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TWO CONSTRUCTIONAL PROBLEMS IN ALIGNED SPACES

Abstract. In this note, aligned spaces are constructed in order to solve two problems posed at the 2nd Oklahoma Conference on Convexity and Related Combinatorial Geometry (1980). The first one is on the sharpness of the inequality $c \leqslant \max\{h, e-1\}$ with c the Carathéodory number, h the Helly number, and e the exchange number. The second problem deals with the relationship between the generalized Radon number and the generalized Helly number, and indicates that a proof of Eckhoff's conjecture is still far away.

1. Convexity spaces. A *convexity space* is a pair $(X, \mathscr{C})$, where X is a nonempty set and $\mathscr{C}$ is a family of subsets of X such that

(A-1) $\emptyset$, $X \in \mathscr{C}$,

(A-2) intersections of sets in $\mathscr{C}$ are again in $\mathscr{C}$.

The family $\mathscr{C}$ is called a *convexity structure* for X and the members of $\mathscr{C}$ are called *$\mathscr{C}$-convex sets*. The *$\mathscr{C}$-hull* of any set S in X, denoted by $\mathscr{C}(S)$, is the intersection of all $\mathscr{C}$-convex sets containing S.

A convexity space $(X, \mathscr{C})$ satisfying the additional axiom

(A-3) unions of upward directed families of sets in $\mathscr{C}$ are again in $\mathscr{C}$

is called an *aligned space* (see [3] and [4]). It is well known that any aligned space has the property that for each $S \subset X$

$$\mathscr{C}(S) = \bigcup \{\mathscr{C}(T)\colon\ T \subset S,\ |T| < \infty\}.$$

(Throughout the note, $|S|$ denotes the cardinality of the set S.) The classical example of an aligned space is $(\boldsymbol{R}^n, \mathrm{conv})$, where conv denotes the family of ordinary convex sets in $\boldsymbol{R}^n$.

Let $(X_1, \mathscr{C}_1)$ and $(X_2, \mathscr{C}_2)$, $X_1 \cap X_2 = \emptyset$, be two convexity (aligned) spaces. The pair $(X_1 \cup X_2, \mathscr{C}_u)$, where

$$\mathscr{C}_u = \{A \cup B\colon\ A \in \mathscr{C}_1,\ B \in \mathscr{C}_2\},$$

is also a convexity (aligned) space and is called the *convex* (*aligned*) *sum space*. Thus the $\mathscr{C}_u$-hull of any S in $X_1 \cup X_2$ is given by

$$\mathscr{C}_u(S) = \mathscr{C}_1(S \cap X_1) \cup \mathscr{C}_2(S \cap X_2).$$

The notion was introduced in [8].

Let $(X, \mathscr{C})$ be a convexity (aligned) space. In [3] it is shown that any *subspace* $(Y, \mathscr{C}_Y)$ with $Y \subset X$ and $\mathscr{C}_Y = \{A \cap Y\colon A \in \mathscr{C}\}$ is also a convexity (aligned) space.

2. Problem 1. We begin with recalling the following definitions.

The *Helly number* of a convexity space $(X, \mathscr{C})$ is the smallest integer h such that, for any finite family $\mathscr{F}$ of convex sets, if each h sets have nonempty intersection, then all the sets in $\mathscr{F}$ have nonempty intersection. The Helly number may be also defined (cf. [7]) to be the smallest integer h such that for any set A with $h+1$ elements the intersection $\bigcap\{\mathscr{C}(A\setminus a)\colon a \in A\}$ is nonempty.

The *Carathéodory number* of a convexity space $(X, \mathscr{C})$ is the smallest integer c such that for any S in X the following holds:

$$\mathscr{C}(S) = \bigcup\{\mathscr{C}(T)\colon T \subset S,\ |T| < c\}.$$

The *exchange number* of a convexity space $(X, \mathscr{C})$ is defined as the smallest integer e such that for each p and A in X with $e \leqslant |A| < \infty$ the following holds:

$$\mathscr{C}(A) \subset \bigcup\{\mathscr{C}(p \cup (A\setminus a))\colon a \in A\}.$$

The well-known relationship between the numbers h, c, and e in aligned spaces is the following inequality due to Sierksma [7]:

$$c \leqslant \max\{h, e-1\}.$$

Combining the so-called *aligned products* and sums Sierksma has shown in [8] the sharpness of this inequality in the case $h \leqslant e-1$. There he noted (and repeated in [9]) that the sharpness in the case $h > e-1$ is an open problem.

In this note we close the problem.

First we remark that it suffices only to consider the case $e > 1$, since $e = 1$ holds if and only if $\mathscr{C} = \{\emptyset, X\}$. We must show that for any $e \geqslant 2$ and $h > e-1$ there exists an aligned space with Carathéodory number $c = h = \max\{h, e-1\}$. To this end we now partition the set N of positive integers into the following subsets:

$$D_1 = \{1\},\quad D_2 = \{2, 3\},\quad D_3 = \{4, 5, 6\},\quad D_4 = \{7, 8, 9, 10\},\quad \ldots$$

Then we take the sequence of aligned spaces $(N, \mathscr{C}^t)$, $t \geqslant 1$, where

$$\mathscr{C}^1 = \{\emptyset, D_1, N\} \cup \{A\colon \exists i,\ i \neq 1,\ A \subsetneqq D_i\},$$

and

$$\mathscr{C}^t = \mathscr{C}^1 \cup \{B = D_1 \cup E\colon E \subsetneqq D_t\} \quad \text{for } t > 1.$$

PROPOSITION 1. *The exchange number of* $(N, \mathscr{C}^t)$ *is equal to* $t+1$.

Proof. The exchange number of $(N, \mathscr{C}^1)$ was determined in [7] and is equal to 2. To prove that $e = t+1$ in the remaining cases we now show that $e > t$. Indeed, for $p = 1$ and D_t we have

$$\mathscr{C}^t(D_t) = N,$$

but

$$\bigcup \{\mathscr{C}^t(p \cup (D_t \setminus q)) \colon q \in D_t\} = D_1 \cup D_t.$$

Thus, in fact, $e \geqslant t+1$. The argument similar to that in [7] gives $e = t+1$ and completes the proof.

PROPOSITION 2. *Let*

$$N_s = \{n \in N \colon 1 \leqslant n \leqslant \sum_{i=1}^{s-1} |D_i| + s\}.$$

The subspace $(N_s, \mathscr{C}_{N_s}^{t-1})$ *with* $2 \leqslant t \leqslant s$ *has the Helly number* $h = s$, *the exchange number* $e = t$, *and the Carathéodory number* $c = s$.

An easy proof is left to the reader.

Theorem 4 in [8] and Proposition 2 lead to the following

THEOREM. *The inequality* $c \leqslant \max\{h, e-1\}$ *is sharp.*

3. Problem 2. A *Radon* τ*-partition* of a set $S \subset X$ is a partition

$$S = S_1 \cup \ldots \cup S_\tau$$

into τ pairwise disjoint subsets such that

$$\bigcap \{\mathscr{C}(S_i) \colon i = 1, \ldots, \tau\} \neq \emptyset.$$

By $D_\tau(S)$ we denote the set of all points p in X for which there exists a Radon τ-partition $S_1, \ldots, S_\tau$ of S with

$$p \in \bigcap \{\mathscr{C}(S_i) \colon i = 1, \ldots, \tau\}.$$

Celebrated results of Radon and Tverberg (see [11]) imply that $D_2(S) \neq \emptyset$ and $D_\tau(S) \neq \emptyset$ for any set $S \subset \boldsymbol{R}^n$ of cardinality $n+2$ and $(\tau-1)(n+1)+1$, respectively. Note that $D_\tau(S)$ is not a convex set in general.

Following [1] we denote the τ*-core* of a set $S \subset X$ by

$$\operatorname{core}_\tau(S) = \bigcap \{\mathscr{C}(S \setminus M) \colon M \subset S, \ |M| \leqslant \tau\}.$$

It is easy to show that $D_\tau(S) \subset \operatorname{core}_{\tau-1}(S)$. The converse is true in special cases only (see [4]). However, the equality

$$\operatorname{conv}(D_\tau(S)) = \operatorname{core}_{\tau-1}(S) \tag{1}$$

holds for $(\boldsymbol{R}^2, \operatorname{conv})$, as shown in [5]. Reay [6], Sierksma [9] and others conjectured that (1) holds for every set S in $\boldsymbol{R}^n$ of cardinality $(\tau-1)(n+1)+1$. This conjecture is not solved yet.

The following problem was posed by Sierksma ([9], Problem 7):

PROBLEM. Find a convexity space $(X, \mathscr{C})$ such that for some $n_0 \geqslant 0$ and each $n \geqslant n_0$ there is a set S in X with $|S| = n$ and

$$\mathscr{C}(D_\tau(S)) \neq \mathrm{core}_{\tau-1}(S). \tag{2}$$

We shall show that the answer to Sierksma's problem is positive. We need the following

PROPOSITION 3. *Let $(X_1, \mathscr{C}_1)$ and $(X_2, \mathscr{C}_2)$ be convexity spaces with $X_1 \cap X_2 = \emptyset$. Then for any $S \subset X_1 \cup X_2$ we have*

(a) $D_\tau(S) = D_\tau(S \cap X_1) \cup D_\tau(S \cap X_2)$,

(b) $\mathrm{core}_\tau(S) = \mathrm{core}_\tau(S \cap X_1) \cup \mathrm{core}_\tau(S \cap X_2)$.

Proof. To show (a), take $p \in D_\tau(S)$. Then there exists a Radon τ-partition $S_1, \ldots, S_\tau$ of S such that

$$p \in \bigcap \{\mathscr{C}_*(S_i)\colon i = 1, \ldots, \tau\}.$$

Further, according to the definition of $\mathscr{C}_*$, we obtain

$$p \in \bigcap \{\mathscr{C}_1(S_i \cap X_1)\colon i = 1, \ldots, \tau\} \cup \bigcap \{\mathscr{C}_2(S_i \cap X_2)\colon i = 1, \ldots, \tau\}.$$

Assume that

$$p \in \bigcap \{\mathscr{C}_1(S_i \cap X_1)\colon i = 1, \ldots, \tau\}.$$

Hence $S_1 \cap X_1, \ldots, S_\tau \cap X_1$ is a Radon τ-partition of $S \cap X_1$ and, obviously, $p \in D_\tau(S \cap X_1)$. To prove the converse inclusion take $p \in D_\tau(S \cap X_2)$. There exists a Radon τ-partition $S_1, \ldots, S_\tau$ of $S \cap X_2$. Obviously, $S_1 \cup (S \cap X_1)$, $S_2, \ldots, S_\tau$ is a Radon τ-partition of S with

$$p \in \mathscr{C}_*(S_1 \cup (S \cap X_1)) \cap \mathscr{C}_*(S_2) \cap \ldots \cap \mathscr{C}_*(S).$$

This means that $p \in D_\tau(S)$, which completes the proof of (a). The proof of (b) is left to the reader.

Now we construct a suitable convexity (aligned) space. Take

$$A = \{(x, y) \in \boldsymbol{R}^2\colon y \geqslant x^2\}.$$

For *every integer* $k \in \boldsymbol{Z}$ let p_k denote the point $(k, k^2) \in \boldsymbol{R}^2$ and let A_k be the closed subset of A lying above the segment joining p_{k-1} and p_{k+1}. On the upper closed half-plane $X_1 = \{(x, y)\colon y \geqslant 0\}$ consider the family $\mathscr{C}$ of sets consisting of $\emptyset$, X_1, all singletons and all possible intersections of the sets A_k's. Clearly, $(X_1, \mathscr{C})$ is a convexity (aligned) space. We give the following examples of $\mathscr{C}$-hulls to make subsequent considerations much more clear:

$$\mathscr{C}(p_k) = p_k; \quad \mathscr{C}(A) = X_1;$$

for $k \neq 0$

$$\mathscr{C}(\{p_i, (k, k^2-1)\}) = X_1;$$

if $I \subset \mathbf{Z}$, $|I| \geqslant 2$, then

$$\mathscr{C}\left(\bigcup\{p_i:\ i\in I\}\right) = \bigcap\{A_k:\ k\in \mathbf{Z}\setminus I\}.$$

Futhermore, take the convex sum space of $(X_1, \mathscr{C})$ and $(\mathbf{R}^2\setminus X_1, \mathrm{conv})$. We show that $(\mathbf{R}^2, \mathscr{C}_u)$ is the desired space. Indeed, for a fixed $\tau \geqslant 2$ we put $n_0 = 2\tau - 1$. Now for each $n \geqslant n_0$ we consider a set S of $\mathbf{R}^2$ such that

$$S \cap X_1 = \{p_{-\tau+1}, \ldots, p_{-1}, p_0, p_1, \ldots, p_{\tau-1}\}$$

and

$$|S \cap (\mathbf{R}^2 \setminus X_1)| = n - 2\tau + 1.$$

It is easy to verify that

$$D_\tau(S \cap X_1) = \emptyset \quad \text{and} \quad \mathrm{core}_{\tau-1}(S \cap X_1) \neq \emptyset.$$

Using Proposition 3 we obtain

$$\mathscr{C}_u(D_\tau(S)) \neq \mathrm{core}_{\tau-1}(S).$$

This shows that the answer to Sierksma's problem is in the affirmative.

The *τ-Radon number* of a convexity space $(X, \mathscr{C})$ is the smallest integer $r(\tau)$ such that each set $S \subset X$ of cardinality at least $r(\tau)$ admits a Radon τ-partition. The well-known *conjecture of Eckhoff* [2] says that

$$r(\tau) \leqslant (r(2)-1)(\tau-1)+1.$$

There are at least two comments in order.

First, from our considerations it follows that even if (1) is shown to be true for $n_0 = (\tau-1)(n+1)+1$ in the case of $(\mathbf{R}^n, \mathrm{conv})$, there are general convexity spaces $(X, \mathscr{C})$ for which such an n_0 and equality (2) do not exist.

Second, if (1) were true, then in fact there would be a new and nice proof of Tverberg's Theorem ([11], [12]). As Eckhoff's conjecture is the "pendant" of Tverberg's Theorem in the general setting of convexity spaces, a similar proof, via $\mathscr{C}(D_\tau(S)) = \mathrm{core}_{\tau-1}(S)$, cannot exist. So the fact that Problem 2 is solved now makes Eckhoff's conjecture not less challenging (see also [10]).

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