ANNALES
POLONICI MATHEMATICI
LII (1990)

Some qualitative problems in the theory of second order partial differential equations

by A. PELCZAR (Kraków)

Abstract. There are presented some results on asymptotic behaviour of solutions of certain second order partial differential equations when the time t is growing up. These results are of the two types: theorems extending the classical Ważewski's retract theorem and stability like statements.

Introduction. We shall consider partial differential equations of the type

$$\partial u/\partial t = f(t, x, u, \partial u/\partial x, \partial^2 u/\partial x^2), \tag{0}$$

where $t \geqslant 0$, x belongs to some (depending on t) interval in $\boldsymbol{R}$, with initial conditions of the type

$$u(0, x) = \varphi(x)$$

and – possibly – some further boundary conditions.

The purpose of the paper is to present some results on the asymptotic behaviour of solutions when the time t is growing up. These results will be of two kinds: theorems extending the classical Ważewski's retract theorem and statements of stability type. The idea, however, is the same in both cases; it is based on the topological method taken from papers [8] and [9]; the same approach has been applied in papers [1]–[3].

The main results of the present paper were presented during the VIIth International Conference on Nonlinear Analysis and Applications (August 1986) in Arlington (Texas, USA) and, in a short form, during ICM 86 (Berkeley) (see [0]).

Equations of type (0) have been investigated by several authors (see for instance Smoller [4] and papers and books quoted there).

We do not examine the existence and uniqueness of solutions of initial-boundary problems under investigation. We do not even discuss the question whether boundary conditions adjoined to the initial ones are necessary for uniqueness (or, on the contrary, are superfluous; this is the case, for instance, when the right-hand side of (0) does not depend on the last variable, which means that we have simply a first order equation). We shall simply assume that the solutions exist and are uniquely determined by the initial (and, possibly, boundary) conditions. In the case of a direct application of the Ważewski's topological method we shall also need the continuous dependence of solutions on the initial data. In this case, the solutions are assumed to be defined in sets depending on the initial data lower semicontinuously.

1. Let α and β be two real-valued functions defined and continuous on $[0, \infty)$ and such that

$$\alpha(t) < \beta(t) \quad \text{for } t \geqslant 0.$$

Let us put

$$\Omega := \{(t,x)\colon\ t \geqslant 0,\ \alpha(t) \leqslant x \leqslant \beta(t)\} \tag{1}$$

and assume that there is given an open subset Ω_0 of $\boldsymbol{R}^2$ such that $\Omega \subset \Omega_0$.

Suppose that

$$f^i\colon\ \Omega_0 \times \boldsymbol{R}^{3n} \to \boldsymbol{R}, \quad i = 1, \ldots, n, \tag{2}$$

are continuous functions. We shall consider the system of second order partial differential equations

$$u_t^i = f^i(t, x, u^1, \ldots, u^n, u_x^1, \ldots, u_x^n, u_{xx}^1, \ldots, u_{xx}^n), \quad i = 1, \ldots, n, \tag{3}$$

with initial-boundary conditions

$$u^i(0,x) = \varphi^i(x) \quad \text{for } x \in [\alpha(0), \beta(0)],\ i = 1, \ldots, n, \tag{4}$$

and

$$u^i(t, \alpha(t)) = \psi_0^i(t), \quad u^i(t, \beta(t)) = \psi_1^i(t) \quad \text{for } t \geqslant 0, \tag{5}$$

where φ^i, ψ_0^i, ψ_1^i are given functions, sufficiently regular, satisfying obvious compatibility conditions

$$\varphi^i(\alpha(0)) = \psi_0^i(0), \quad \varphi^i(\beta(0)) = \psi_1^i(0), \quad i = 1, \ldots, n. \tag{6}$$

Some additional conditions imposed on f^i, φ^i, ψ_0^i and ψ_1^i will be formulated in the sequel.

In order to exclude any misunderstanding we wish to establish precisely the definition of solution: we say that a vector function $u = (u^1, \ldots, u^n)$ is a *saturated solution of problem* (3)–(4)–(5) (with given functions α, β and φ^i, ψ_0^i, ψ_1^i) if and only if:

$u^1, \ldots, u^n$ are defined (or shortly: u is defined) in a set of the form

$$\Omega_0 \cap ([0,b) \times \boldsymbol{R}), \tag{7}$$

where b is either a positive number or infinity (in symbols: $b \in (0, \infty]$);

$u^1, \ldots, u^n$ are of the class C^1 in the set (7) (clearly, $u_t^i(0,x)$ denotes the right-hand derivative: $\lim[u^i(t,x)-u^i(0,x)]/t$ as $t \to 0$, $t > 0$) and moreover, $u^1,\ldots,u^n$ have their second order derivatives $u^1_{xx},\ldots,u^n_{xx}$ in the set

$$\{(t,x)\colon\ 0 \leqslant t \leqslant b,\ \alpha(t) < x < \beta(t)\}, \tag{8}$$

for every (t,x) belonging to the set (8), the equalities

$$\frac{\partial u^i}{\partial t}(t,x) = f^i\Big(t,x,u^1(t,x),\ldots,u^n(t,x),\frac{\partial u^1}{\partial x}(t,x),\ldots,\frac{\partial u^n}{\partial x}(t,x),\\ \frac{\partial^2 u^1}{\partial x^2}(t,x),\ldots,\frac{\partial^2 u^n}{\partial x^2}(t,x)\Big), \qquad i = 1,\ldots,n, \tag{9}$$

are satisfied, conditions (4) are fulfilled and we have

$$u^i(t,\alpha(t)) = \psi^i_0(t), \quad u^i(t,\beta(t)) = \psi^i_1(t) \quad \text{for } t \in [0,b),\ i = 1,\ldots,n; \tag{10}$$

b is maximal in the sense that if $b < \infty$, then it is impossible to find $c > b$ and $v = (v^1,\ldots,v^n)$ defined in (8) with b replaced by c, satisfying all conditions required above with respect to u (with b replaced by c), such that

$$v^i(t,x) = u^i(t,x) \quad \text{for } i = 1,\ldots,n,\ t \in [0,b),\ x \in [\alpha(t),\beta(t)]$$

(in other words: it is impossible to extend u over a set essentially larger than (8)).

If u is a saturated solution of (3)–(4)–(5), then we shall denote by $b[u]$ the number b which appears in the definition of the set (8), or we put $b[u] := \infty$ when the domain of existence of u covers the real half line $[0,\infty) \times \{0\}$.

Observe that we do not exclude here a case where the boundary conditions (5) are superfluous in the sense that the values of solutions of (3) on the curves $x = \alpha(t)$ and $x = \beta(t)$ are uniquely determined by the initial values for $t = 0$. Such a situation occurs if f^i do not depend on the last variables, that is if we in fact consider partial differential equations of the first order assuming conditions sufficient for the uniqueness of solutions of suitable Cauchy problems. There are well-known examples of such conditions in that special case of (3) which can be reduced to

$$u^i_t = g^i(t,x,u^1,\ldots,u^n,u^i_x), \quad i = 1,\ldots,n, \tag{11}$$

considered together with suitable initial conditions of type (4).

We refer to fundamental papers by Szarski [5] and Ważewski [6], [7].

2. Now assume that there are given differentiable functions

$$\lambda^i, \mu^i\colon\ [0,\infty) \to \boldsymbol{R}, \quad i = 1,\ldots,n,$$

(by derivatives at zero we mean right-hand derivatives) such that

$$\lambda^i(t) < \mu^i(t) \quad \text{for } t \geqslant 0,\ i = 1,\ldots,n. \tag{12}$$

We shall admit the following assumption.

ASSUMPTION H. There are two sets N_1 and N_2 of positive integers such that

(13) $$N_1 \cap N_2 = \emptyset \quad \text{and} \quad N_1 \cup N_2 = \{1, \ldots, n\}$$

(we do not exclude the cases $N_1 = \emptyset$ or $N_2 = \emptyset$) and, moreover,

(a) there are continuous functions

$$\xi_1^i, \xi_2^i, \eta_1^i, \eta_2^i \colon [0, \infty) \to \mathbf{R}, \qquad i = 1, \ldots, n,$$

and

$$r \colon [0, \infty) \to (0, \infty)$$

such that

for every $t \geqslant 0$ and every $s \in (0, r(t))$ we have:

$$\big(t+s, \alpha(t) + \xi_k^i(t)\, s\big) \in \Omega \quad \text{for } i \in N_1,\ k = 1, 2,$$

$$\big(t+s, \beta(t) + \eta_k^i(t)\, s\big) \in \Omega \quad \text{for } i \in N_1,\ k = 1, 2;$$

for every $t > 0$ and every $s \in (0, r(t))$:

$$\big(t-s, \alpha(t) - \xi_k^i(t)\, s\big) \in \Omega \quad \text{for } i \in N_2,\ k = 1, 2,$$

$$\big(t-s, \beta(t) - \eta_k^i(t)\, s\big) \in \Omega \quad \text{for } i \in N_2,\ k = 1, 2.$$

(b) For $j \in N_1$:

if $(t, x, u, v, w) = (t, x, u_1, \ldots, u_n, v_1, \ldots, v_n, w_1, \ldots, w_n) \in \Omega \times \mathbf{R}^{3n}$ and $u_j = \lambda^j(t)$, $v_j = 0$, $w_j \geqslant 0$, then

(14) $$f^j(t, x, u, v, w) < (\lambda^j)'(t);$$

if $(t, x, u, v, w) \in \Omega \times \mathbf{R}^{3n}$ and $u_j = \mu^j(t)$, $v_j = 0$, $w_j \leqslant 0$, then

(15) $$f^j(t, x, u, v, w) > (\mu^j)'(t);$$

if $(t, x, u, v, w) \in \Omega \times \mathbf{R}^{3n}$ and $x = \alpha(t)$, $u_j = \lambda^j(t)$, $v_j \geqslant 0$, then

(16) $$f^j(t, x, u, v, w) < (\lambda^j)'(t) - v_j \xi_1^j(t);$$

if $(t, x, u, v, w) \in \Omega \times \mathbf{R}^{3n}$ and $x = \beta(t)$, $u_j = \lambda^j(t)$, $v_j \leqslant 0$, then

(17) $$f^j(t, x, u, v, w) < (\lambda^j)'(t) - v_j \eta_1^j(t);$$

if $(t, x, u, v, w) \in \Omega \times \mathbf{R}^{3n}$ and $x = \alpha(t)$, $u_j = \mu^j(t)$, $v_j \leqslant 0$, then

(18) $$f^j(t, x, u, v, w) > (\mu^j)'(t) - v_j \xi_2^j(t);$$

if $(t, x, u, v, w) \in \Omega \times \mathbf{R}^{3n}$ and $x = \beta(t)$, $u_j = \mu^j(t)$, $v_j \geqslant 0$, then

(19) $$f^j(t, x, u, v, w) > (\mu^j)'(t) - v_j \eta_2^j(t).$$

(c) For $j \in N_2$, there are assumed analogous conditions with all inequalities reversed; e.g. instead of (14) we demand

$$f^j(t, x, u, v, w) > (\lambda^j)'(t)$$

for $j \in N_2$, $u_j = \lambda^j(t)$, $v_j = 0$, $w_j \geqslant 0$; instead of (16) we assume

$$f^j(t, x, u, v, w) > (\lambda^j)'(t) - v_j \xi_1^j(t)$$

for $j \in N_2$, $x = \alpha(t)$, $u_j = \lambda^j(t)$, $v_j \geqslant 0$, etc.

If $N_1 \neq \emptyset$, we may assume without loss of generality that $N_1 = \{1, \ldots, k\}$, $N_2 = \{k+1, \ldots, n\}$ (or $N_2 = \emptyset$ if $k = n$).

3. Suppose that there are families Φ^i of real-valued functions defined and differentiable in $[\alpha(0), \beta(0)]$ (by derivatives at $\alpha(0)$ and $\beta(0)$ we mean corresponding one-side derivatives). We assume that for $\varphi^i \in \Phi^i$

(20) $$\lambda^i(0) \leqslant \varphi^i(x) \leqslant \mu^i(0) \quad \text{for } x \in [\alpha(0), \beta(0)],\ i \in N_1,$$

and

(21) $$\lambda^i(0) < \varphi^i(x) < \mu^i(0) \quad \text{for } x \in [\alpha(0), \beta(0)],\ i \in N_2$$

and, moreover, for every i the family Φ^i is connected with respect to the usual topology induced in the space $C([\alpha(0), \beta(0)], \boldsymbol{R})$ of real continuous functions on $[\alpha(0), \beta(0)]$ by the classical maximum norm

$$\|\varphi^j\| := \max\{|\varphi^j(x)|\colon\ \alpha(0) \leqslant x \leqslant \beta(0)\}.$$

Suppose also that there are families Ψ_0^i, Ψ_1^i of real-valued functions such that, for $\psi_0^i \in \Psi_0^i$ and $\psi_1^i \in \Psi_1^i$,

(22) $$\lambda^i(0) \leqslant \psi_0^i(0) \leqslant \mu^i(0), \quad \lambda^i(0) \leqslant \psi_1^i(0) \leqslant \mu^i(0) \quad \text{if} \quad i \in N_1,$$

(23) $$\lambda^i(0) < \psi_0^i(0) < \mu^i(0), \quad \lambda^i(0) < \psi_1^i(0) < \mu^i(0) \quad \text{if} \quad i \in N_2,$$

and, moreover,

(24) if $\tilde{\psi}_0^i(0) = \psi_0^i(0)$ for $i = 1, \ldots, n$
or $\tilde{\psi}_1^i(0) = \psi_1^i(0)$ for $i = 1, \ldots, n$,
then

$$\tilde{\psi}_0^i(t) = \psi_0^i(t) \quad \text{and} \quad \tilde{\psi}_1^i(t) = \psi_1^i(t) \quad \text{for } i = 1, \ldots, n,\ t \geqslant 0,$$

(25) for every $\varphi = (\varphi^1, \ldots, \varphi^n) \in \Phi := \Phi^1 \times \ldots \times \Phi^n$ there exists a pair $(\psi_0, \psi_1) = ((\psi_0^1, \ldots, \psi_0^n), (\psi_1^1, \ldots, \psi_1^n))$ belonging to $\Psi_0 \times \Psi_1 := (\Psi_0^1 \times \ldots \times \Psi_0^n) \times (\Psi_1^1 \times \ldots \times \Psi_1^n)$ such that

$$\varphi^i(\alpha(0)) = \psi_0^i(0), \quad \varphi^i(\beta(0)) = \psi_1^i(0), \quad i = 1, \ldots, n.$$

Remark 1. In view of (24), for every $\varphi \in \Phi$ there exists exactly one pair (ψ_0, ψ_1) such that the equalities required in (25) are satisfied.

4. Consider a fixed positive number b. Let z be a real-valued function defined and continuous in the set (7) and let b^0 be a number belonging to the interval $[0, b)$. We say that the function z satisfies the *condition* $C_i[b^0; b]$ if and only if

$$\lambda^i(t) < z(t,x) < \mu^i(t) \quad \text{for } (t,x)\in\Omega\cap([0,b^0)\times \mathbf{R}) \tag{26}$$

and there are two connected subsets E_1 and E_2 of the set (7) such that for every $t\in[b^0, b)$ there exist $x, \tilde{x}\in[\alpha(t), \beta(t)]$ for which

$$(t,x)\in E_1 \quad \text{and} \quad (t,\tilde{x})\in E_2,$$

and, moreover,

$$z(s,w)\leqslant \lambda^i(s) \text{ for } (s,w)\in E_1 \quad \text{and} \quad z(s,y)\geqslant \mu^i(s) \text{ for } (s,y)\in E_2. \tag{27}$$

5. THEOREM 1. *Suppose that* f^i, λ^i, μ^i $(i = 1,\dots,n)$, α *and* β *are functions satisfying the conditions from Sections* 1 *and* 2, *and that* Φ^i, Ψ_0^i, Ψ_1^i *are the function families introduced in Section 3:*

Assume that for every $\varphi = (\varphi^1,\dots,\varphi^n)\in\Phi = \Phi^1\times \quad \times\Phi^n$ *there exists exactly one saturated solution* $u[\varphi] = (u^1[\varphi],\dots,u^n[\varphi])$ *of problem* (3)–(4)–(5), *with* ψ_0^i *and* ψ_1^i *uniquely determined by* φ^i (see (25) and Remark 1). *Assume that the mapping*

$$\varphi\to u[\varphi] \tag{28}$$

is continuous in the sense that for every $\varphi\in\Phi$, *every* $\varepsilon > 0$ *and every* $c\in[0, b(\varphi))$, where

$$b(\varphi) := b[u[\varphi]], \tag{29}$$

there exists $\sigma > 0$ *such that:*

if $\tilde{\varphi} = (\tilde{\varphi}^1,\dots,\tilde{\varphi}^n)\in\Phi$ *and* $|\varphi^i(x) - \tilde{\varphi}^i(x)| < \delta$ *for* $x\in[\alpha(0), \beta(0)]$, $i = 1,\dots,n$, *then*

$$c < b(\tilde{\varphi}) \tag{30}$$

and the inequalities

$$|u^i[\varphi](t,x) - u^i[\tilde{\varphi}](t,x)| < \varepsilon, \quad i = 1,\dots,n, \tag{31)}$$

are satisfied for $(t,x)\in\Omega\cap([0,c]\times\mathbf{R})$.

Finally assume the following condition:

(E) *for every* $i\in N_1$ *there exist* $\varphi, \tilde{\varphi}\in\Phi$ *such that there are* $t\in[0, b(\varphi))$, $\tilde{t}\in[0, b(\tilde{\varphi}))$ *for which*

$$\lambda^j(s) < u^j[\varphi](s,x) < \mu^j(s) \quad \text{for } s\in[0,t),\ j\in N_1, x\in[\alpha(s), \beta(s)], \tag{32}$$

$$\lambda^j(s) < u^j[\tilde{\varphi}](s,x) < \mu^j(s) \quad \text{for } s\in[0,\tilde{t}),\ j\in N_1, x\in[\alpha(s), \beta(s)], \tag{33}$$

and

$$\lambda^i(t) = u^i[\varphi](t,x), \quad u^i[\tilde{\varphi}](\tilde{t},\tilde{x}) = \mu^i(\tilde{t}) \tag{34}$$

for some $x\in[\alpha(t), \beta(t)]$ *and some* $\tilde{x}\in[\alpha(\tilde{t}), \beta(\tilde{t})]$.

Then the following alternative holds true:

(J) *there exists* $\varphi \in \Phi$ *such that*

$$\lambda^i(t) < u^i[\varphi](t,x) < \mu^i(t), \quad i = 1,\ldots,n, \tag{35}$$

for $t \in [0, b(\varphi))$, $x \in [\alpha(t), \beta(t)]$ *or*

(JJ) *there is a non-empty subset* M_1 *of* N_1 *such that for every* $j \in M_1$ *there exists* $\hat{\varphi} = (\hat{\varphi}^1, \ldots, \hat{\varphi}^n) \in \Phi$ *for which the j-th coordinate function* $\hat{u}^j$ *of the saturated solution* $\hat{u} := u[\hat{\varphi}]$ *of problem* (3)–(4)–(5) (with $\varphi = \hat{\varphi}$ and suitable functions ψ_0 and ψ_1 given by $\hat{\varphi}$) *satisfies the condition* C_j $[b^0, \hat{b}]$ *with some* $b_0 \in [0, \hat{b})$, *where* $\hat{b} = b[\hat{u}]$, *and for every* $k \in N_1 \backslash M_1$ *there exists* $\check{\varphi} = (\check{\varphi}^1, \ldots, \check{\varphi}^n) \in \Phi$ *for which the k-th coordinate function* $\check{u}^k$ *of the saturated solution* $\check{u} := u[\check{\varphi}]$ *of* (3)–(4)–(5) (with $\varphi = \check{\varphi}$ and suitable ψ_0 and ψ_1 given by $\check{\varphi}$) *satisfies the condition*

$$\lambda^k(t) < \check{u}^k(t,x) < \mu^k(t) \tag{36}$$

for $t \in [0, b[\check{u}])$, $x \in [\alpha(t), \beta(t)]$.

Moreover, the following condition (JJJ) *holds true:*

(JJJ) *for every* $m \in N_2$, *we have*

$$\lambda^m(t) < u^m[\varphi](t,x) < \mu^m(t) \tag{37}$$

for every $\varphi \in \Phi$, *every* $t \in [0, b(\varphi))$ *and* $x \in [\alpha(t), \beta(t)]$.

Proof (outline). We first prove (JJJ). Suppose the contrary. Then for every $\varphi \in \Phi$ and every $m \in N_2$, the set

$$\{t \in [0, b(\varphi)) \colon \lambda^m(s) < u^m[\varphi](s,x) < \mu^m(s) \text{ for } s \in [0,t),\ x \in [\alpha(s), \beta(s)]\} \tag{38}$$

(which is obviously not empty because of (23)) is bounded from above. Denote by t^0 its least upper bound (for fixed φ and $m \in N_2$).

We have $t^0 > 0$ and

$$\lambda^m(t^0) \leqslant u^m[\varphi](t^0,x) \leqslant \mu^m(t^0) \quad \text{for } x \in [\alpha(t^0), \beta(t^0)] \tag{39}$$

and for some $x_0 \in [\alpha(t^0), \beta(t^0)]$

$$u^m[\varphi](t^0, x_0) = \lambda^m(t^0) \tag{40}$$

or for some $x^0 \in [\alpha(t^0), \beta(t^0)]$

$$u^m[\varphi](t^0, x^0) = \mu^m(t^0). \tag{41}$$

We have the following six possible cases:

1. $x_0 = \alpha(t^0)$, 2. $x_0 \in (\alpha(t^0), \beta(t^0))$, 3. $x_0 = \beta(t^0)$, 4. $x^0 = \alpha(t^0)$,
5. $x^0 \in (\alpha(t^0),\ \beta(t^0))$, 6. $x^0 = \beta(t^0)$.

In case (40) the function

$$x \mapsto u^m[\varphi](t^0, x) \tag{42}$$

attains its local minimum at the point x_0, whereas in case (41) the function (42) has a local maximum at x^0. So we have for instance

$$u^m[\varphi](t^0, x_0) = \lambda^m(t^0), \quad u_x^m[\varphi](t^0, x_0) \geqslant 0 \quad \text{in case 1,}$$

$$u^m[\varphi](t^0, x_0) = \lambda^m(t^0), \quad u_x^m[\varphi](t^0, x_0) = 0$$

and

$$u_{xx}^m[\varphi](t^0, x_0) \geqslant 0 \quad \text{if case 2 occurs,}$$

etc., and finally

$$u^m[\varphi](t^0, x^0) = \mu^m(t^0), \quad u_x^m[\varphi](t^0, x^0) \geqslant 0 \quad \text{in case 6}$$

(here $u_x^m[\varphi]$ denotes, of course, the derivative $\frac{\partial}{\partial x} u^m[\varphi]$; the same for $u_{xx}^m[\varphi]$).

Now we apply Assumption H in order to get a contradiction in every case. If, for instance, case 1 occurs, then we get

$$f^m(t^0, x_0, u[\varphi](t^0, x_0), u_x[\varphi](t^0, x_0), u_{xx}[\varphi](t^0, x_0))$$
$$> (\lambda^m)'(t^0) - u_x^m[\varphi](t^0, x_0), (\xi_1^m t^0)$$

(where, of course, $u_x[\varphi] = (u_x^1[\varphi], \ldots, u_x^m[\varphi])$), because of condition (C).

This gives

$$u_t^m[\varphi](t^0, x_0) + u_x^m[\varphi](t^0, x_0)\, \xi_1^m(t^0) > (\lambda^m)'(t^0).$$

So the left-hand derivative of the function g defined in the interval $(-r(t^0), 0]$ (see condition (a) of Assumption H) by the formula

$$g(s) := u^m[\varphi](t^0 + s, \alpha(t^0) + \xi_1^m(t^0) s)$$

at the point $s = 0$ is strictly greater than the derivative of the function h defined by

$$h(s) := \lambda^m(t^0 + s)$$

at the same point $s = 0$. Thus, by virtue of the fact that $g(0) = h(0)$, we obtain the inequality

$$g(s) < h(s) \tag{43}$$

for s belonging to some interval $(-\delta, 0)$ (with some $\delta \in (0, r(t^0))$). This gives

$$u^m[\varphi](t, x) < \lambda^m(t) \tag{44}$$

for $t = t^0 + s$, $x = \alpha(t^0) + \xi_1^m(t^0) s$, $s \in (-\delta, 0)$, which contradicts the definition of t^0 (cf. (38)).

We can apply similar arguments in all cases (in cases 2 and 5 the situation is even simpler since we get inequalities between $u_t^m[\varphi]$ and $(\lambda^m)'$ or $(\mu^m)'$). In all cases we are led to a contradiction. The proof of (JJJ) is finished.

In order to prove the alternative (J) or (JJ), let us first assume that the set M_1 defined in condition (JJ) is empty. We claim that condition (J) holds true in that case. Suppose the contrary. So for every $\varphi \in \Phi$ there is a point (t, x) with $t \in [0, b(\varphi))$, $x \in [\alpha(t), \beta(t)]$ for which

$$u^i[\varphi](t,x) \leqslant \lambda^i(t) \tag{45}$$

for some $i \in \{1, \ldots, n\}$, or for some $i \in \{1, \ldots, n\}$:

$$u^i[\varphi](t,x) \geqslant \mu^i(t) \tag{46}$$

for some $t \in [0, b(\varphi))$, $x \in [\alpha(t), \beta(t)]$.

Observe that for given i we can have either (45) or (46). This follows from the assumption that $M_1 = \emptyset$. By virtue of (JJJ) we may assume that such an index i for which one of conditions (45) or (46) is satisfied belongs to N_1.

We now put for $\varphi \in \Phi$

$$t^0(\varphi) := \inf\{t \geqslant 0\colon \text{ there is } i \in N_1 \text{ such that (45) or (46) holds true for some } x \in [\alpha(t), \beta(t)]\}. \tag{47}$$

Obviously, for every $\varphi \in \Phi$ there is $i \in N_1$ such that

$$u^i[\varphi]\big(t^0(\varphi), y\big) = \lambda^i(t^0(\varphi)) \tag{48}$$

for some $y \in \big[\alpha\big(t^0(\varphi)\big), \beta\big(t^0(\varphi)\big)\big]$ or

$$u^i[\varphi]\big(t^0(\varphi), z\big) = \mu^i(t^0(\varphi)) \tag{49}$$

for some $z \in \big[\alpha(t^0(\varphi)), \beta\big(t^0(\varphi)\big)\big]$ and, moreover, for every j,

$$\lambda^j(t) < u^j[\varphi](t,x) < \mu^j(t) \tag{50}$$

for $0 \leqslant t < t^0(\varphi)$, $x \in [\alpha(t), \beta(t)]$.

LEMMA 1. *If for some $\varphi \in \Phi$, some $i \in N_1$ and a certain $t \geqslant 0$ we have*

$$u^i[\varphi](t,x) = \lambda^i(t) \quad \textit{for some } x \in [\alpha(t), \beta(t)] \tag{51}$$

(resp.

$$u^i[\varphi](t,x) = \mu^i(t) \quad \textit{for some } x \in [\alpha(t), \beta(t)]) \tag{52}$$

and for every $y \in [\alpha(t), \beta(t)]$

$$\lambda^i(t) \leqslant u^i[\varphi](t,y) \leqslant \mu^i(t), \tag{53}$$

then there exists a pair (s^*, ξ) of real numbers such that $s^* > 0$, the segment

$$\{(t+s, x+\xi s)\colon\ s \in [0, s^*)\} \tag{54}$$

is contained in $\Omega \cap \big([0, b(\varphi)) \times \boldsymbol{R}\big)$ and

$$u^i[\varphi](t+s, x+\xi s) < \lambda^i(t+s) \quad \textit{for } s \in (0, s^*) \tag{55}$$

(resp.

$$u^i[\varphi](t+s, x+\xi s) > \mu^i(t+s) \quad \textit{for } s \in (0, s^*)). \tag{56}$$

In order to prove this lemma we use Assumption H (condition (b)). We discuss six cases, as in the proof of (JJJ), but with opposite inequalities. If, for instance, (48) occurs with $x \in (\alpha(t), \beta(t))$, we apply inequality (14) and get

$$(57) \qquad u_t^i[\varphi](t,x) = f^i(t,x,u[\varphi](t,x), u_x[\varphi](t,x), u_{xx}[\varphi](t,x)) < (\lambda^i)'(t)$$

since $u_{xx}^i[\varphi](t,x) \geqslant 0$ and $u_x^i[\varphi](t,x) = 0$ (the function $u^i[\varphi](t,\cdot)$ attains its local minimum at the point x). From (57) we obtain (55) with $\xi = 0$. We omit further details.

Let us come back to the main line of our proof. We shall define a mapping which associates with every $\varphi \in \Phi$ a finite sequence of integers.

Let $\varphi \in \Phi$ be fixed. We put

$$(58) \qquad q(\varphi) := (q_1(\varphi), \ldots, q_k(\varphi)),$$

where

$$q_i(\varphi) = -1 \quad \text{iff (48) occurs;}$$

$$q_i(\varphi) = 0 \quad \text{iff for every } y \in [\alpha(t^0(\varphi)), \beta(t^0(\varphi))]:$$
$$\lambda^i(t^0(\varphi)) < u^i[\varphi](t^0(\varphi), y) < \mu^i(t^0(\varphi));$$

$$q_i(\varphi) = 1 \quad \text{iff (49) occurs.}$$

LEMMA 2. *The mapping*

$$q\colon \Phi \to R^k$$

is continuous.

Proof. It is enough to observe that the continuity of the mapping (28) implies the following statement: if, for some φ and some $i \in N_1$, condition (48) (resp. (49)) is satisfied, then there exists $\delta > 0$ such that for every $\tilde{\varphi} \in \Phi$ satisfying the condition

$$|\varphi^i(y) - \tilde{\varphi}^i(y)| < \delta \quad \text{for } i = 1, \ldots, n, \ y \in [\alpha(0), \beta(0)],$$

we have

$$u^i[\tilde{\varphi}](t^0(\tilde{\varphi}), x) = \lambda^i(t^0(\tilde{\varphi}))$$

(resp.

$$u^i[\tilde{\varphi}](t^0(\tilde{\varphi}), x) = \mu^i(t^0(\tilde{\varphi}))),$$

for some $x \in [\alpha(t^0(\tilde{\varphi})), \beta(t^0(\tilde{\varphi}))]$.

We get this from Lemma 1 (by virtue of the fact that $M_1 = \emptyset$). This ends the proof of Lemma 2.

Observe that $q(\varphi)$ belongs (for every $\varphi \in \Phi$) to the set

$$(59) \qquad Q := \{(i_1, \ldots, i_k)\colon i_j \in \{-1, 0, 1\}, \ j = 1, \ldots, k\}.$$

Clearly if P is a subset of Q having at least two distinct elements, then P cannot be the image of a connected set under a continuous mapping. Assumption (E) of the theorem implies that $P := q(\varphi)$ has more than one element. So we get a contradiction by virtue of Lemma 2, since Φ was supposed to be connected.

This contradiction finishes the proof of (J) if $M_1 = \emptyset$.

Now suppose that $M_1 \neq \emptyset$. We first prove that for every $j \in M_1$ the condition $C_j[b^0, b(\varphi)]$ is satisfied for $u^j[\hat{\varphi}]$ with some $\hat{\varphi} \in \Phi$. We omit details; they are presented in a similar situation in paper [3].

It is also not difficult to prove that for every $k \in N_1 \backslash M_1$ there is $\check{\varphi} \in \Phi$ such that (36) is fulfilled. Indeed, if we suppose the contrary, then we can define a mapping

$$\varphi \mapsto \tilde{q}_k(\varphi),$$

where $\tilde{q}_k(\varphi) = -1$ if

$$\lambda^k(t_k^0(\varphi)) = u^k[\varphi](t_k^0(\varphi), x)$$

and $\tilde{q}_k(\varphi) = 1$ if

$$u^k[\varphi](t_k^0(\varphi), x) = \mu^k(t_k^0(\varphi))$$

for some $x \in [\alpha(t_k^0(\varphi)), \beta(t_k^0(\varphi))]$, where

$$t_k^0(\varphi) := \inf\{t \geqslant 0\colon\ u^k[\varphi](t,x) \leqslant \lambda^k(t) \text{ for some } x \in [\alpha(t), \beta(t)]$$
$$\text{or } u^k[\varphi](t,x) \geqslant \mu^k(t) \text{ for some } x \in [\alpha(t), \beta(t)]\}.$$

The mapping $\tilde{q}$ is continuous and we obtain a contradiction similarly to the proof of (J) in the case $M_1 = \emptyset$. The proof of Theorem 1 is complete.

Remark 2. If α and β are constant functions, $\alpha(t) = c < d = \beta(t)$, then we can consider solutions defined in the set

$$[0, b) \times [c, d] \qquad (b \leqslant \infty) \tag{60}$$

and assume that they are of class C^1 in

$$[0, b) \times (c, d), \tag{61}$$

continuous in (60), and have the first order continuous derivatives with respect to t in (60), right-hand (left-hand) derivatives with respect to x at the points of the form (t, c) (resp. (t, d)), $t \geqslant 0$, and have second order continuous derivatives with respect to x in the set (61). In that case a natural assumption is that the functions ξ_k^i and η_k^i from condition (a) of Assumption H vanish identically.

EXAMPLE. Let $n = 1$, $\alpha(t) = c < d = \beta(t)$, $\lambda(t) = \lambda^0 < \mu^0 = \mu(t)$ (λ^0, μ^0 being constants),

$$f(t, x, u, v, w) = g(t, x, u) + h(t, x, u)v + k(t, x, u, v)w,$$

where g, h, k are continuous functions such that

$$g(t,x,\lambda^0) < 0 \quad \text{for } t \geqslant 0,\ x \in [c,d],$$
$$g(t,x,\mu^0) > 0 \quad \text{for } t \geqslant 0,\ x \in [c,d],$$
$$h(t,c,\lambda^0) \leqslant 0 \quad \text{for } t \geqslant 0,$$
$$h(t,c,\mu^0) \leqslant 0 \quad \text{for } t \geqslant 0,$$
$$h(t,d,\lambda^0) \geqslant 0 \quad \text{for } t \geqslant 0,$$
$$h(t,d,\mu^0) \geqslant 0 \quad \text{for } t \geqslant 0,$$
$$k(t,x,\lambda^0,0) \geqslant 0 \quad \text{for } t \geqslant 0,\ x \in (c,d),$$
$$k(t,x,\mu^0,0) \geqslant 0 \quad \text{for } t \geqslant 0,\ x \in (c,d),$$
$$k(t,c,\lambda^0,v) = k(t,d,\lambda^0,v) = k(t,c,\mu^0,v) = k(t,d,\mu^0,v) = 0.$$

In this case, Assumption H is satisfied with $N_1 = \{1\}$, $N_2 = \emptyset$ and $\xi_k, \eta_k = 0$, $k = 1,2$ (cf. Remark 2).

Remark 3. If we know that for every $\varphi \in \Phi$ the inequality

$$u_x^i[\varphi](t,x) < \mu^i(t) - \lambda^i(t) \tag{62}$$

holds for every $t \in [0, b(\varphi))$, $x \in [\alpha(t), \beta(t)]$, then under the assumptions of Theorem 1 we have only condition (J). Estimates of type (62) are possible if we reduce (3) to a system of form (11); conditions sufficient for estimates of $u_x^i[\varphi]$ can be found for instance in [5], [6], [7].

6. In the present section we shall use the method applied in [1] and [2] in the theory of first order partial differential equations.

Assume that there are two sets $\tilde{N}_1$ and $\tilde{N}_2$ of integers such that

$$\tilde{N}_1 \cap \tilde{N}_2 = \emptyset, \qquad \tilde{N}_1 \cup \tilde{N}_2 = \{1,\ldots,n\}.$$

They will play roles similar to those played by N_1 and N_2 in Assumption H. Without loss of generality we may assume that

$$\tilde{N}_1 = \{1,\ldots,k\}, \quad \text{and} \quad \tilde{N}_2 = \{k+1,\ldots,n\}$$

(or $N_2 = \emptyset$ and so $k = n$).

Now we assume that there are positive constants L_i, $i = 1,\ldots,k$ and continuous functions:

$$\beta_0, N_0^i, M_0^i\colon [0,\infty) \to (0,\infty), \qquad i = 1,\ldots,k,$$

such that

$$N_0^i(0) = \mu^i(0), \qquad i = 1,\ldots,k, \tag{63}$$

where μ^i ($i = 1,\ldots,n$) are functions introduced in Section 2.

Write for brevity

$$K_i := \mu^i(0), \quad i = 1,\dots,n,\ \gamma := \beta_0(0). \tag{64}$$

Let us also put

$$\beta_1^i(t) := \gamma\cdot\mu^i(t)\cdot(N_0^i(t))^{-1}, \quad i = 1,\dots,k, \tag{65}$$

and

$$\beta^i(t) := \min(\beta_1^i(t), \beta_0(t)) \quad t \geqslant 0,\ i = 1,\dots,k. \tag{66}$$

Finally we define

$$\Omega_i := \{(t,x)\colon\ t \geqslant 0,\ |x| \leqslant \beta^i(t)\}, \tag{67}$$

$$\Omega_* := \{(t,x)\colon\ t \geqslant 0,\ |x| \leqslant \beta_0(t)\}, \tag{68}$$

$$\beta(t) := \max(\beta^1(t),\dots,\beta^k(t),\beta_0(t)), \quad t \geqslant 0, \tag{69}$$

and

$$\alpha(t) := -\beta(t), \quad t \geqslant 0. \tag{70}$$

Suppose all the conditions imposed on f^i $(i = 1,\dots,n)$ in Section 1 are satisfied; now Ω_0 contains the set Ω defined by (1) with α and β given by (69) and (70).

We suppose that the following assumption, a modification of Assumption H from Section 2, is satisfied:

Assumption $\tilde{H}$.

($\tilde{a}$) There are continuous functions

$$\xi_1^i, \xi_2^i, \eta_1^i, \eta_2^i\colon\ [0,\infty)\to \boldsymbol{R}, \quad i = 1,\dots,n,$$

and

$$r\colon\ [0,\infty)\to(0,\infty)$$

such that:

For every $t \geqslant 0$ and every $s\in(0,r(t))$ we have

$$(t+s, -\beta^i(t)+\xi_k^i(t)s)\in\Omega_i \quad \text{for } i\in\tilde{N}_1,\ k = 1,2,$$
$$(t+s, \beta^i(t)+\eta_k^i(t)s)\in\Omega_i \quad \text{for } i\in\tilde{N}_1,\ k = 1,2.$$

For every $t > 0$ and every $s\in(0,r(t))$

$$(t-s, -\beta_0(t)+\xi_k^i(t)s)\in\Omega_* \quad \text{for } i\in\tilde{N}_2,\ k = 1,2,$$
$$(t-s, \beta_0(t)+\eta_k^i(t)s)\in\Omega_* \quad \text{for } i\in\tilde{N}_2,\ k = 1,2.$$

Conditions ($\tilde{b}$) and ($\tilde{c}$) are the same as (b) and (c) in Assumption H with only one change: Ω has to be replaced by Ω_j for $j\in\tilde{N}_1$ and by Ω_* if $j\in\tilde{N}_2$.

Let

$$\hat{\Phi} = \hat{\Phi}^1 \times \quad \times \hat{\Phi}^k$$

be the class of all vector valued C^1 functions $\hat{\varphi} = (\hat{\varphi}^1, \ldots, \hat{\varphi}^k)$ defined in $[-\gamma, \gamma]$ such that

(71) $$|\hat{\varphi}^i(x)| \leqslant K_i, \quad |(\hat{\varphi}^i)'(x)| \leqslant K_i \quad \text{for } i \in \tilde{N}_1,\ |x| \leqslant \gamma,$$

(72) $$|(\hat{\varphi}^i)'(x) - (\hat{\varphi}^i)'(\tilde{x})| \leqslant L_j|x - \tilde{x}| \quad \text{for } j \in \tilde{N}_1,\ x, \tilde{x} \in [-\gamma, \gamma].$$

We now assume that

(73) $$K_i \cdot \mu^i(t) \cdot M_0^i(t) \leqslant L_i(N_0^i(t))^2 \quad \text{for } t \geqslant 0,\ i = 1, \ldots, k,$$

LEMMA 3. *Let $t \geqslant 0$ be fixed and let functions*

$$z^i\colon [-\beta^i(t), \beta^i(t)] \to \boldsymbol{R}, \quad i = 1, \ldots, k,$$

be given. Assume that they are of class C^1 (here, as elsewhere, by derivatives at the endpoints we mean the corresponding one-sided derivatives) *and satisfy*:

(74) $$|z^i(x)| \leqslant \mu^i(t) \qquad \textit{for } i = 1, \ldots, k,\ |x| \leqslant \beta^i(t),$$

(75) $$|(z^i)'(x)| \leqslant N_0^i(t) \qquad \textit{for } i = 1, \ldots, k,\ |x| \leqslant \beta^i(t),$$

(76) $$|(z^i)'(x) - (z^i)'(\tilde{x})| \leqslant M_0^i(t)\,|x - \tilde{x}| \quad \textit{for } i = 1, \ldots, k,\ |x|, |\tilde{x}| \leqslant \beta^i(t).$$

Let functions $w^{i,z,t}$ be defined by the formulae

(77) $$w^{i,z,t}(x) := K_i \cdot (\mu^i(t))^{-1} z\left(\frac{\beta^i(t)}{\gamma} \cdot x\right), \quad x \in [-\gamma, \gamma],\ i = 1, \ldots, k.$$

Then

$$w^{z,t} := (w^{1,z,t}, \ldots, w^{k,z,t}) \in \Phi.$$

The proof of this Lemma consists in an elementary computation and simple estimations; details are given in [1], [2]. It is clear that the set $\hat{\Phi}$ is compact in the Banach space $C([-\gamma, \gamma], \boldsymbol{R}^k)$ of all continuous functions from $[-\gamma, \gamma]$ into $\boldsymbol{R}^k$, provided with the usual (maximum) norm inducing the topology of uniform convergence, as well as in the space $C^1([-\gamma, \gamma], \boldsymbol{R}^k)$ of C^1-functions provided with the topology of uniform convergence together with first order derivatives. It is also convex. So, it has the fixed point property by virtue of the classical Schauder fixed point theorem.

Let us now put

(78) $$\hat{\Phi}^* := \{\varphi \in \hat{\Phi}\colon \text{there are } x \in [-\gamma, \gamma] \text{ and } i \in \tilde{N}_1 \text{ such that } |\varphi^i(x)| = K_i\}.$$

LEMMA 4. *It is impossible to find a retraction $\hat{\Phi} \to \hat{\Phi}^*$, either in $C([-\gamma, \gamma], \boldsymbol{R}^k)$, or in $C^1([-\gamma, \gamma], \boldsymbol{R}^k)$.*

Proof. We repeat the classical reasoning (compare for instance [1] or [2]). Suppose that there is a retraction

$$\varrho\colon \hat{\Phi} \to \hat{\Phi}^*.$$

Then for any fixed point z of the mapping $-\varrho$ we must have: $z \in \hat{\Phi}^*$ and so $z = -\varrho(z) = -z$ which gives $z = 0$; however, this is impossible since $0 \notin \hat{\Phi}^*$.

THEOREM 2. *Suppose that f^i, μ^i $(i = 1,\dots,n)$, β^j $(j = 1,\dots,k)$, β_0 are functions satisfying conditions introduced above. Assume that $\Phi = \hat{\Phi} \times \check{\Phi}$, where $\hat{\Phi}$ is as above and $\check{\Phi} = \check{\Phi}^{k+1} \times \dots \times \check{\Phi}^n$ is a family of systems $\check{\varphi} = (\check{\varphi}^{k+1},\dots,\check{\varphi}^n)$ of C^1-functions defined in $[-\gamma,\gamma]$ such that*

$$(79) \qquad |\check{\varphi}^j(x)| \leqslant \mu^j(0), \quad j = k+1,\dots,n, \ |x| \leqslant \gamma.$$

Assume that $\Psi_0 = \Psi_0^1 \times \dots \times \Psi_0^n$ and $\Psi_1 = \Psi_1^1 \times \dots \times \Psi_1^n$ are families of systems $\psi_0 = (\psi_0^1,\dots,\psi_0^n)$, $\psi_1 = (\psi_1^1,\dots,\psi_1^n)$ of continuous functions defined in $[0,\infty)$ such that

$$|\psi_0^i(0)| \leqslant \mu^i(0), \quad |\psi_1^i(0)| \leqslant \mu^i(0), \quad i = 1,\dots,n;$$

If $\psi_0^i(0) = \tilde{\psi}_0^i(0)$, $i = 1,\dots,n$ $(\psi_1^i(0) = \tilde{\psi}_1^i(0)$, $i = 1,\dots,n)$, then $\psi_0^i = \tilde{\psi}_0^i$ for $i = 1,\dots,n$ $(\psi_1^i = \tilde{\psi}_1^i$, $i = 1,\dots,n)$.

For every $\varphi \in \Phi$ there exists exactly one pair $(\psi_0,\psi_1) \in \Psi_0 \times \Psi_1$ such that

$$\varphi(-\gamma) = \psi_0(0), \quad \varphi(\gamma) = \psi_1(0)$$

(cf. Section 3).

Assume that for every $\varphi \in \Phi$ there exists exactly one saturated solution $u[\varphi]$ of problem (3)–(4)–(5) with ψ_0^i and ψ_1^i uniquely determined by φ^i and that the mapping (28) is continuous in the sense as in the assumption of Theorem 1.

Then, for every $\check{\varphi} \in \check{\Phi}$ there exists $\hat{\varphi} \in \hat{\Phi}$ such that, for $\varphi = (\hat{\varphi}^1,\dots,\hat{\varphi}^k;$ $\check{\varphi}^{k+1},\dots,\check{\varphi}^n)$, the solution $u[\varphi]$ satisfies the conditions

$$(80) \qquad |u^i[\varphi](t,x)| < \mu^i(t)$$

for $i = 1,\dots,n$, $t \in [0, b(\varphi))$, $|x| \leqslant \beta^i(t)$ if $i \in \tilde{N}_1$ and $|x| \leqslant \beta_0(t)$ if $i \in \tilde{N}_2$.

Proof. Let $\check{\varphi} = (\check{\varphi}^{k+1},\dots,\check{\varphi}^n)$ be given. We have to show that there exists $\hat{\varphi} \in \hat{\Phi}$ such that the solution $u[\varphi]$ for $\varphi = (\hat{\varphi};\check{\varphi})$ satisfies (80). Suppose the contrary.

So we can define $t^0(\hat{\varphi})$ similarly to the proof of Theorem 1:

$$(81) \qquad t^0(\hat{\varphi}) := \inf\{t \geqslant 0\colon \text{there is } i \text{ such that } |u^i[\varphi](t,x)| = \mu^i(t) \text{ for some } |x| \leqslant \beta^i(t)\}.$$

Observe that for every $i \in \tilde{N}_2$ we have

$$|u^i[\varphi](t,x)| < \mu^i(t)$$

for each $t \geqslant 0$, $|x| \leqslant \beta_0(t)$; the proof of this statement is practically the same as the proof of (JJJ) in Theorem 1. So we can put in the definition of $t^0(\hat{\varphi})$: $i \in \tilde{N}_1$. Using the arguments presented in [1] and [2] (and applied implicitly in a simple situation in the proof of the statement (J) of Theorem 1 when $M_1 = \emptyset$) we prove that the mapping

$$(82) \qquad \hat{\varphi} \to w^{j,z,t^0},$$

where $z := u[\varphi]$ $(\varphi = \hat{\varphi};\check{\varphi}))$, $t^0 = t^0(\hat{\varphi})$, is for every $j \in \tilde{N}_1$ continuous in C^1-topology (and in C^0-topology as well). We omit details since it is enough to modify very slightly the reasoning given in [1], [2]. Now it is easy to observe that the mapping

$$\hat{\varphi} \to (w^{1,z,t^0}, \ldots, w^{k,z,t^0})$$

(see (82)) is a retraction of $\hat{\Phi}$ onto $\hat{\Phi}^*$; however, the existence of such a retraction is impossible (see Lemma 4). This contradiction finishes the proof.

Remark 4. We have considered above the C^0- and C^1-topologies. It would suffice to consider only one of them, yet we wished to underline that both of them are useful; the C^1-topology is of interest in the theory of first order equations since the estimates usually assumed to ensure existence and uniqueness theorem involve Lipschitz conditions for derivatives.

Remark 5. Theorem 1 generalizes a result presented in [1], [2].

7. The methods used above, especially in the proof of the statement (JJJ) of Theorem 1, can be also applied with respect to some stability questions.

Let us again consider problem (3)–(4)–(5) under the assumptions introduced in Section 1. Let Φ, Ψ_0, Ψ_1 be as in Section 3. Suppose that $\{\lambda_\varrho\}$ and $\{\mu_\varrho\}$, $\varrho > 0$, are families of differentiable vector-valued functions: $\lambda_\varrho = (\lambda_\varrho^1, \ldots, \lambda_\varrho^n)$, $\mu_\varrho = (\mu_\varrho^1, \ldots, \mu_\varrho^n)$ defined in $[0, \infty)$ and such that

$$\lambda_\varrho^i(t) < \lambda_\delta^i(t) < \mu_\delta^i < \mu_\varrho^i(t) \quad \text{for } 0 < \varrho < \delta \tag{83}$$

and for $t \geqslant 0$, $i = 1, \ldots, n$.

Assume, moreover, that for every $\varepsilon > 0$ there exists $\delta \in (0, \varepsilon)$ such that for $\lambda^i = \lambda_\delta^i$ and $\mu^i = \mu_\delta^i$ the conditions of Assumption H are satisfied with $N_1 = \emptyset$. This means, in particular, that for any $\varepsilon > 0$ there is $\delta > 0$ such that: if $(t, x, u, v, w) \in \Omega \times \mathbf{R}^{3n}$ and $u_j = \lambda_\delta^j(t)$, $v_j = 0$, $w_j \geqslant 0$, then

$$f^j(t, x, u, v, w) > (\lambda_\delta^j)'(t);$$

If $u_j = \mu_\delta^j(t)$, $v_j = 0$, $w_j \leqslant 0$, then

$$f^j(t, x, u, v, w) < (\mu_\delta^j)'(t);$$

If $x = \alpha(t)$, $u_j = \lambda_\delta^j(t)$, $v_j \geqslant 0$, then

$$f^j(t, x, u, v, w) < (\lambda_\sigma^j)'(t) - v_j \xi_1^{j,\delta}(t),$$

with $\xi_1^{i,\delta}$ being a suitable function such that

$$(t - s, \alpha(t) - \xi_1^{i,\delta}(t) \cdot s) \in \Omega \quad \text{for } s \in (0, r(t)),$$

etc.

Let us now propose a definition of a stability-like condition. Suppose that $\tilde{u} = (\tilde{u}^1, \ldots, \tilde{u}^n)$ is a saturated solution of (3)–(4)–(5) with some $\varphi^i, \psi_0^i, \psi_1^i$, such that

$$\lambda_\varrho^i(t) < \tilde{u}^i(t, x) < \mu_\varrho^i(t) \tag{84}$$

for every $\varrho > 0$, $i = 1, \ldots, n$, $t \geqslant 0$, and $x \in [\alpha(t), \beta(t)]$. We say that the solution u is $(\{\lambda_\varrho\}, \{\mu_\varrho\})$*-stable with respect to* Φ if and only if for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $\varphi = (\varphi^1, \ldots, \varphi^n) \in \Phi = \Phi^1 \times \ldots \times \Phi^n$ satisfies the conditions

$$|\tilde{u}^i(0,x) - \varphi^i(x)| < \delta \qquad \text{for } i = 1, \ldots, n, \; x \in [\alpha(0), \beta(0)],$$

then

$$\lambda_\varepsilon^i(t) < u^i[\varphi](t,x) < \mu_\varepsilon^i(t), \qquad i = 1, \ldots, n, \qquad x \in [\alpha(t), \beta(t)].$$

It is clear that the idea of the proof of (JJJ) in Theorem 1 gives almost immediately the following

THEOREM 3. *Assume that* f^i, β, Φ, Ψ_0, Ψ_1 *are as in Sections* 1–3, *and* $\{\lambda_\varrho\}$, $\{\mu_\varrho\}$ *are as above. Then every saturated solution* $\tilde{u}$ *of* (3)–(4)–(5) *satisfying for every* ϱ *condition* (84) $(i = 1, \ldots, n)$ *is* $(\{\lambda_\varrho\}, \{\mu_\varrho\})$*-stable.*

References

[0] A. Pelczar, *Some qualitative problems in the theory of partial differential equations*, Proceedings of the VII-th International Conference on Nonlinear Analysis and Applications, Arlington (Texas), 1986, ed. V. Lakshmikantham, Lecture Notes in Pure and Appl. Math. 109, Marcel Dekker, New York–Basel 1987.

[1] —, *Some generalizations of the retract theorem of T. Ważewski with applications to ordinary and partial differential equations of the first order*, Ann. Polon. Math. 29 (1974), 15–59.

[2] —, *A local version of the generalized retract theorem of Ważewski with applications in the theory of partial differential equations of first order*, ibid. 36 (1979), 11–28.

[3] —, *The Ważewski method applied to some second order partial parabolic equations*, Universitatis Iagellonicae Acta Math. (to appear).

[4] J. Smoler, *Shock Waves and Reaction-Diffusion Equations*, New York–Heidelberg–Berlin 1983.

[5] J. Szarski, *Characteristics and Cauchy problem for nonlinear partial differential equations of first order*, University of Kansas, Dept. of Math., Studies in Eigenvalue Problems, Technical Rep. 21, Lawrence, Kansas, 1959.

[6] T. Ważewski, *Sur l'appréciation de domaine d'existence des intégrales de l'équation aux dérivées partielles du premier ordre*, Ann. Soc. Polon. Math. 14 (1935), 149–177.

[7] —, *Über die Bedingungen der Existenz der Integrale partieller Differentialgleichungen erster Ordnung*, Math. Z. 43 (1936), 522–532.

[8] —, *Sur un principe topologique de l'examen de l'allure asymptotique des intégrales des équations différentielles ordinaires*, Ann. Soc. Polon. Math. 20 (1947), 279–313.

[9] —, *Sur une méthode topologique de l'examen de l'allure asymptotique des intégrales des équations différentielles*, Proc. Intern. Congr. of Mathematicians, 1954, Groningen–Amsterdam 1956, 132–139.

INSTITUTE OF MATHEMATICS, JAGIELLONIAN UNIVERSITY
KRAKÓW, POLAND

Reçu par la Rédaction le 17.06.1987