ANNALES
POLONICI MATHEMATICI
XLIX (1988)

Propagation of weak singularities on characteristic surfaces of non-constant multiplicity

by PIOTR RYBKA (Warszawa)

Abstract. We discuss the propagation of weak singularities of quasi-linear non-elliptic first-order systems along focal curves on characteristic surfaces with non-constant multiplicity. Appropriate system of transport equations is constructed.

1. Introduction. Weak singularities of solutions of hyperbolic systems were investigated in [2], [9] and other papers, but only for characteristic surfaces of constant multiplicity. In the present paper, we propose a method of the construction of transport equations for an arbitrary non-elliptic quasi-linear partial differential system of first order, with which we treat the following question:

Given a solution with a non-zero singularity, does the singularity vanish at a point of multiplicity change of a characteristic surface if this takes place along a focal curve?

Preliminary results are contained in [10].

Related problems were considered for pseudodifferential operators (cf. [3], [5], [7]). In particular, it was shown that the intersection of bicharacteristics does not generate singularity of a solution. In this note we establish the analogical result for quasi-linear systems.

2. Transport equations. Consider a system

$$L(x, \partial)u = \sum_{i=1}^{m} A_i(x, u)\frac{\partial u}{\partial x_i} + B(x, u) = 0, \qquad m \geqslant 3, \tag{1}$$

where $u\colon D \to \boldsymbol{R}^l$, D is a region in $\boldsymbol{R}^m$, $A_i(x, u)$, $i = 1, \ldots, m$, are real-valued matrices, $B(x, u)$, $A_i(x, u)$ are C^2 functions. We assume that $u(x)$ has a weak discontinuity, i.e.,

$$u \in C(D) \cap C^2(\overline{D^+}) \cap C^2(\overline{D^-}), \tag{2}$$

where C is a characteristic surface, D, D^+, D^- are regions in $\boldsymbol{R}^m$ such that

$D = D^+ \cup C \cup D^-$, C is of class C^3, $n(x)$ is a normal vector to C at x, and $\partial u/\partial n$ has a jump across C.

Let

$$A(x, \xi) = \sum_{i=1}^{m} A_i(x, u(x))\xi_i, \qquad \xi = (\xi_1, \ldots, \xi_m), \ \xi \in \boldsymbol{R}^m.$$

$A(x, n(x))$ is called a *characteristic matrix*. Let

$$\Lambda(x, u) = \{\xi \in \boldsymbol{Q}^m\colon \ \det A(x, \xi) = 0\},$$

and assume $\Lambda(x, u) \neq \{0\}$; let

$$M_\xi(x) = \ker A^T(x, \xi), \qquad \Gamma_\xi(x) = \ker A(x, \xi).$$

Denote by $\left[\frac{\partial u}{\partial n}(x)\right]$ the jump of normal derivative of u on C. It is known that

$$\left[\frac{\partial u}{\partial n}(x)\right] \in \Gamma_{n(x)}(x).$$

Suppose that $n(x)$ belongs to $\Lambda_1(x)$, a subset of $\Lambda(x, u)$, that is a submanifold of $\boldsymbol{R}^m$ of codimension 1 given by

$$\Lambda_1(x) = \{\xi\colon \ \Phi(x, u(x), \xi) = 0\}.$$

Here $\Phi(x, u, \xi)$ is a twice continously differentiable function and $\operatorname{grad}_\xi \Phi(x, u(x), \xi) \neq 0$ on $\Lambda_1(x)$. Our characteristic surface C is foliated by focal curves defined by

$$\frac{dx}{dt} = \operatorname{grad}_\xi \Phi(x, u(x), n(x));$$

obviously, the right-hand side is continously differentiable.

We define the multiplicity of the characteristic surface C at x as a number $\dim M_{n(x)}(x) = \dim \Gamma_{n(x)}(x) = k(x)$.

Let U be an open set of C such that in U the multiplicity of C is constant and equal to k. It is easy to construct functions $\mu^i, \gamma^i\colon U \to \boldsymbol{R}^l$ so that $\mu^i, \gamma^i \in C^1(U, \boldsymbol{R}^l)$, $i = 1, \ldots, k$, $M_{n(x)}(x)$ is the span of $\mu^i(x)$, $\Gamma_{n(x)}(x)$ is the span of $\gamma^i(x)$ (under an additional condition we shall construct μ^i, γ^i with a special property later on). $\left[\frac{\partial u}{\partial n}(x)\right]$ belongs to $\Gamma_{n(x)}(x)$, so

$$\left[\frac{\partial u}{\partial n}(x)\right] = \sum_{i=1}^{k} \sigma_i(x)\gamma^i(x). \tag{3}$$

We have

$$\langle \mu^j(x), L(x, \partial)u\rangle = \sum_{p=1}^{l} \frac{\partial u}{\partial v^{jp}} + \langle \mu^j(x), B(x, u(x))\rangle,$$

where $v_i^{jp} = \langle \mu^j, a^{pi} \rangle$, a^{pi} is the p-th column of the matrix A_i, $p = 1, \ldots, l$. The vectors v^{jp} are tangent to C.

We extend $\mu^i(x)$, $\gamma^i(x)$, $i = 1, \ldots, k$, in a neighbourhood $V \subset \boldsymbol{R}^m$ of U so that they are differentiable. After having computed the jump of the normal derivative of $\langle \mu^j, L(x, \partial) u \rangle$, and taking into account formula (3) and the fact that for $w \in T_x C$ if $\|n(x)\| = 1$, then

$$\frac{\partial}{\partial w}\left[\frac{\partial u}{\partial n}\right] = \left[\frac{\partial^2 u}{\partial w \partial n}\right],$$

we obtain the following system of transport equations

$$\text{(4)} \qquad \sum_{j=1}^{k} \frac{\partial \sigma_j}{\partial w^{rj}} = -\sum_{j=1}^{k} S_{rj}(x)\sigma_j, \qquad r = 1, \ldots, k,$$

where $w^{rj} = (w_1^{rj}, \ldots, w_m^{rj})$, $w_i^{rj} = \langle \mu^r, A_i \gamma^j \rangle$ and

$$\text{(5)} \quad S_{rj}(x) = \sum_{i=1}^{m} \langle \mu^r, D_1 A_i(x, u(x))(n) n_i \gamma^i \rangle +$$

$$+ \sum_{i=1}^{m} \left\langle \mu^r, D_2 A_i(x, u(x))\left(\frac{\partial u^+}{\partial n}\right)\gamma^j \right\rangle - \sum_{i=1}^{m} \left\langle \mu^r, D_2 A_i(x, u(x))\left(\frac{\partial u^-}{\partial n}\right)\gamma^j \right\rangle +$$

$$+ \sum_{p=1}^{l} \frac{\partial \gamma_p^j}{\partial v^{rp}} + \langle \mu^r, D_2 B(x, u(x))\gamma^j \rangle,$$

$$\gamma^j = (\gamma_1^j, \ldots, \gamma_l^j), \qquad n(x) = (n_1, \ldots, n_m).$$

D_1, D_2 denote the partial derivatives with respect to x and u, respectively.

Under our assumption on $A_1(x)$ we obtain (cf. [2]) that w^{rj} and $\operatorname{grad}_\xi \Phi(x, u(x), n(x))$ are parallel:

$$w^{rj} = h_{rj} \frac{\operatorname{grad}_\xi \Phi}{\|\operatorname{grad}_\xi \Phi\|};$$

hence (4) takes the form

$$\text{(6)} \qquad H\dot{\sigma} = -S\sigma,$$

where $H = (h_{rj})$, $S = (S_{rj})$, and the dot denotes the derivative with respect to arc length. We see that singularities propagate in the bicharacteristic direction along focal curves.

3. Weak discontinuity and change of multiplicity. We are interested in the characteristic surface on which multiplicity changes in the following way. There exists a submanifold X of C, of codimension $\geqslant 1$, such that for $x \in C \setminus X$ the multiplicity $k(x)$ is constant and equal to k, and for $x \in X$, $k(x)$ is greater than k. If B is focal curve, then $B \cap X = \{x_0\}$, i.e., the multiplicity changes along B. A simple example of a system for which focal curves have this property will be given in Section 4. Conversely, for some solutions of the system as below, the multiplicity may be constant along focal curves and X contains focal curves. In this case, transport equations do not give any information about propagation of a weak singularity. The system in question reads

$$\frac{\partial p}{\partial x}-k\left(\frac{\partial \vartheta}{\partial x}\cos 2\vartheta+\frac{\partial}{\partial y}\sin 2\vartheta\right)-\varrho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=0,$$

$$\frac{\partial p}{\partial y}-k\left(\frac{\partial \vartheta}{\partial x}\sin 2\vartheta-\frac{\partial \vartheta}{\partial y}\cos 2\vartheta\right)-\varrho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=0,$$

$$\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\sin 2\vartheta+\left(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)\cos 2\vartheta=0, \qquad \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0.$$

It describes the perfectly plastic flow (von Mises model; cf. [8]).

Let $x(t)$, $t\in|t_1, t_2|$, be a C^2 parametrization of B, $x_0 = x(t_0)$. Set $T(x) = A(x, n(x))$. Assume that for $x\in C$ the matrix $T(x)$ is normal. We will use the theory of perturbations of linear operators (cf. [6]) applied to $T(x)$ regarded as an operator in C^l. Let $P(t)$ denote the orthogonal projection onto $\Gamma_{n(x(t))}x(t) = \ker T(x(t))$. Then

$$P(t) = \frac{-1}{2\pi i}\int_\gamma R(t, \zeta)\,d\zeta,$$

where $\gamma\subset C$ is a closed positively oriented curve, say a circle, in the resolvent set enclosing 0 but no other eigenvalue of $T(x(t))$, $R(t, \zeta) = \left(T(x(t))-\zeta\right)^{-1}$. It is known that $P(t)$ is continously differentiable in $(t_1, t_2)\setminus\{t_0\}$ (cf. [6]).

THEOREM. *Suppose the following conditions:*

(i) *the characteristic matrix $T(x)$ is normal;*

(ii) *there exists a submanifold X of C so that, for $x\in C\setminus X$, the multiplicity $k(x)$ is constant and equal to k, and for $x\in X$, $k(x)$ is greater than k;*

(iii) *$\frac{d}{dt}P(t)$ is integrable over (t_1, t_2);*

(iv) *there exists i_0, $1\leqslant i_0\leqslant m$, such that the operator $A_{i_0}\left(x(t), u(x(t))\right)$ restricted to $\operatorname{Im} P(t)$ is not singular, and, moreover, there exists $\varepsilon>0$ such that $\varrho(0, \sigma(PA_{i_0}P))\geqslant\varepsilon$ for $t\in(t_1, t_2)\setminus\{t_0\}$, where $\sigma(PA_{i_0}P)$ denotes the spectrum of $PA_{i_0}P$ considered in $\operatorname{Im} P$;*

(v) *$B\cap X = \{x(t_0)\}$, $\dot{x}(t_0)\notin T_{x(t_0)}X$.*

Then for the jump

$$j(t) = \left[\frac{\partial u}{\partial n}(x(t))\right]$$

we have:

If, for some $t^* \in (t_1, t_2)$, $j(t^*) = 0$, *then* $j(t) = 0$ *for all* $t \in (t_1, t_2)$.

Remark. Observe that condition (iii) imply that $P(t)$ is continous in (t_1, t_2). $j(t)$ is eigenvector of $T(x(t))$ corresponding to the eigenvalue 0. We require the continuity of $P(t)$ to ensure the existence of continous eigenvector of $T(x(t))$ corresponding to the eigenvalue 0, which should be non-zero at t_0.

Proof. Suppose for a moment that we have showed that H is invertible, H^{-1} has bounded norm, and S has integrable entries. Then (6) takes the form

$$\dot{\sigma} = -H^{-1} S\sigma$$

and has a unique solution by well-known result on ordinary differential equations (see [4]). The theorem follows.

The invertibility will be shown using a special construction of μ^i (see Lemma 1).

Now we construct an orthonormal basis $\gamma^i(x)$, $i = 1, \ldots, k$, of $\Gamma_n(x)$ such that $\gamma^i \in C^1(C \setminus X, R^l)$. Without loss of generality one may assume codim $X = 1$. It is known (cf. [6]) that if T is a normal operator in Hilbert space C^l, i.e., $T^* T = TT^*$, then P defined as

$$P = \frac{-1}{2\pi i} \int_\gamma R(\zeta)\, d\zeta,$$

where γ is a positively-oriented circle small radius r such that $\{z \in C\colon |z| < r\} \cap \sigma(T) = \{0\}$, is an orthogonal projection onto the kernel of T. We define an operator S_1 called the *reduced resolvent* of T at 0,

$$S_1 = \frac{1}{2\pi i} \int_\gamma \frac{R(\zeta)}{\zeta}\, d\zeta,$$

where γ is as described above. Since none of the eigenvalues of $T(x)$ coalesce to 0 for $x \in C \setminus X$, $P(x)$ and $S_1(x)$ are continuously differentiable in $C \setminus X$. We recall that $T(x)$ belongs to class C^2. Note that

$$\frac{\partial}{\partial x_i} P(x) = -P(x) \frac{\partial}{\partial x_i} T(x) S_1(x) - S_1(x) \frac{\partial}{\partial x_i} T(x) P(x),$$

which shows that $\dfrac{\partial^2}{\partial x_i\, \partial x_j} P(x)$ exist in $C \setminus X$ (cf. [6]).

For $x \in C$, we may write

$$x = \eta(t, y), \quad t \in (t_1, t_2),\ y \in X,$$

where $\eta(t, y)$ is a solution of the equation

$$\dot{\eta} = \frac{\operatorname{grad}_\xi \Phi(\eta, u(\eta), n(\eta))}{\|\operatorname{grad}_\xi \Phi(\eta, u(\eta), n(\eta))\|} \quad \text{and} \quad \eta(t_0) = y.$$

Obviously, $\eta(t, y)$ is continuously differentiable and

$$\frac{\partial^2}{\partial t\, \partial y_i} \eta(t, y), \quad \frac{\partial^2}{\partial y_i\, \partial t} \eta(t, y)$$

exist and are equal.

If $P(t)$ is an orthogonal projection continuously depending on $t \in (a, b)$, then $\dim \operatorname{Im} P(t) = \text{const} = k$. If $\varphi_1, \ldots, \varphi_k$ form a basis of $\operatorname{Im} P(t^*)$ for some $t^* \in (a, b)$, then $\varphi_i(t) = U(t)\varphi_i$, $i = 1, \ldots, k$, form a basis of $\operatorname{Im} P(t)$, $t \in (a, b)$, where $U(t)$ is a solution of the equation

$$\dot{Y} = Q(t)\, Y, \quad Y(t^*) = \mathrm{Id}; \tag{7}$$

$$Q(t) = \left[\frac{d}{dt} P(t), P(t)\right] = \frac{d}{dt} P(t) P(t) - P(t) \frac{d}{dt} P(t).$$

For each $t \in (a, b)$, $U(t)$ is a unitary operator, and if $\varphi_1, \ldots, \varphi_k$ are orthonormal, then $\varphi_1(t), \ldots, \varphi_k(t)$ are orthonormal too (see [6]). We shall employ (7) and the parametrization $x = \eta(t, y)$ to construct γ^i.

Let $(a, b) = (t_1, t_0)$; we modify equation (7) introducing dependence of $Q(t)$ on $y \in X$:

$$\dot{Y} = Q(t, y)\, Y, \quad Y(t^*) = \mathrm{Id}, \quad Q(t, y) = \left[\frac{\partial}{\partial t} P(\eta(t, y)), P(\eta(t, y))\right]; \tag{8}$$

obviously, $Q(t, y)$ is continuously differentiable. The solution $U(t, y)$ of this equation is a C^1 function. We set

$$\gamma^i(t, y) = U(t, y)\varphi_i,$$

where φ_i form an orthonormal basis of $\operatorname{Im} P(t^*) = \ker T(x(t^*))$, $t^* \in (t_1, t_0)$. The case of interval (t_0, t_2) is considered in the same way.

It is important that $\gamma^i(t, y)$ are solutions of equation (8).

In the following lemma we establish the existence of μ^i with a useful property.

LEMMA 1. *Given $\gamma^i(t, y)$, $i = 1, \ldots, k$, there exist $\mu^i(t, y)$, $i = 1, \ldots, k$, and $\delta > 0$ so that $\mu^i(t, y)$ form a basis of $M(t, y) = M_{n(\eta(t,y))}(\eta(t, y))$ and $\|\mu^i\| = 1$, $\langle \mu^i, A_{i_0}\gamma^j \rangle = c_j \delta_{ij}$, $c_j \geqslant \delta > 0$ for $y = y_0$, where $x(t_0) = \eta(t_0, y_0)$.*

Proof. Under assumption (iv) the vectors $\{PA_{i_0}\gamma^j\}_{j=1,\dots,k}$ are linearly independent. We set $\Gamma_i(t, y) = \operatorname{span}_{j \neq i}\{P(\eta(t, y)) A_{i_0}(\eta(t, y))\gamma^j(t, y)\}$. If μ belongs to $\Gamma_i^\perp \cap M(t, y)$ and $\mu \neq 0$, then $\langle \mu, PA_{i_0}\gamma^i\rangle \neq 0$. Let $\mu^i(t, y)$ be such that $\|\mu^i\| = 1$, $\mu^i(t, y) \in \Gamma_i^\perp \cap M(t, y)$, and $\langle \mu^i, PA_{i_0}\gamma^i\rangle > 0$; clearly, $\mu^i(t, y)$ are continuously differentiable and form a basis of $M(t, y)$. $\varrho(PA_{i_0}\gamma^i, \Gamma_i)$ denotes the distance between the vector $PA_{i_0}\gamma^i$ and the set Γ_i.

We have $c_i = \langle \mu^i, PA_{i_0}\gamma^i\rangle = \varrho(PA_{i_0}\gamma^i, \Gamma_i)$; hence

$$c_i = \sqrt{\frac{G(PA_{i_0}\gamma^j)_{j=1,\dots,k}}{G(PA_{i_0}\gamma^j)_{j=1,\dots,k,j\neq i}}} = \frac{\sqrt{G(PA_{i_0}\gamma^j)_{j=1,\dots,k}}}{\prod_{j\neq i}\|PA_{i_0}\gamma^j\|}.$$

If c_i converged to 0 for $t_n \to t_0$, then $\sqrt{G(PA_{i_0}\gamma^j)} = |\det PA_{i_0}P\gamma^j|$ would tend to 0, contrary to (iv). The proof of the lemma is complete.

The special construction of μ^i allows us to show that H is invertible. Indeed, observe that

$$w^{ij} = h_{ij}b, \qquad b = \operatorname{grad}_\xi \Phi(x, u(x), n(x))/\|\operatorname{grad}_\xi \Phi(x, u(x), n(x)\|.$$

Thus $w_{i_0}^{ij} = c_j\delta_{ij}$, $c_j \neq 0$, and therefore H is diagonal. Since h_{ii} are equal to $\varepsilon_i\|w^{ii}\|$, $\varepsilon_i = \pm 1$, we have $|\det H| \geq \delta^k$ and $\|H^{-1}\| \leq \delta^{-1}$. Now the transport equation takes the form

$$\dot{\sigma} = H^{-1}S\sigma.$$

We start investigating the integrability of S over (t_1, t_0). We observe that all terms in the expression defining S_{rj} are integrable, except possibly for

$$\partial\gamma_p^j/\partial v^{rp}. \tag{9}$$

We fix j, p, r. We observe that

$$v^{rp} = \langle v^{rp}, b\rangle b + (v^{rp} - \langle v^{rp}, b\rangle b) = \alpha(t, y)b(t, y) + v(t, y);$$

if $v \neq 0$ then the vectors v and b are linearly independent. We rewrite (9) as

$$\frac{\partial\gamma^j}{\partial(\alpha b + v)} = \alpha(t, y)\dot{\gamma}^j(t, y) + \frac{\partial\gamma^j}{\partial v},$$

where clearly $\alpha(t, y)$ is bounded, and

$$\alpha\dot{\gamma}^j(t, y) = \alpha(t, y)Q(t, y)\gamma^j(t, y).$$

By assumption (iii) it follows that $Q(t, y)$ is integrable over (t_1, t_2), thus $\alpha(t, y)\dot{\gamma}^j(t, y)$ is integrable too. We have to investigate $\partial\gamma^j/\partial v$. Consider the vector field $v(t, y)$ in $C \setminus X$, and the equation

$$\frac{d\eta}{ds} = v(\eta) \quad \text{and} \quad \eta(0, t) = x(t). \tag{10}$$

According to (v), for each $t \in (t_1, t_0)$ there exists a neighbourhood $U_t \subset (t_1, t_0)$, $t \in U_t$, and $\varepsilon_t > 0$ such that if $t' \in U_t$, then $\eta(s, t')$ is defined over $(-\varepsilon_t, \varepsilon_t)$. Moreover, $\partial^2\eta/\partial s\partial t$ and $\partial^2\eta/\partial t\partial s$ exist and are equal. Now it is easy to find $\varepsilon > 0$ and

$$\varphi\colon (t_1, t_0) \times (-\varepsilon, \varepsilon) \to C$$

such that φ is continuously differentiable, $\varphi(s, t) \notin \dot{X}$, and

$$\frac{\partial}{\partial s}\varphi(0, t) = v(x(t)), \qquad \frac{\partial}{\partial t}\varphi(0, t) = \dot{x}(t) = b(x(t)), \qquad \frac{\partial^2}{\partial t\partial s}\varphi = \frac{\partial^2}{\partial s\partial t}\varphi.$$

We see that

$$\frac{\partial\gamma^j(t, y_0)}{\partial v(t, y_0)} = \frac{\partial}{\partial s}f(0, t),$$

where $f(s, t) = \gamma^j(\varphi(s, t))$.

The idea of the solution is quite simple: we want to replace $f(s, t)$ by an analytic function $F(s, t)$ of s such that

$$F(0, t) = f(0, t), \qquad \frac{\partial}{\partial s}F(0, t) = \frac{\partial}{\partial s}f(0, t)$$

and $F(s, t)$ is integrable over (t_1, t_2). We would like to know if $\int_{t_1}^{t_0} F(s, t)\,dt$ is analytic and

$$\frac{\partial}{\partial s}\int_{t_1}^{t_0} F(s, t)\,dt = \int_{t_1}^{t_0}\frac{\partial}{\partial s}F(s, t)\,dt.$$

For this purpose we replace $T(\varphi(s, t))$ by an operator analytically depending on s. We set

$$T_0(s, t) = T(\varphi(s, t)), \qquad T_1(s, t) = T_0(0, t) + s\frac{\partial}{\partial s}T_0(0, t).$$

T_0, T_1 are continuously differentiable, $T_0(0, t) = T_1(0, t)$,

$$\frac{\partial}{\partial s}T_0(0, t) = \frac{\partial}{\partial s}T_1(0, t) \quad \text{and} \quad \frac{\partial^2}{\partial s\partial t}T_1(s, t) = \frac{\partial^2}{\partial t\partial s}T_1(s, t).$$

We denote by $P_0(s, t)$ the eigenprojection for the eigenvalue 0 of $T_0(s, t)$ and by $P_1(s, t)$ the total projection for 0-group of $T_1(s, t)$. One can prove that $P_1(s, t)$ depends analytically on s (cf. [6]). We have

$$P_0(0, t) = P_1(0, t) \quad \text{and} \quad \frac{\partial}{\partial s}P_0(0, t) = \frac{\partial}{\partial s}P_1(0, t).$$

LEMMA 2. *There exist functions* $\bar{\gamma}^i\colon \{|z| < \varepsilon\} \times (t_1, t_0) = \Omega \to C^l$, $i = 1, \ldots, k$, *such that* $\bar{\gamma}^i \in C^1(\Omega, C^l)$, $\bar{\gamma}^i(s, t)$ *for each* $t \in (t_1, t_0)$ *is analytic at* s, $\bar{\gamma}^i(0, t) = \gamma^i(0, t)$, $\bar{\gamma}^i$ *are bounded, and*

$$\frac{\partial}{\partial s}\bar{\gamma}^i(0, t) = \frac{\partial}{\partial s}\gamma^i(0, t).$$

Proof. γ^i are the solution of equation (7) and $P(t) = P_0(0, t)$,

$$\dot{P}(t) = \frac{\partial}{\partial t}P_0(0, t).$$

Consider the equation

(11) $\quad \dot{Y} = Q_1(s, t)Y, \qquad Y(s, t^*) = \mathrm{Id},$

$$\text{where } Q_1(s, t) = \left[\frac{\partial}{\partial t}P_1(s, t),\ P_1(s, t)\right]$$

is continuous at s, t and analytic at s. It is known that the solution $\bar{U}(s, t)$ of (11) analytically depends on s; $U(s, t)$ is a unitary operator. We set $\bar{\gamma}^i(s, t) = \bar{U}(s, t)\varphi_i$. Equality $\bar{U}(0, t) = U(\varphi(0, t))$ is obvious. Since

$$\frac{\partial}{\partial s}Q_1(0, t) = \frac{\partial}{\partial s}Q(0, t),$$

we have

$$\frac{\partial}{\partial s}\bar{U}(0, t) = \frac{\partial}{\partial s}U(\varphi(0, t)).$$

The proof of the lemma is complete.

We will prove now that there exists the integral

$$\int_{t_1}^{t_0}\frac{\partial\bar{\gamma}^i}{\partial s}(s, t)\,dt,$$

that is, that

$$\int_{t_1}^{t_0}\left|\frac{\partial\gamma^i}{\partial s}(x(t))\right|dt$$

is finite. We use the following known fact (cf. [1]).

LEMMA 3. *Let* V *be an open subset of* C *and let* $V \times [0, 1]^M(z, t) \to F(z, t) \in C$ *be a bounded function which is holomorphic at* z *for each* t *and is Riemann integrable at* t *for each* $z \in V$.

Then $G(z) = \int_0^1 F(z, t)\,dt$ *is holomorphic in* V, *and, for each positive integer* *i.* $D_1^{(j)} F(z, t)$ *is Riemann integrable for* $z \in V$ *and we have*

$$G^{(j)}(z) = \int_0^1 D_1^{(j)} F(z, t)\, dt.$$

We apply this lemma to $F(z, t) = \tilde{\gamma}_p(z, t)$. The proof of the theorem is now complete.

4. Example. We now exhibit a simple example of a multiplicity change along a focal curve for which all assumptions of our theorem hold. Consider

$$\sum_{i=1}^{3} A_i(x, u) \frac{\partial u}{\partial x_i} = 0,$$

where

$$A_1(x, u) = \begin{bmatrix} 0 & f(u)a(x) & 1 \\ f(u)a(x) & a(x) & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad A_2(x, u) = \begin{bmatrix} 0 & f(u) & 1 \\ f(u) & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

$$A_3(x, u) = \begin{bmatrix} 0 & f(u)a^2(x) & 1 \\ f(u)a^2(x) & a^2(x) & 0 \\ 1 & 0 & 0 \end{bmatrix};$$

f, a are real valued non-constant C^2 functions. Moreover,

$$A(x, \xi) = \begin{bmatrix} 0 & f(u)\Phi^- & \Phi^+ \\ f(u)\Phi^- & \Phi^- & 0 \\ \Phi^+ & 0 & 0 \end{bmatrix},$$

where $\Phi^-(x, u, \xi) = a(x)\xi_1 + \xi_2 + a_2 + a^2(x)\xi_3$, $\Phi^+(x, u, \xi) = \xi_1 + \xi_2 + \xi_3$, $\det A(x, \xi) = -\Phi^{+2}(x, u, \xi)\Phi^-(x, u, \xi)$; we have $\Lambda(x, u) = \Lambda^+(x) \cup \Lambda^-(x)$,

$$\Lambda^+(x) = \{\xi \in \boldsymbol{R}^3 \colon \Phi^+(x, u, \xi) = 0\}, \quad \Lambda^-(x) = \{\xi \in \boldsymbol{R}^3 \colon \Phi^-(x, u, \xi) = 0\}.$$

One computes

$$\Lambda^+(x) \cap \Lambda^-(x) = \{\xi \in \boldsymbol{R}^3 \colon \xi = t\big(1 + a(x), a(x), -1\big), t \in \boldsymbol{R}\}.$$

Let u satisfy $L(x, \partial)u = 0$. We will prove that there exist a characteristic surface C and its submanifold X such that condition (v) of our theorem is fulfilled. The normal vector $n(x)$ will belong to $\Lambda^+(x)$. C is constructed from the solution of the following system

$$\frac{dx}{dt} = \frac{\partial}{\partial \xi}\Phi^+\big(x, u(x), \xi\big), \quad \frac{d\xi}{dt} = -\frac{\partial}{\partial x}\Phi^+\big(x, u(x), \xi\big),$$

$$x(0, s) = l(s), \quad \xi(0, s) = \xi_0(s),$$

and

$$\frac{dl}{ds}(s) \times \xi_0(s) \in \Lambda^+ \cap \Lambda^- \ (l(s)).$$

We will compute a normal vector to C at $x(t, s)$:

$$n(x(t, s)) = \frac{\partial}{\partial s} x(t, s) \times \frac{\partial}{\partial t} x(t, s) / \left\| \frac{\partial}{\partial s} x(t, s) \times \frac{\partial}{\partial t} x(t, s) \right\|;$$

clearly,

$$\frac{\partial}{\partial t} x = \frac{\partial}{\partial \xi} \Phi^+.$$

We set $y = \partial x/\partial s$, $\eta = \partial \xi/\partial s$; it is known that (y, η) satisfies the system

$$\frac{dy}{dt} = \frac{\partial^2}{\partial x \partial \xi} \Phi^+ y + \frac{\partial^2}{\partial^2 \xi} \Phi^+,$$

$$\frac{d\eta}{dt} = -\frac{\partial^2}{\partial x^2} \Phi^+ y - \frac{\partial^2}{\partial x \partial \xi} \Phi^+,$$

$$y(0) = \frac{dl}{ds}(s), \qquad \eta(0) = \frac{d}{ds} \xi_0(s),$$

but $D^2 \Phi^+ = 0$; hence

$$y(t) = \frac{dl}{ds}(s).$$

Therefore, $n(x(t, s))$ is equal to $n(x(0, s))$. Let us compute multiplicity of C: if $t = 0$ then $n(x(t, s)) \in \Lambda^+ \cap \Lambda^- \ (l(s))$ and $k(x(0, s)) = 3$; if $t \neq 0$ then

$$n(x(t, s)) \in \Lambda^+ (l(s)) \quad \text{if} \quad da\left(\frac{\partial}{\partial \xi} \Phi^+\right) \neq 0,$$

and $k(x(t, s)) = 1$. Therefore, X is a curve $l(s)$ and assumption (v) holds. Let B be any bicharacteristic $x(t, s_0)$ (in the sense used in [2]); we set $t_0 = 0$; for $\xi \in \Lambda^+(x)$ we have $\ker A(x, \xi) = \boldsymbol{R}(0, 0, 1)$, eigenprojection is equal to $P(t) v = (0, 0, v_3)$ and (iii) holds. We set $i_0 = 2$ and see that

$$PA_2 Pv = (0, 0, v_3), \qquad \det PA_2 P = 1,$$

therefore (iv) holds too.

Acknowledgement. I would like to express my thanks to Professor Marek Burnat for his encouragement and valuable suggestions in writing this paper.

References

[1] R. B. Burckel, *An Introduction to Classical Complex Analysis*, vol. 1, Birkhauser Verlag, Basel 1979.

[2] R. Courant and D. Hilbert, *Methods of Mathematical Physics*, vol. II, Interscience, New York 1962.

[3] N. Hanges, *Propagation of singularities for a class of operators with double characteristics*, in: *Seminar on Singularities of Solution of Linear Partial Differential Equations*, L. Hörmander (ed.), Princeton University Press, Princeton 1979.

[4] Ph. Hartman, *Ordinary Differential Equations*, Wiley, New York 1964.

[5] V. Ya. Ivriĭ, *Wave front set of solutions of symmetric pseudodifferential systems*, Dokl. Akad. Nauk SSSR 233, 6 (1977), 1035–1038 (in Russian).

[6] T. Kato, *Perturbation Theory for Linear Operators*, Springer-Verlag, Berlin 1976.

[7] V. M. Petkov, *Propagation des singularités pour des systèmes hyperboliques à caractéristique de multiplicité variable*, C. R. Acad. Sci. 30 (1977), 183–185.

[8] W. Prager, *Introduction to Mechanics of Continua*, Ginn and Company, Boston 1961.

[9] B. L. Rozhdestvenskiĭ and N. N. Yanenko, *Quasi-linear Systems and its Application to Gas Dynamics*, Nauka, Moscow 1978 (in Russian).

[10] P. Rybka, *The behaviour of weak singularities on characteristic surfaces with multiplicity change*, Bull. Polish Acad. Sci. Math. 32 (1984), 675–679.

Reçu par la Rédaction le 09.02.1987