{\mathbf{b}} = X_{\mathbf{a}} \cup (X_{\mathbf{u}} - x), \quad \text{and} \quad Y_{\mathbf{b}} = Y_{\mathbf{a}} \cup (Y_{\mathbf{u}} - y),$$

where $X_{\mathbf{u}}$ and $Y_{\mathbf{u}}$ are as in 2.5.

If $a, b \notin X_{\mathbf{b}}$ and $c, d \notin Y_{\mathbf{b}}$, then for $H = [a, b) \times [c, d)$ we have

$$m(H) = m(H+\mathbf{a}) = m(H+\mathbf{a}+\mathbf{u}) = m(H+\mathbf{b}),$$

that is, $\mathbf{b} \in A$.

2.7. Now we are going to complete the proof of the Lemma. We put $X = \bigcup (X_{\mathbf{a}} - p)$ and $Y = \bigcup (Y_{\mathbf{a}} - q)$, where the unions extend over all vectors $\mathbf{a}$ with rational coordinates and rational numbers p and q. Suppose that $a \notin X$ and $c \notin Y$. We show that

$$(15) \quad m\left(\left[a, a+\frac{1}{n}\right) \times \left[c, c+\frac{1}{n}\right)\right)$$
$$= m\left(\left[a+\frac{i-1}{n}, a+\frac{i}{n}\right) \times \left[c+\frac{j-1}{n}, c+\frac{j}{n}\right)\right) = \frac{1}{n^2} \quad (i, j = 1, 2, \ldots).$$

Indeed, $a \notin X$ and $c \notin Y$ imply

$$a, a+\frac{1}{n} \notin X_{\mathbf{a}} \quad \text{and} \quad c, c+\frac{1}{n} \notin Y_{\mathbf{a}}, \quad \text{where } \mathbf{a} = \left(\frac{i-1}{n}, \frac{j-1}{n}\right),$$

whence

$$(16) \quad m\left(\left[a, a+\frac{1}{n}\right) \times \left[c, c+\frac{1}{n}\right)\right)$$
$$= m\left(\left[a+\frac{i-1}{n}, a+\frac{i}{n}\right) \times \left[c+\frac{j-1}{n}, c+\frac{j}{n}\right)\right) \quad \text{for every } i, j.$$

On the other hand,

$$[a, a+1) \times [c, c+1)$$
$$= \bigcup \left\{\left[a+\frac{i-1}{n}, a+\frac{i}{n}\right) \times \left[c+\frac{j-1}{n}, c+\frac{j}{n}\right) : 1 \leqslant i, j \leqslant n\right\}$$

and $m\big([a, a+1) \times [c, c+1)\big) = 1$ by assumption. These equalities and (16) imply (15). Finally, if p and q are arbitrary positive rationals, then (15) obviously gives

$$m\big([a, a+p) \times [c, c+q)\big) = pq,$$

which proves the Lemma.

3. Proof of Theorems 1 and 2.

3.1. Let $n = 2$ and let m be a non-negative and additive set function defined on J_2 for which (iii)* holds with $n = 2$. By the definition of Jordan measure and by the monotonicity of m it is enough to prove that for every interval $H = [a, b) \times [c, d)$ the equality $m(H) = (b-a)(d-c)$ holds.

Since $m \geqslant 0$, the Lemma is applicable and we get the countable sets X and Y. Let $\varepsilon > 0$ be arbitrary. Then there exist intervals

$$H' = [a', b') \times [c', d') \quad \text{and} \quad H'' = [a'', b'') \times [c'', d'')$$

such that $H' \subset H \subset H''$, $a', a'' \notin X$, $c', c'' \notin Y$, the differences $b'-a'$, $d'-c'$, $b''-a''$, and $d''-c''$ are rational, and

$$b''-a''-\varepsilon < b-a < b'-a'+\varepsilon, \qquad d''-c''-\varepsilon < d-c < d'-c'+\varepsilon.$$

By (1),

$$(b'-a')(d'-c') \leqslant m(H) \leqslant (b''-a'')(d''-c'')$$

and, consequently,

$$(b-a-\varepsilon)(d-c-\varepsilon) < m(H) < (b-a+\varepsilon)(d-c+\varepsilon).$$

Since ε is arbitrary, we have $m(H) = (b-a)(d-c)$, which proves Theorem 1 for $n = 2$.

3.2. Let m be a σ-additive set function on B_2 and suppose that $m(Q) = 1$ for every unit square Q. By the σ-additivity of m and λ_2, it is enough to prove that $m(X) = \lambda_2(X)$ for any bounded $X \in B_2$.

m is bounded from below, since it is bounded either from above or from below (this is true for every σ-additive set function defined on a σ-field; see [3], 3.3.2, p. 17) and, obviously, $m(R^2) = \infty$. Hence, by the Lemma, there are countable sets $X, Y \subset R^1$ such that $a \notin X$ and $c \notin Y$ imply

$$m\big([a, a+p) \times [c, c+q)\big) = pq$$

for all positive rationals p and q.

Fix a bounded open set $U \subset R^2$ and put

$$E = \{X \in B_2 \colon X \subset U \text{ and } m(X) = \lambda_2(X)\}.$$

As m is σ-additive and bounded on the σ-ring $\{X \in B_2 \colon X \subset U\}$ (cf. 2.1), E is a monotone class. Hence, in order to prove that $E = \{X \in B_2 \colon X \subset U\}$, it is enough to show that E contains any open set $G \subset U$. As easily seen, G can be represented as a countable disjoint union of intervals $[a, b) \times [c, d)$ with $a \notin X$, $c \notin Y$, and $b-a$, $d-c$ rational. It follows that $G \in E$, and Theorem 2 is proved for $n = 2$.

3.3. We prove Theorems 1 and 2 for every n by induction. Let $n > 2$ and assume that Theorems 1 and 2 are true for $n-1$. Sets of the form

$a\left(\bigtimes_{j=1}^{n}[0,1)\right)$, where a is a rigid motion of R^n, will be called *unit cubes of dimension* n.

It is easy to see that the set function $m\big(a([0,1)\times X)\big)$ satisfies the conditions of Theorems 1 and 2. Hence, by the inductive hypothesis,

$$m\big(a([0,1)\times X)\big) = \iota_{n-1}(X)$$

for any $X \in J_{n-1}\cap B_{n-1}$ and any rigid motion a of R^n.

Given $a, b \in R^1$ with $a < b$ and a unit cube Q of dimension $n-1$, there is a such that

$$a\left(\bigtimes_{j=1}^{n-1}[0,1)\times[a,b)\right) = Q\times[a,b).$$

It follows that

$$m\big(Q\times[a,b)\big) = \iota_{n-1}\Big[\bigtimes_{j=2}^{n-1}[0,1)\times[a,b)\Big] = b-a.$$

Hence, applying again the inductive hypothesis for the set function $(b-a)^{-1}m\big(X\times[a,b)\big)$, we get

$$m\big(X\times[a,b)\big) = (b-a)\,\iota_{n-1}(X) \quad \text{for every } X\in J_{n-1}\cap B_{n-1}.$$

In particular, for every interval

$$H = \bigtimes_{j=1}^{n}[a_j, b_j)$$

we have

$$(17) \qquad m(H) = \prod_{j=1}^{n}(b_j-a_j).$$

Now, if m is non-negative and additive on J_n, then (17), obviously, gives Theorem 1. Finally, if m is σ-additive on B_n, then the assertion of Theorem 2 follows from (17) by an argument similar to that given in 3.2.

4. Remarks.

4.1. The theorems are not valid for $n = 1$. In fact, let $f(x)$ be a continuous and increasing function for which $f(x+1) \equiv f(x)+1$. Then for the Lebesgue-Stieltjes measure λ_f generated by f we have

$$\lambda_f\big([a, a+1)\big) = f(a+1)-f(a) = 1$$

(i.e., every unit interval is of measure 1) but if $f(x)$ is not of the form $x+c$, then λ_f is not equal to ι_1 on J_1.

4.2. The question arises for what families $\mathcal{Q}$ of unit cubes of dimension n it is true that if m is a non-negative and additive set function on J_n $(n \geqslant 2)$ and $m(Q) = 1$ for every $Q \in \mathcal{Q}$, then $m = \iota_n$. It follows from the proof of the Lemma that there exists a countable $\mathcal{Q}$ for which the statement above is true.

On the other hand, if m is the direct product of the above-mentioned λ_f and λ_{n-1}, then m is σ-additive on L_n and

$$m\Big(\bigtimes_{i=1}^{n} [a_i, a_i+1)\Big) = 1$$

for every $a_1, a_2, \ldots, a_n$ but $m \neq \lambda_n$.

4.3. Theorems 1, 2 and 3 remain true if condition (iii)* is replaced by the following:

(iii)** *If a is an arbitrary rigid motion of R^n, then*

$$m\Big(a\Big(\bigtimes_{i=1}^{n} [0, 1]\Big)\Big) = 1.$$

In fact, in the proof of the Lemma it would be enough to use $m(Q) = 1$ for unit squares of continuity only, and it is known that the measure of every interval of continuity equals that of its closure.

4.4. In the following we show that Theorem 1 fails to remain valid if we drop the condition $m \geqslant 0$.

Denote by $\mathscr{R}_n$ the linear space generated by the characteristic functions of Jordan measurable sets, and by Q_n its subspace generated by the characteristic functions of unit cubes of dimension n. A standard Hamel basis argument shows that there exists a linear functional $L\colon \mathscr{R}_n \to R^1$ such that $L(f) = 0$ for all $f \in Q_n$ and $L(\chi_A) = 1$ for some $A \in J_n$. (Moreover, we can choose L to be bounded in the linear space $\mathscr{R}_n$ endowed with the supremum norm $\|\cdot\|_\infty$. This is a consequence of the fact that $\|\chi_A - f\|_\infty \geqslant \frac{1}{2}$ whenever $f \in Q_n$ and $A \in J_n$ does not belong to the ring generated by the family of unit cubes of dimension n.)

Let $m(X) = \iota_n(X) + L(\chi_X)$ for every $X \in J_n$. Then m is additive on J_n and $m(Q) = 1$ for any unit cube Q, since $\chi_Q \in Q_n$, and thus $L(\chi_Q) = 0$. Nevertheless, $m \neq \iota_n$. If L is chosen to be bounded, then m is bounded from below and, as it easy to see, is of bounded variation on every set $X \in J_n$.

4.5. Theorem 3 fails to remain true if we suppose m to be only additive (but we retain conditions $m \geqslant 0$ and $m(Q) = 1$ for unit cubes).

In fact, let J_n' denote the field generated by J_n and let

$$m(X) = \begin{cases} \iota_n(X) & \text{if } X \in J_n, \\ \infty & \text{if } R^n \setminus X \in J_n. \end{cases}$$

Then m is non-negative and additive on J_n'. Hence, by the theorem of Łoś and Marczewski ([5], Theorem 2), for every c, $0 \leqslant c \leqslant 1$, there exists a non-negative and additive set function m' defined on all subsets of R^n for which $m'(X) = m(X)$ if $X \in J_n'$ (consequently, $m(Q) = 1$ for any unit cube Q) and $m(I) = c$, where I denotes the set of rational points of the unit cube $\mathop{\mathsf{X}}\limits_{i=1}^{n} [0, 1)$.

4.6. In this section we deal with the following generalization of Theorem 3:

PROPOSITION. *Let m be a σ-additive set function defined on L_n ($n \geqslant 2$) for which* (iii)* *holds. Then $m = \lambda_n$.*

We show that this proposition is consistent with the axioms of set theory (ZFC).

THEOREM 4. *Let $n \geqslant 2$ and let $2^{[0,1]}$ denote the σ-field of all subsets of $[0, 1]$. The following statements are equivalent:*

(1) *There exists a measure μ defined on $2^{[0,1]}$ such that $\mu(X) = \lambda_1(X)$ for every $X \in L_1$, $X \subset [0, 1]$.*

(2) *There exists a σ-additive set function m defined on L_n such that* (iii)* *holds and $m \neq \lambda_n$.*

Proof. Let μ be a measure satisfying the conditions of (1). μ cannot be invariant under translations (this can be shown by the argument of [4], p. 93). Hence there exists $h \in [0, 1]$ such that for the measure μ_h defined by

$$\mu_h(X) = \mu\big(([0, 1-h]\cap X)+h\big)+\mu\big(([1-h, 1]\cap X)+h-1\big)$$

we have $\mu_h \not\equiv \mu$. On the other hand, $\mu_h(X) = \mu(X)$ for every $X \subset [0, 1]$ $X \in L_1$.

Let $H \subset [0, 1]$ be such that $\mu_h(H) \neq \mu(H)$ and let

$$m(X) = \lambda_n(X)+\mu_h(X\cap[0, 1])-\mu(X\cap[0, 1])$$

for every $X \in L_n$. Then $H \in L_n$, m is σ-additive on L_n and $m(H) = \mu_h(H) - \mu(H) \neq 0 = \lambda_n(H)$. In addition, for every Q we have $Q\cap[0, 1] \in L_1$, and hence $m(Q) = \lambda_n(Q) = 1$, i.e. (iii)* holds. This proves (2).

Now we suppose (2) and prove (1), Let m be a σ-additive set function defined on L_n and satisfying (iii)* and let $H \in L_n$ be such that $m(H) \neq \lambda_n(H)$. Let $\{Q_i\}$ be a sequence of disjoint unit cubes of dimension n with

$$\bigcup_{i=1}^{\infty} Q_i = R^n$$

and let Q_i be such that $m(H\cap Q_i) \neq \lambda_n(H\cap Q_i)$.

Since $H\cap Q_i \in L_n$, we have $H\cap Q_i = A\cup B$, where $\lambda_n(A) = 0$ and $B \in B_n$. By Theorem 2, $m(B) = \lambda_n(B)$ and thus $m(A) \neq \lambda_n(A) = 0$, and we can suppose $m(A) > 0$. The subsets of A belong to L_n since $\lambda_n(A) = 0$; hence m is defined on every subset of A. m is finite-valued on the subsets of A, since $A \subset Q_i$ and $m(Q_i) = 1$ by assumption. In addition, for every $x \in A$ we have $m(\{x\}) = \lambda_n(\{x\}) = 0$ by Theorem 2.

It follows that there exists a probability measure ν on $2^{[0,1]}$ vanishing at points. Put $f(x) = \nu([0, x])$ for $x \in [0, 1]$. As easily seen, $\lambda_1(X)$

$= \nu\big(f^{-1}(X)\big)$ for $X \in L_1$, $X \subset [0,1]$ (cf. [6], p. 56, $(^{**}_{*})$). Hence $\mu(X) = \nu\big(f^{-1}(X)\big)$ for $X \subset [0,1]$ is as desired, and Theorem 4 is proved.

It is well known that if the cardinality of continuum is less than the first weakly inaccessible cardinal, then statement (1) in Theorem 4 does not hold (see [8], Satz (A)). That is, in this case our proposition is valid, and so it is consistent with the axioms of set theory (ZFC). On the other hand, if statement (1) in Theorem 4 is consistent with the axioms of ZFC, then so is the following statement: "there exists a measurable cardinal", and vice versa (see [7], Theorem 2, p. 398). It follows that the proof of the Proposition would yield a proof of the non-existence of measurable cardinals.

We remark that with the additional condition $m \ll \lambda_n$ (i.e. m is absolutely continuous with respect to λ_n) the Proposition follows immediately from Theorem 2. Another way of proving this assertion (for $n = 2$) is the following.

If $m \ll \lambda_2$, then, by the Radon-Nikodym theorem, there exists a measurable function $f\colon R^2 \to R^1$ such that

$$m(X) = \int_X f\, d\lambda_2 \quad \text{for every } X \in L_2.$$

Hence, by (iii)*,

$$\int_Q (f-1)\, d\lambda_2 = m(Q) - 1 = 0$$

for every unit square Q. If f is continuous, then, by a theorem of Christov [2], we have $f \equiv 1$. In the general case we can apply Christov's theorem to f convoluted with smooth functions with compact support and we get $f-1 = 0$ a.e. in R^2 (see [9], Sections 1 and 7). Hence

$$m(X) = \int_X 1\, d\lambda_2 = \lambda_2(X).$$

4.7. The interesting result belonging to this topic is Chakalov's theorem [1] by which there exist $a \neq 0$ and $b \neq 0$ such that

$$\iint_K \sin(ax+by)\, dx\, dy = 0$$

for every circle K with radius 1 (1). This implies that there are measures on J_2 for which $m(K) = \pi$ for every circle K with radius 1 and $m \neq \iota_2$. Indeed,

$$m(X) = \iota_2(X) + \iint_X \sin(ax+by)\, dx\, dy$$

is a measure on J_2, and $m(K) = \iota_2(K) = \pi$ whenever K is a circle with radius 1 but $m \neq \iota_2$.

(1) A similar example was earlier constructed by P. Szymański and W. Wolibner in Annales de la Société Polonaise de Mathématique 13 (1934), p. 134.

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