COLLOQUIUM MATHEMATICUM

VOL. LIV 1987 FASC. 1

THE BOCHNER–KOLMOGOROV EXTENSION THEOREM FOR SEMI-SPECTRAL MEASURES

BY

JAN STOCHEL (KRAKÓW)

1. Preliminaries. In 1953, Marczewski [6] introduced a purely set-theoretical notion of compact paving and showed how it can be used to produce non-trivial measures. Recall that a paving $\mathscr{K}$ on the set X is said to be *compact* if for every sequence $(K_n)_{n=1}^{\infty} \subset \mathscr{K}$ such that

$$\bigcap_{n=1}^{\infty} K_n = \emptyset$$

there exists m such that

$$\bigcap_{n=1}^{m} K_n = \emptyset.$$

It is well known that if a paving $\mathscr{K}$ is compact, then its closure under finite unions is also compact (cf. [6] and [10]).

A slight but useful variation of Marczewski's definition was recently suggested by Mallory [5]. We say that a paving $\mathscr{K}$ on the set X is *monocompact* if for every decreasing sequence $(K_n)_{n=1}^{\infty} \subset \mathscr{K}$ such that $\bigcap_{n=1}^{\infty} K_n = \emptyset$ there exists m such that $K_m = \emptyset$. Obviously, a compact paving is monocompact, but the converse is not true. Contrary to compact pavings, monocompact ones are not closed under finite unions (cf. [14]).

2. Projective cones. Let H be a complex Hilbert space and let $\mathscr{A}$ be a semi-algebra of subsets of the set X. Denote by $L(H)$ the algebra of all bounded linear operators on H, and by $\sigma(\mathscr{A})$ the smallest σ-algebra containing $\mathscr{A}$. By a *semi-spectral measure* in H on $\mathscr{A}$ we mean a positive operator-valued, weakly σ-additive set function $F\colon \mathscr{A} \to L(H)$. A semi-spectral measure $F\colon \mathscr{A} \to L(H)$ is said to be *spectral* if

$$F(A \cap B) = F(A)F(B) \quad \text{and} \quad F(A) = F(A)^* \quad \text{for all } A, B \in \mathscr{A}.$$

Notice that every semi-spectral (spectral) measure $F\colon \mathscr{A} \to L(H)$ extends in a unique way to a semi-spectral (spectral) measure ${}^{\sigma}F\colon \sigma(\mathscr{A}) \to L(H)$ (cf. [1]).

Let us denote by $\boldsymbol{C}$ and $\boldsymbol{R}_+$ the complex number field and the set of all non-negative real numbers, respectively. Identifying $\boldsymbol{C}$ with $L(\boldsymbol{C})$, one can treat a *positive measure* on $\mathscr{A}$ (i.e., positive, finite, σ-additive set function $\mu\colon \mathscr{A} \to \boldsymbol{R}_+$) as a semi-spectral measure in $\boldsymbol{C}$. A triplet $(X, \mathscr{A}, \mu)$, μ being a positive measure on a semi-algebra $\mathscr{A}$, will be called a *measure space*. We say that a paving $\mathscr{K}$ on X *approximates* a positive measure μ on $\mathscr{A}$ if for each $A \in \mathscr{A}$ and for each $\varepsilon > 0$ there exist $K \in \mathscr{K}$ and $B \in \mathscr{A}$ such that $B \subset K \subset A$ and $\mu(A \setminus B) < \varepsilon$.

For the purpose of this paper, we call

$$\boldsymbol{X} = (X_i, \mathscr{A}_i, F_i, f_{ij}, X, f_i, I)$$

a *projective cone of semi-spectral measures* in H (in short, a *projective cone over* H) if the following conditions hold true:

(1) $(I, \leqslant)$ is an upward directed set;

(2) for each $i \in I$, F_i is a semi-spectral measure in H on a semi-algebra $\mathscr{A}_i$ of subsets of the set X_i;

(3) for each $i \in I$, f_{ii} is the identity map on X_i;

(4) for all $i \leqslant j$, $f_{ij}\colon X_j \to X_i$ is a measurable measure-preserving surjective map (i.e., $f_{ij}^{-1}(\mathscr{A}_i) \subset \mathscr{A}_j$ and $F_i = F_j \circ f_{ij}^{-1}$);

(5) $f_{ij} f_{jk} = f_{ik}$ for all $i \leqslant j \leqslant k$;

(6) $f_i\colon X \to X_i$ is a map such that $f_{ij} f_j = f_i$ for all $i \leqslant j$.

The projective cone $\boldsymbol{X}$ over H is said to be *convergent* if there exists a unique semi-spectral measure F in H defined on the σ-algebra $\sigma\big(\bigcup_{i\in I} f_i^{-1}(\mathscr{A}_i)\big)$ such that

(7) $F_i = F \circ f_i^{-1}$ for each $i \in I$.

The measure F will be called the *limit measure* of $\boldsymbol{X}$.

Notice that if $\boldsymbol{X} = (X_i, \mathscr{A}_i, F_i, f_{ij}, X, f_i, I)$ is a projective cone over H, then

$${}^{\sigma}\boldsymbol{X} = \big(X_i, \sigma(\mathscr{A}_i), {}^{\sigma}F_i, f_{ij}, X, f_i, I\big)$$

is also a projective cone over H. Moreover, in virtue of the equality

$$\sigma\big(\bigcup_{i\in I} f_i^{-1}(\mathscr{A}_i)\big) = \sigma\big(\bigcup_{i\in I} f_i^{-1}\big(\sigma(\mathscr{A}_i)\big)\big),$$

${}^{\sigma}\boldsymbol{X}$ is convergent if and only if $\boldsymbol{X}$ is convergent. In the sequel we will say "*projective cone of measure spaces*" instead of "projective cone over $\boldsymbol{C}$".

By a *projective system* over H we mean a projective cone

$$(X_i, \mathscr{A}_i, F_i, f_{ij}, X_I, f_{iI}, I)$$

over H, where X_I and f_{iI} are defined as follows:

$$X_I = \{(x_i)_{i\in I}\colon\ x_i \in X_i \text{ and } f_{ij}(x_j) = x_i \text{ for all } i \leqslant j\},$$

$$f_{iI}(x) = x_i \quad \text{for } x \in X_I,\ x = (x_j)_{j\in I},\ i \in I.$$

Consider a projective cone X over H. Let $i_1 \leqslant i_2 \leqslant \ldots$ be an increasing sequence on I and let $A_1, A_2, \ldots$ be sets with $A_n \subset X_{i_n}$, $n \geqslant 1$. Following Topsøe [14] we say that (A_n) is *subconsistent* if $f_{i_n i_m}(A_m) \subset A_n$ for all $n \leqslant m$ or, equivalently, if $f_{i_n i_{n+1}}(A_{n+1}) \subset A_n$ for all $n \geqslant 1$. If all maps f_i, $i \in I$, are surjective, then (A_n) is subconsistent if and only if $(f_{i_n}^{-1}(A_n))$ is decreasing. We call (B_n) *subordinated* to (A_n) if $B_n \subset A_n$, $n \geqslant 1$. The sequence (A_n) is *consistent* if $f_{i_n i_m}(A_m) = A_n$ for all $n \leqslant m$.

The following notion is due to Bochner [3]. We say that a projective cone X over H satisfies the condition of *sequential maximality* if for each sequence $i_1 \leqslant i_2 \leqslant \ldots$ from I and for each consistent sequence $(x_n)_{n=1}^{\infty}$ with $x_n \in X_{i_n}$, $n \geqslant 1$, there exists $x \in X$ such that $f_{i_n}(x) = x_n$ for all n. It is easy to see that if X satisfies the condition of sequential maximality, then all maps f_i, $i \in I$, are surjective.

The notion of sequential maximality condition can be weakened to almost sequential maximality (cf. [9], Definition 4.5). We say that a projective cone $(X_i, \mathscr{A}_i, \mu_i, f_{ij}, X, f_i, I)$ of measure spaces satisfies the *sequential almost maximality condition* if for every $\varepsilon > 0$ and every sequence $i_1 \leqslant i_2 \leqslant \ldots$ from I there exists a subconsistent sequence $A_n \in \mathscr{A}_{i_n}$, $n \geqslant 1$, such that

(8) $\mu_{i_n}(X_{i_n} \setminus A_n) \leqslant \varepsilon$ for each $n \geqslant 1$;

(9) for any consistent sequence (x_n) subordinated to (A_n), there exists $x \in X$ such that $f_{i_n}(x) = x_n$ for every $n \geqslant 1$.

Let $X = (X_i, \mathscr{A}_i, \mu_i, f_{ij}, X, f_i, I)$ be a projective cone of measure spaces and let $J = (i_n)_{n=1}^{\infty}$ be an increasing sequence on I. Denote by X_J the set of all consistent sequences (x_n) with $x_n \in X_{i_n}$, $n \geqslant 1$, and by f_{nJ} the map on X_J into X_{i_n} defined as follows:

$$f_{nJ}(x) = x_n \quad \text{for } x \in X_J,\ x = (x_n)_{n=1}^{\infty}.$$

Following Musiał (cf. [9]) we say that X is *sequentially convergent* if, for every increasing sequence $J = (i_n)_{n=1}^{\infty}$ from I, the projective cone

$$X_J = (X_{i_n}, \mathscr{A}_{i_n}, \mu_{i_n}, f_{i_n i_m}, X_J, f_{nJ}, N)$$

is convergent ($N = \{1, 2, \ldots\}$). It is obvious that X is sequentially convergent if and only if ${}^{\sigma}X$ is sequentially convergent. Thus (cf. [9], Theorem 4.3) every sequentially convergent projective cone X over C such that ${}^{\sigma}X$ satisfies the sequential almost maximality condition is convergent.

3. Operator properties of the projective limit. Our first result shows how some operator properties of a given convergent projective cone X over H affect those of the limit measure of X.

PROPOSITION 1. *Let $F\colon \mathscr{B} \to L(H)$ be the limit of a convergent projective cone*

$$X = (X_i, \mathscr{A}_i, F_i, f_{ij}, X, f_i, I)$$

over H. Then

(i) *if all measures F_i, $i \in I$, are spectral, then F is also a spectral measure;*

(ii) *if an operator $T \in L(H)$ commutes with every one from $\bigcup_{i\in I} F_i(\mathscr{A}_i)$, then it commutes with every operator from $F(\mathscr{B})$.*

Proof. Denote by F_a the restriction of F to the algebra $\mathscr{A}_a$ generated by the semi-algebra

$$\mathscr{A} = \bigcup_{i\in I} f_i^{-1}(\mathscr{A}_i)$$

($\mathscr{A}_a$ consists of all finite disjoint unions of sets from $\mathscr{A}$). Observe that if all measures F_i, $i \in I$, are spectral, then F_a is also a spectral measure. Similarly, if an operator $T \in L(H)$ commutes with every operator from $\bigcup_{i\in I} F_i(\mathscr{A}_i)$, then it commutes with every operator from $F_a(\mathscr{A}_a)$. Since $\mathscr{B} = \sigma(\mathscr{A}_a)$ and $F = {}^{\sigma}F_a$, Proposition 1 follows from the well-known results of Berberian (cf. [1], Theorem 7, p. 15, and Theorem 12, p. 31).

COROLLARY 1. *Let $X = (X_i, \mathscr{A}_i, F_i, f_{ij}, X, f_i, I)$ be a convergent projective cone over H with the limit measure $F\colon \mathscr{B} \to L(H)$. Denote by $\mathscr{V}_i$ and $\mathscr{V}$ the von Neumann algebras generated by $F_i(\mathscr{A}_i)$ and $F(\mathscr{B})$, respectively. Then $\mathscr{V} = \varinjlim \mathscr{V}_i$, i.e., $\mathscr{V}_i \subset \mathscr{V}_j \subset \mathscr{V}$ for all $i \leqslant j$, and $\mathscr{V}$ is the von Neumann algebra generated by $\bigcup_{i\in I} \mathscr{V}_i$ (or, equivalently, $\mathscr{V}$ is the closure of $\bigcup_{i\in I} \mathscr{V}_i$ in the strong operator topology).*

Proof. Since

$$F_i(\mathscr{A}_i) = F_j f_{ij}^{-1}(\mathscr{A}_i) \subset F_j(\mathscr{A}_j) = F f_j^{-1}(\mathscr{A}_j) \subset F(\mathscr{B}),$$

we have $\mathscr{V}_i \subset \mathscr{V}_j \subset \mathscr{V}$ for all $i \leqslant j$. Thus we have only to show that $\mathscr{V}$ is the von Neumann algebra generated by $\bigcup_{i\in I} \mathscr{V}_i$. The key idea of the proof is to use the von Neumann double commutant theorem.

Let $\mathscr{F}$ be an arbitrary family in $L(H)$, closed under the operation of taking adjoints. The commutant $\mathscr{F}'$ of $\mathscr{F}$ is the totality of all operators from

$L(H)$ which commute with every one from $\mathscr{F}$. The null space $\ker \mathscr{F}$ of $\mathscr{F}$ is the set of all $h \in H$ such that $Th = 0$ for every $T \in \mathscr{F}$. Denote by $P_{\mathscr{F}}$ the orthogonal projection of H onto $H \ominus \ker \mathscr{F}$, the orthogonal complement of $\ker \mathscr{F}$ in H. Recall that the von Neumann double commutant theorem states that the von Neumann algebra $W^*(\mathscr{F})$ generated by $\mathscr{F}$ equals

$$\{T \in \mathscr{F}'':\ TP_{\mathscr{F}} = P_{\mathscr{F}} T = T\}.$$

Moreover, $\ker \mathscr{F} = \ker W^*(\mathscr{F})$.

Consider now an arbitrary semi-spectral measure E in H on a semi-algebra $\mathscr{A}$. Then $\ker E(\mathscr{A}) = \ker E(X)$. Indeed, if $E(X)h = 0$, then

$$\begin{aligned}\|E(A)h\| &\leqslant \|E(A)^{1/2}\|\,\|E(A)^{1/2}h\| = \|E(A)^{1/2}\|\big(E(A)h,\, h\big)^{1/2}\\ &\leqslant \|E(A)^{1/2}\|\big(E(X)h,\, h\big)^{1/2} = 0 \quad \text{for all } A \in \mathscr{A}.\end{aligned}$$

The second part of Proposition 1 states that

$$\{\bigcup_{i \in I} F_i(\mathscr{A}_i)\}' = F(\mathscr{B})'.$$

Thus we can write

$$F(\mathscr{B})'' \supset \mathscr{V}'' \supset (\bigcup_{i \in I} \mathscr{V}_i)'' \supset \big(\bigcup_{i \in I} F_i(\mathscr{A}_i)\big)'' = F(\mathscr{B})''$$

and, consequently,

$$F(\mathscr{B})'' = (\bigcup_{i \in I} \mathscr{V}_i)''.$$

Since $F(X) = Ff_j^{-1}(X_j) = F_j(X_j)$, we have

$$\begin{aligned}\ker F(\mathscr{B}) &= \ker F(X) = \ker F_j(X_j) = \ker F_j(\mathscr{A}_j) = \ker \mathscr{V}_j = \bigcap_{i \in I} \ker \mathscr{V}_i\\ &= \ker(\bigcup_{i \in I} \mathscr{V}_i)\end{aligned}$$

for each $j \in I$. This means that

$$P_{F(\mathscr{B})} = P_{\bigcup_{i \in I} \mathscr{V}_i}.$$

Summing up,

$$\mathscr{V} = W^*\big(F(\mathscr{B})\big) = W^*(\bigcup_{i \in I} \mathscr{V}_i).$$

This completes the proof.

Notice that the conclusion of Corollary 1 remâins still true if we take into consideration the von Neumann algebras $W^*\big(F(\mathscr{A}_i) \cup \{\mathrm{id}_H\}\big)$, $i \in I$, and $W^*\big(F(\mathscr{B}) \cup \{\mathrm{id}_H\}\big)$ instead of $\mathscr{V}_i = W^*\big(F(\mathscr{A}_i)\big)$, $i \in I$, and $\mathscr{V} = W^*\big(F(\mathscr{B})\big)$, respectively ($\mathrm{id}_H$ stands for the identity operator on H).

To state the second corollary we need the notion of minimal dilation. Let F be a semi-spectral measure in H defined on a semi-algebra $\mathscr{A}$ on X. A triplet $(\tilde{H}, R, \tilde{F})$ is a *minimal dilation* of F if $\tilde{H}$ is a Hilbert space, R: $H \to \tilde{H}$

is a bounded linear operator, and $\tilde{F}$ is a normalized spectral measure in $\tilde{H}$ on $\mathscr{A}$ (i.e., $\tilde{F}(X) = \mathrm{id}_{\tilde{H}}$) such that

(10) $\quad F = R^* \tilde{F} R,$

(11) $\quad \tilde{H} = \bigvee \{\tilde{F}(A) RH \colon A \in \mathscr{A}\} =$ the smallest subspace of $\tilde{H}$ containing the union $\bigcup \{\tilde{F}(A) RH \colon A \in \mathscr{A}\}$.

Recall that the Naimark dilation theorem (cf. [8], Theorem 4, p. 30) states that every semi-spectral measure in H defined on $\mathscr{A}$ has a minimal dilation. Moreover, its minimal dilations are determined up to unitary equivalence, i.e., if $(\tilde{H}, R, \tilde{F})$ and $(\tilde{H}_1, R_1, \tilde{F}_1)$ are minimal dilations of F, then there exists a unitary isomorphism $U \colon \tilde{H} \to \tilde{H}_1$ such that

$$U\tilde{F} = \tilde{F}_1 U, \tag{12}$$

$$UR = R_1. \tag{13}$$

Corollary 2. *Let X be a convergent projective cone over H with the limit measure F. Let $(\tilde{H}, R, \tilde{F})$ be a minimal dilation of F. Then for each $T \in \bigcap_{i \in I} F_i(\mathscr{A}_i)'$ there exists a unique operator $\tilde{T} \in \tilde{F}(\mathscr{B})'$ such that $\tilde{T}R = RT$ and $\|\tilde{T}\| \leqslant \|T\|$.*

Notice that Corollary 2 is a simple consequence of the second part of Proposition 1 and the lifting theorem (cf. [8], Theorem 1, p. 40).

Let now $(\tilde{H}, R, \tilde{F})$ be a minimal dilation of a semi-spectral measure F in H defined on a semi-algebra $\mathscr{A}$ on X. Since ${}^{\sigma}\tilde{F}$ is a spectral measure and ${}^{\sigma}F = R^* \, {}^{\sigma}\tilde{F} R$ on $\mathscr{A}$, we have

(14) $\quad {}^{\sigma}F = R^* \, {}^{\sigma}\tilde{F} R,$

(15) $\quad \tilde{H} = \bigvee \{{}^{\sigma}\tilde{F}(A) RH \colon A \in \mathscr{A}\}.$

In other words, $(\tilde{H}, R, {}^{\sigma}\tilde{F})$ is a minimal dilation of ${}^{\sigma}F$, which satisfies condition (15). Since minimal dilations are determined up to unitary equivalence, we infer that for any minimal dilation $(\tilde{H}_1, R_1, E)$ of ${}^{\sigma}F$ the space $\tilde{H}_1$ equals $\bigvee \{E(A) R_1 H \colon A \in \mathscr{A}\}$. Using the notion of dilation extension (cf. [12], Proposition 3), one can say that ${}^{\sigma}F$ is a dilation extension of F (for another proof of this fact see [13], Appendix). Summing up we have proved the following

Proposition 2. *Let X be a convergent projective cone over H with the limit measure $F \colon \mathscr{B} \to L(H)$. Define the maps*

$$g_{ij} \colon \mathscr{A}_i \to \mathscr{A}_j, \ i \leqslant j, \quad \text{and} \quad g_i \colon \mathscr{A}_i \to \mathscr{B}, \ i \in I,$$

by

$$g_{ij} = f_{ij}^{-1} \quad \text{and} \quad g_i = f_i^{-1}.$$

Then the function F is the limit of the inductive cone (F_i, g_{ij}, g_i, I) of dilatable *functions (i.e.,* g_{ii} *is the identity map on* $\mathscr{A}_i$ *for each* $i \in I$, $g_{ik} = g_{jk} g_{ij}$ *for all* $i \leqslant j \leqslant k$, $g_i = g_j g_{ij}$ *for all* $i \leqslant j$, $F_i = F_j g_{ij}$ *for all* $i \leqslant j$, $F_i = F g_i$ *for each* $i \in I$ *and* F *is a dilation extension of the restriction of* F *to* $\mathscr{A} = \bigcup_{i \in I} g_i(\mathscr{A}_i)$*).*

The connections between minimal dilations of F_i, $i \in I$, and minimal dilations of F are described in [12]. In particular, the first part of Proposition 1 follows from Proposition 4 of [12].

4. Convergence of projective cones over C**.** The following lemma gives necessary and sufficient conditions for a projective cone of measure spaces to be convergent. This is essentially due to Topsøe (cf. [14], Lemmas 1 and 2). A similar result was obtained by the author in his doctoral thesis written under the supervision of Professor W. Mlak in 1978.

LEMMA 1. *Let* $X = (X_i, \mathscr{A}_i, \mu_i, f_{ij}, X, f_i, I)$ *be a projective cone of measure spaces such that*

(i) *for each* $i \in I$, $\mathscr{A}_i$ *is an algebra,*

(ii) *for each* $i \in I$, f_i *is a surjective map.*

Assume that, for each $i \in I$*, a paving* $\mathscr{K}_i$ *on* X_i *approximates* μ_i*. Then* X *is convergent if and only if for each* $\varepsilon > 0$*, for any sequence* $i_1 \leqslant i_2 \leqslant \ldots$ *on* I*, for any sequence* (K_n) *with* $K_n \in \mathscr{K}_{i_n}$*, and for any sequence* (B_n) *with* $B_n \in \mathscr{A}_{i_n}$ *such that*

$$f_{i_1}^{-1}(K_1) \supset f_{i_1}^{-1}(B_1) \supset f_{i_2}^{-1}(K_2) \supset f_{i_2}^{-1}(B_2) \supset \ldots,$$

and

$$\mu_{i_n}(B_n) \geqslant \varepsilon \quad \text{for each } n \geqslant 1,$$

we have

$$\bigcap_{n=1}^{\infty} f_{i_n}^{-1}(B_n) \neq \emptyset.$$

If, additionally, I *has a countable cofinal set* $(j_n)_{n=1}^{\infty}$ *such that* $j_1 \leqslant j_2 \leqslant \ldots$*, then we can take into consideration only one sequence* $i_n = j_n$, $n \geqslant 1$.

In this section we give sufficient conditions for a projective cone of measure spaces to be convergent. The first result within this general frame was obtained by Choksi [4] and Métivier [7] (see also [11]). Our result is related to Theorem 3 of [14].

PROPOSITION 3. *Let* $X = (X_i, \mathscr{A}_i, \mu_i, f_{ij}, X, f_i, I)$ *be a projective cone of measure spaces such that* ${}^{\sigma}X$ *satisfies the sequential almost maximality condition. Let* $\mathscr{K}_i$, $i \in I$, *be a family of pavings such that*

(i) $f_{ij}(\mathscr{K}_j) \subset \mathscr{K}_i$ *for all* $i \leqslant j$;

(ii) *for all* $i \leqslant j$ *and for each* $x \in X_i$*, the paving* $f_{ij}^{-1}(x) \cap \mathscr{K}_j$ *is compact;*

(iii) *for each* $i \in I$, $\mathscr{K}_i$ *approximates* μ_i;

(iv) *for each* $i \in I$, $\mathscr{K}_i$ *is contained in the* ${}^{\sigma}\mu_i$*-completion of* $\sigma(\mathscr{A}_i)$.

Then X *is convergent.*

Proof. Notice that without loss of generality we may assume that

(ii$_*$) each $\mathscr{A}_i$ is an algebra and each paving $f_{ij}^{-1}(x) \cap \mathscr{K}_j$ is monocompact.

If not, then one can take into consideration a new projective cone

$$\tilde{X} = (X_i, \tilde{\mathscr{A}}_i, \tilde{\mu}_i, f_{ij}, X, f_i, I)$$

and a new family $\tilde{\mathscr{K}}_i$, $i \in I$, where $\tilde{\mu}_i$ is a unique extension of μ_i to a measure on the algebra $\tilde{\mathscr{A}}_i$ generated by $\mathscr{A}_i$, and $\tilde{\mathscr{K}}_i$ is the closure of $\mathscr{K}_i$ under finite unions. Then $\tilde{X}$ and $\tilde{\mathscr{K}}_i$, $i \in I$, satisfy the assumptions of Proposition 3 and condition (ii$_*$). Moreover, X is convergent if and only if $\tilde{X}$ is convergent.

Assume that (ii$_*$) holds. First we show that X is sequentially convergent. To begin with let we choose an arbitrary increasing sequence $J = (i_n)_{n=1}^{\infty}$ and consider the projective cone

$$X_J = (X_{i_n}, \mathscr{A}_{i_n}, \mu_{i_n}, f_{i_n i_m}, X_J, f_{nJ}, N).$$

Since each $f_{i_n i_m}$ is surjective, so is each f_{nJ}. Thus we can apply the second part of Lemma 1 to the cone X_J. Let (K_n) be a sequence with $K_n \in \mathscr{K}_{i_n}$ and let (B_n) be a sequence with $B_n \in \mathscr{A}_{i_n}$ such that

$$f_{1J}^{-1}(K_1) \supset f_{1J}^{-1}(B_1) \supset f_{2J}^{-1}(K_2) \supset f_{2J}^{-1}(B_2) \supset \dots$$

and $\mu_{i_n}(B_n) \geqslant \varepsilon$ for each $n \geqslant 1$, where ε is a positive real number. Since each f_{nJ} is a surjective map, the sequence (K_n) is subconsistent and the sequence $(f_{i_n i_m}(K_m))_{m=n}^{\infty}$ is decreasing. Let us denote by $\tilde{K}_n$, $n \geqslant 1$, the set $\bigcap_{m=n}^{\infty} f_{i_n i_m}(K_m)$. Since conditions (i) and (ii$_*$) hold, the sequence $(\tilde{K}_n)$ is consistent (cf. [4]).

Now we show that, for each $n \geqslant 1$, $\tilde{K}_n$ is a non-empty set. Denote by $(X_i, \mathscr{B}_i, \nu_i)$ the ${}^\sigma\mu_i$-completion of $(X_i, \sigma(\mathscr{A}_i), {}^\sigma\mu_i)$, $i \in I$. It is easy to see that $f_{ij}^{-1}(\mathscr{B}_i) \subset \mathscr{B}_j$ and $\nu_j \circ f_{ij}^{-1} = \nu_i$ for all $i \leqslant j$. Since each f_{nJ} is a surjective map, we obtain

$$f_{i_n i_m}^{-1}(f_{i_n i_m}(K_m)) \supset K_m \supset B_m \quad \text{for all } n \leqslant m.$$

Moreover, by (i) and (iv), K_n and $\tilde{K}_n$ belong to $\mathscr{B}_{i_n}$ for each $n \geqslant 1$. Thus

$$\nu_{i_n}(\tilde{K}_n) = \lim_{m\to\infty} \nu_{i_n}(f_{i_n i_m}(K_m)) = \lim_{m\to\infty} \nu_{i_m}\big(f_{i_n i_m}^{-1}(f_{i_n i_m}(K_m))\big)$$
$$\geqslant \lim_{m\to\infty} \nu_{i_m}(B_m) = \lim_{m\to\infty} \mu_{i_m}(B_m) \geqslant \varepsilon > 0.$$

This means that $\tilde{K}_n \neq \emptyset$.

Summing up, $(\tilde{K}_n)$ is a consistent sequence of non-empty sets subordinated to (K_n). It is easy to construct a consistent sequence $x = (x_n)$ subordinated to $(\tilde{K}_n)$, hence also to (K_n). This means that $x \in X_J$ and $f_{nJ}(x) = x_n \in K_n$ for every

$n \geqslant 1$. Thus

$$x \in \bigcap_{n=1}^{\infty} f_{nJ}^{-1}(K_n) = \bigcap_{n=1} f_{nJ}^{-1}(B_n).$$

In virtue of Lemma 1, X_J is convergent. Since the sequence J was chosen to be arbitrary, X is sequentially convergent. By the sequential almost maximality condition, X is convergent. This completes the proof.

PROPOSITION 4. *If I has a countable cofinal subset, X is a projective system of measure spaces and $\mathcal{K}_i$, $i \in I$, is a family of pavings which fulfil conditions* (i)–(iv) *of Proposition* 3, *then X is convergent.*

Proof. First observe that if I has a countable cofinal subset, then it has a countable cofinal subset $J = (i_n)_{n=1}^{\infty}$ such that $i_1 \leqslant i_2 \leqslant \ldots$ It is an observation of Musiał (cf. [9], Proposition 2.3 (ii)) that X is convergent if and only if X_J is convergent. But the last statement can be proved in the same way as it was done in the proof of Proposition 3.

Remark. Notice that condition (ii) of Proposition 3 is automatically satisfied if each map f_{ij} is injective.

COROLLARY 3. *Let I be an upward directed set and let $\mathcal{A}_i$, $i \in I$, be a family of semi-algebras on the set X such that $\mathcal{A}_i \subset \mathcal{A}_j$ for all $i \leqslant j$. Let μ_i, $i \in I$, be a consistent family of measures (i.e., μ_i: $\mathcal{A}_i \to \boldsymbol{R}_+$ and $\mu_i = \mu_j|_{\mathcal{A}_j}$ for all $i \leqslant j$). If $\mathcal{K}_i$, $i \in I$, is a family of pavings such that each $\mathcal{K}_i$ approximates μ_i, $\mathcal{K}_j \subset \mathcal{K}_i$ for all $i \leqslant j$ and $\mathcal{K}_i$ is contained in the ${}^{\sigma}\mu_i$-completion of $\sigma(\mathcal{A}_i)$ for each $i \in I$, then there exists a unique measure μ on $\sigma\big(\bigcup_{i \in I} \mathcal{A}_i\big)$ such that $\mu_i = \mu|_{\mathcal{A}_i}$ for every $i \in I$.*

5. Convergence of projective cones over H. This section deals with projective cones over a complex Hilbert space H. First we find out the connection between convergence of such cones and suitable families of projective cones of measure spaces.

Given a projective cone $X = (X_i, \mathcal{A}_i, F_i, f_{ij}, X, f_i, I)$ over H and an arbitrary vector $h \in H$, we define the projective cone X_h of measure spaces as follows:

$$X_h = \big(X_i, \mathcal{A}_i, (F_i(\cdot)h, h), f_{ij}, X, f_i, I\big).$$

The following simple lemma enables us to reduce the problem of convergence of projective cones over H to the case where H is one-dimensional.

LEMMA 2. *A projective cone X over H is convergent if and only if X_h is convergent for every $h \in H$ such that $\|h\| = 1$.*

Proof. Suppose that X_h is convergent for each $h \in H$ such that $\|h\| = 1$. Denote by μ_h, $\|h\| = 1$, the limit measure of X_h and define an operator-valued set-function F on the semi-algebra

$$\mathcal{A} = \bigcup_{i \in I} f_i^{-1}(\mathcal{A}_i)$$

by the formula $F(f_i^{-1}(A)) = F_i(A)$, where $A \in \mathscr{A}_i$ and $i \in I$. To prove correctness of the definition, suppose that $f_i^{-1}(A) = f_j^{-1}(B)$, where $A \in \mathscr{A}_i$, $B \in \mathscr{A}_j$ and $i, j \in I$. Then for each $h \in H$ such that $\|h\| = 1$ we have

$$(F_i(A)h, h) = \mu_h(f_i^{-1}(A)) = \mu_h(f_j^{-1}(B)) = (F_j(B)h, h).$$

This means that $F_i(A) = F_j(B)$. Since $(F(\cdot)h, h) = \mu_h|_{\mathscr{A}}$, $\|h\| = 1$, the set function $(F(\cdot)h, h)$ is σ-additive for each $h \in H$. In other words, F is a weakly σ-additive set function. Thus ${}^{\sigma}F$ is the limit measure of X. The converse is obvious. This completes the proof.

Proposition 3, Proposition 4, and Lemma 2 can be used to produce some sufficient conditions for projective cones over H to be convergent. In particular, we obtain the following

COROLLARY 4. *Let X be a projective cone over H which fulfils the sequential maximality condition. Let $\mathscr{K}_i$, $i \in I$, be a family of pavings which satisfies conditions* (i) *and* (ii) *of Proposition* 3. *If additionally, for each $i \in I$, $\mathscr{K}_i \subset \mathscr{A}_i$ and, for all $i \in I$ and $h \in H$, $\mathscr{K}_i$ approximates $(F_i(\cdot)h, h)$, then X is convergent.*

Notice that if all the sets X_i, $i \in I$, are Polish spaces and all maps f_{ij} and f_i are continuous, then every projective cone $(X_i, \mathscr{B}(X_i), F_i, f_{ij}, X, f_i, I)$ over H is convergent ($\mathscr{B}(X_i)$ stands for the family of all Borel subsets of X_i).

The next result is an operator version of the Jessen theorem. Let $(X_\omega, \mathscr{A}_\omega)$, $\omega \in \Omega$, be a family of measurable spaces (i.e., $\mathscr{A}_\omega$ is a σ-algebra of subsets of the set X_ω, $\omega \in \Omega$) indexed by the set Ω. Denote by $I(\Omega)$ the set of all finite subsets of Ω, directed by inclusion. If $i \subset \Omega$, then X_i stands for the Cartesian product $\prod_{\omega \in i} X_\omega$, $\mathscr{S}_i$ stands for the semi-algebra of all cylinder sets on X_i (i.e., $\mathscr{S}_i$ consists of all sets of the form $(\prod_{\omega \in j} A_\omega) \times X_{i \setminus j}$, where $A_\omega \in \mathscr{A}_\omega$ for each $\omega \in j$ and $j \in I(\Omega)$, $j \subset i$), and $\mathscr{A}_i$ stands for the σ-algebra $\sigma(\mathscr{S}_i)$.

Given a family X_ω, $\omega \in \Omega$, of sets, we define the maps $f_{ij}\colon X_j \to X_i$, $i \subset j$, and $f_i\colon X_\Omega \to X_i$, $i \in I(\Omega)$, by

$$f_{ij}\big((x_\omega)_{\omega \in j}\big) = (x_\omega)_{\omega \in i}, \tag{16}$$

$$f_j\big((x_\omega)_{\omega \in \Omega}\big) = (x_\omega)_{\omega \in j}, \tag{17}$$

where $x_\omega \in X_\omega$, $\omega \in j$, and $i \subset j$.

COROLLARY 5. *Let $(X_\omega, \mathscr{A}_\omega)$, $\omega \in \Omega$, be a family of measurable spaces and let $F_\omega\colon \mathscr{A}_\omega \to L(H)$, $\omega \in \Omega$, be a family of normalized semi-spectral measures satisfying the following condition:*

(i) *$F_\omega(A)F_\varrho(B) = F_\varrho(B)F_\omega(A)$ for all $\omega, \varrho \in \Omega$ such that $\omega \neq \varrho$ and for all $A \in \mathscr{A}_\omega$ and $B \in \mathscr{A}_\varrho$.*

Assume that for each $\omega \in \Omega$ there exists a compact paving $\mathscr{K}_\omega \subset \mathscr{A}_\omega$ such that

(ii) *for each $\omega \in \Omega$ and for each $h \in H$, $\mathscr{K}_\omega$ approximates $(F_\omega(\cdot)h, h)$.*

Then

1° *There exists one and only one semi-spectral measure* $F\colon \mathscr{A}_\Omega \to L(H)$ *such that*

$$F\Big(\big(\prod_{\omega\in i} A_\omega\big)\times X_{\Omega\setminus i}\Big) = \prod_{\omega\in i} F_\omega(A_\omega) \tag{18}$$

for all $A_\omega\in\mathscr{A}_\omega$, $\omega\in i$, *and* $i\in I(\Omega)$.

2° *If all the measures* F_ω, $\omega\in\Omega$, *are spectral, then* F *is also a spectral measure.*

3° $\bigcap_{\omega\in\Omega} F_\omega(\mathscr{A}_\omega)' \subset F(\mathscr{A}_\Omega)'$.

The measure F is called a *product* of semi-spectral measures F_ω, $\omega\in\Omega$.

Proof. 1° For $i\in I(\Omega)$ we define the paving $\mathscr{K}_i$ on X_i as follows:

$$\mathscr{K}_i = \Big\{\prod_{\omega\in i} K_\omega\colon K_\omega\in\mathscr{K}_\omega \text{ for } \omega\in i\Big\}.$$

It is easy to see that $\mathscr{K}_i$ is a compact subpaving of the semi-algebra $\mathscr{S}_i$. Let us define the function $F_i\colon \mathscr{S}_i\to L(H)$, $i\in I(\Omega)$, by the formula

$$F_i\big(\prod_{\omega\in i} A_\omega\big) = \prod_{\omega\in i} F_\omega(A_\omega), \qquad A_\omega\in\mathscr{A}_\omega,\ \omega\in i.$$

Using the same arguments as in [1] (Lemma 2, p. 90), one can show that F_i is finitely-additive and that, for each $h\in H$, $\mathscr{K}_i$ approximates $(F_i(\cdot)h, h)$, $i\in I(\Omega)$. Thus, for each $h\in H$, $(F_i(\cdot)h, h)$ is σ-additive and, consequently, F_i is a semi-spectral measure. Summing up, the projective cone $X = (X_i, \mathscr{S}_i, F_i, f_{ij}, X_\Omega, f_i, I(\Omega))$, where the maps f_{ij} and f_i are defined by (16) and (17), respectively, satisfies all the assumptions of Corollary 4. Therefore, X is convergent. Denote by F the limit measure of X. Since

$$\sigma\Big(\bigcup_{i\in I(\Omega)} f_i^{-1}(\mathscr{S}_i)\Big) = \sigma(\mathscr{S}_\Omega) = \mathscr{A}_\Omega$$

and

$$F\Big(\big(\prod_{\omega\in i} A_\omega\big)\times X_{\Omega\setminus i}\Big) = F\Big(f_i^{-1}\big(\prod_{\omega\in i} A_\omega\big)\Big) = F_i\big(\prod_{\omega\in i} A_\omega\big) = \prod_{\omega\in i} F_\omega(A_\omega)$$

for all $A_\omega\in\mathscr{A}_\omega$, $\omega\in i$ and $i\in I(\Omega)$, the measure F satisfies (18).

2° and 3°. If all the measures F_ω, $\omega\in\Omega$, are spectral, then the measure F_i is spectral for each $i\in I(\Omega)$. If

$$T\in \bigcap_{\omega\in\Omega} F_\omega(\mathscr{A}_\omega)',$$

then $T\in F_i(\mathscr{S}_i)'$ for each $i\in I(\Omega)$. Thus we can apply Proposition 1 to the cone X. This completes the proof.

It follows from Corollary 5 that if X_ω, $\omega\in\Omega$, is a family of Polish spaces, then there always exists a product of commuting normalized semi-spectral measures F_ω: $\mathscr{B}(X_\omega) \to L(H)$ (cf. [2]). Notice that the assumption (ii) of Corollary 5 cannot be omitted even if Ω is a two-point set and F_ω, $\omega \in \Omega$, are spectral measures. Namely, one can construct two commuting normalized spectral measures without product measure.

Acknowledgment. The author is grateful to the referee for many helpful suggestions.

REFERENCES

[1] S. K. Berberian, *Notes on Spectral Theory*, New York 1966.

[2] Yu. M. Berezanskiĭ, *Selfadjoint Operators in Spaces of Infinite Number of Variables* (in Russian), Kiev 1978.

[3] S. Bochner, *Harmonic Analysis and Probability Theory*, Berkeley 1955.

[4] J. R. Choksi, *Inverse limits of measure spaces*, Proc. London Math. Soc. 8 (1958), pp. 321–342.

[5] D. Mallory, *Extension of set functions to measures and applications to inverse limit measures*, Canad. Math. Bull. 18 (1975), pp. 547–553.

[6] E. Marczewski, *On compact measures*, Fund. Math. 40 (1953), pp. 113–124.

[7] M. Métivier, *Limites projectives de mesures. Martingales. Applications*, Ann. Mat. Pura Appl. 63 (1963), pp. 225–352.

[8] W. Mlak, *Dilations of Hilbert space operators. General theory*, Dissertationes Math. 153 (1978).

[9] K. Musiał, *Projective limits of perfect measure spaces*, Fund. Math. 110 (1980), pp. 163–189.

[10] J. Pfanzagl and W. Pierlo, *Compact systems of sets*, Lecture Notes in Math. 16 (1966).

[11] M. M. Rao, *Projective limits of probability spaces*, J. Multivariate Anal. 1 (1971), pp. 28–57.

[12] J. Stochel, *Dilations of inductive limits of operator functions*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 28 (1980), pp. 143–148.

[13] — *The Fubini theorem for semi-spectral integrals and semi-spectral representations of some families of operators*, Universitatis Iagellonicae Acta Math. 26 (1987), pp. 17–27.

[14] F. Topsøe, *Approximating pavings and construction of measures*, Colloq. Math. 42 (1979), pp. 377–385.

INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
BRANCH IN KRAKÓW

Reçu par la Rédaction le 1.3.1981;
en version modifiée le 27.12.1983