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BREAKDOWN PROCESSES OF SYSTEMS IN PARALLEL

1. This paper deals with system of facilities composed of n identical facilities, each with independent breakdown processes having exponentially distributed working and breakdown times. The facilities are connected in such a manner that the whole system is in working order if and only if at least i of the component facilities are in working order. Generally, breakdown time may be understood either as the repair time of a failed facility or as the replacement time of a failed facility or as the time necessary for automatic regeneration, etc. In the case $i = 1$ the facilities are connected in parallel in the usual way, as e.g. communication lines in telefony. Then for the system to be in working order a working order of any of the facilities is necessary and sufficient. If in a system in parallel a reserve of several facilities being in working order is necessary for the whole system to be in working order then we have the above defined breakdown system. In the extreme case $i = n$ the facilities are said to be connected in series.

We are interested in working and breakdown time distributions of the system under stationary conditions i.e. in the working time distribution from the end of one breakdown period to the beginning of the next breakdown period and in the breakdown time distribution from the end of the working period to the beginning of the next working period.

The expected values of those distributions have been given by Sedyakin in [7] in a general case of facilities each of them having an arbitrary working time distribution and breakdown time distribution. For the unnecessarily identical facilities having exponentially distributed working and breakdown times Kozlov and Ushakov in [5] have given the Laplace transform of the density of the working time distribution in the form of a quotient of two determinants (which is quite sufficient for numerical investigation of the facility system by the use of computers) and the expected value of this distribution. A similar result is known for new systems beginning work from the working order state of all facilities (see also [1], [2], [6]).

Assuming that the facilities in the system are the same we can find in a different manner the Laplace transforms of the density and the distribution function of the working and breakdown times of the system, and present them in a form convenient for calculating the higher moments of the distributions in question. This paper is a continuation of [3] and [4] where special cases $i = 1$ and $i = n$ have been analyzed.

2. Let us consider n identical facilities with independent breakdown processes $a^{(j)}(t)$ for $j = 1, 2, \ldots, n$, $-\infty < t < \infty$, defined in [3] in the following way

$$(1) \qquad a^{(j)}(t) = \begin{cases} 1 & \text{for} \quad Z_{2k}^{(j)} \leqslant t < Z_{2k+1}^{(j)}, \\ 0 & \text{for} \quad Z_{2k-1}^{(j)} \leqslant t < Z_{2k}^{(j)}, \end{cases} \quad k = 0, \pm 1, \pm 2, \ldots,$$

where the intervals $Z_{2k+1}^{(j)} - Z_{2k}^{(j)} = X_k^{(j)}$, called the working times of the j-th facility, are independent random variables having an exponential distribution with parameter λ, the intervals $Z_{2k}^{(j)} - Z_{2k-1}^{(j)} = Y_k^{(j)}$, called the breakdown times of the j-th facility, are independent random variables having an exponential distribution with parameter μ; and the working and breakdown times are independent.

Let us form a new process

$$(2) \qquad \beta_n(t) = \sum_{j=1}^{n} a^{(j)}(t),$$

which denotes a number of facilities in working order at the moment t. If $a^{(j)}(t)$ represents also the rate of production of the facility at the moment t then the process $\beta_n(t)$ is the rate of production of the system. Due to the former assumptions the process $\beta_n(t)$ is a stationary Markov process having the states $0, 1, \ldots, n$ and the transition probabilities

$$a_{k,l}(\tau) = P\big(\beta_n(t+\tau) = l \mid \beta_n(t) = k\big) = o(\tau) \quad \text{for} \quad |k-l| > 1,$$

$$a_{k,k-1}(\tau) = k\lambda\tau + o(\tau),$$

$$a_{k,k+1}(\tau) = (n-k)\mu\tau + o(\tau),$$

$$a_{k,k}(\tau) = 1 - [k\lambda + (n-k)\mu]\tau + o(\tau),$$

where $0 \leqslant k \leqslant n$.

We shall make a symbolic recording of these probabilities in the form of the matrix of transition rates

$$A = \begin{pmatrix} -n\mu & n\mu & 0 & \ldots & 0 & 0 & 0 \\ \lambda & -(n-1)\mu-\lambda & (n-1)\mu & \ldots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \ldots & (n-1)\lambda & -\mu-(n-1)\lambda & \mu \\ 0 & 0 & 0 & \ldots & 0 & n\lambda & -n\lambda \end{pmatrix}.$$

The probabilities of the states of the proces $\beta_n(t)$ are as follows

$$p_k = P(\beta_n(t) = k) = \binom{n}{k} p^k q^{n-k} \tag{3}$$

where $p = \mu/(\lambda+\mu)$, $q = 1-p$, and the time of the process $\beta_n(t)$ to be in state k is a random variable of exponential distribution with parameter $k\lambda+(n-k)\mu$.

As an interesting point let us consider a chain $N(r)$ defined as the sequence of states $\{\beta_n(t_r+0)\}$, where t_r are the discontinuity points of the process $\beta_n(t)$. The chain $N(r)$ is a stationary Markov chain with the transition probabilities

$$a_{k,k+1} = P(N(r+1) = k+1 \,|\, N(r) = k) = \frac{(n-k)\mu}{k\lambda+(n-k)\mu},$$

$$a_{k,k-1} = P(N(r+1) = k-1 \,|\, N(r) = k) = \frac{k\lambda}{k\lambda+(n-k)\mu}$$

for $0 \leqslant k \leqslant n$.

The probabilities of the states of the chain $p_k = P(N(r) = k)$ fulfill the system of equations

$$p_k = p_{k-1}a_{k-1,k}+p_{k+1}a_{k+1,k} \quad \text{for} \quad 0 \leqslant k \leqslant n.$$

The solutions of this system of equations are probabilities

$$p_k = \frac{1}{2n}\binom{n}{k}[kq+(n-k)p]p^{k-1}q^{n-k-1}.$$

We shall define the breakdown process for the system as follows

$$a_{i,n}(t) = \begin{cases} 1 & \text{for} \quad \beta_n(t) \geqslant i, \\ 0 & \text{for} \quad \beta_n(t) < i, \end{cases}$$

which means that the system is in working order if and only if at least i facilities of the n are in working order. It is easy to check that the process $a_{i,n}(t)$ is well defined which means that the sequence of consecutive intervals of working periods is a sequence of independent random variables with the same distribution, that the sequence of consecutive intervals of breakdown periods is a sequence of independent random variables with the same distribution, and that the working and breakdown periods of the system are independent. It is due to the fact, that the process $\beta_n(t)$ is a Markov process, therefore if, for example, at a particular moment the system passes from working order to breakdown order the continuation of the process $a_{i,n}(t)$ does not depend on its run up to this moment.

3. We shall investigate the working time distributions of the system. We shall mak euse of the well known method of differential equations applied, for instance, in Sandler's book [6]. We are considering the process $\beta_n(t)$ on the states $i-1, i, \ldots, n$ assuming that the state $i-1$ is absorbing and the matrix of transition rates for this process is of the form

$$B = \begin{pmatrix} 0\ 0 \ldots 0 \\ \ldots\ldots\ldots \\ A^{(i,i-1)} \end{pmatrix}$$

where $A^{(i,i-1)}$ denotes the matrix of elements in matrix A in the rows $i, i+1, \ldots, n$ and in columns $i-1, i, \ldots, n$.

It is known that the probabilities of the states for the Markov process

$$P(t) = [P_{i-1}(t), P_i(t), \ldots, P_n(t)]$$
$$= [P(\beta_n(t) = i-1), P(\beta_n(t) = i), \ldots, P(\beta_n(t) = n)]$$

fulfill the system of differential equations

$$P'(t) = P(t)B, \tag{4}$$

which should be solved with some initial condition $P(0)$. Passing to the Laplace transforms $\mathscr{L}(P(t)) = P^*(s)$ and applying the known formula $\mathscr{L}(P'(t)) = sP^*(s) - P(0)$ the system becomes a system of linear equations

$$P^*(s)[sI - B] = P(0). \tag{5}$$

We are interested in the probability $P_{i-1}(t)$ which is the probability for the system to be in breakdown order from the moment $t = 0$ until the moment t. This probability is usually calculated for the following initial condition:

A. $P_n(0) = 1$ which means that at the starting point all facilities are in working order, for example, when the system is a new one. Probability $P_{i-1}(t)$ is then the distribution function of the working time for a new system. The solution of the system (5) for $P^*_{i-1}(s)$ is (see [5] p. 90, [6] p. 195)

$$P^*_{i-1}(s) = \det U_n^{(i)}(s)/s \det W_n^{(i)}(s), \tag{6}$$

where

$$U_n^{(i)}(s) = \begin{pmatrix} 0 & -i\lambda & 0 & 0 & \ldots & 0 & 0 \\ 0 & s+i\lambda+(n-i)\mu & -(i+1)\lambda & 0 & \ldots & 0 & 0 \\ 0 & -(n-i)\mu & s+(i+1)\lambda+(n-i-1)\mu & -(i+2)\lambda & \ldots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \ldots & s+(n-1)\lambda+\mu & -n\lambda \\ 1 & 0 & 0 & 0 & \ldots & -\mu & s+n\lambda \end{pmatrix} \tag{7}$$

and

(8) $$W_n^{(i)}(s) = \begin{pmatrix} s+i\lambda+(n-i)\mu & -(i+1)\lambda & 0 & \dots & 0 & 0 \\ (n-i)\mu & s+(i+1)\lambda+(n-i-1)\mu & -(i+2)\lambda & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & s+(n-1)\lambda+\mu & -n\lambda \\ 0 & 0 & 0 & \dots & -\mu & s+n\lambda \end{pmatrix}.$$

It is easy to see that

(9) $$\det U_n^{(i)}(s) = i(i+1)\dots n\lambda^{n-i+1}.$$

The denominator in the formula (6) will be dealt with later.

B. In this paper attension is paid to the following initial condition:

$$P_j(0) = p_j/(p_i+p_{i+1}+\dots+p_n) \quad \text{for} \quad j = i, i+1, \dots, n$$

which means that the moment $t = 0$ has been chosen at random in the working period of the system. To be more exact we reject from the process $\beta_n(t)$ the time intervals where $\beta_n(t) < i$, so receiving a stationary process with state probabilities $P_j(0)$. Probability $P_{i-1}(t)$, the Laplace transform of which fulfills the system (5), is the distribution function of the system working time which has still remained after the moment $t = 0$. The complete working time of the system is then a random variable $X_{i,n}$ with the distribution $F_{i,n}(x)$ involved in the formula

$$\frac{1}{EX_{i,n}}\int_0^x \big(1-F_{i,n}(u)\big)du = P_{i-1}(x).$$

Therefrom we calculate

$$F_{i,n}(x) = 1-EX_{i,n}P'_{i-1}(x),$$

where

$$EX_{i,n} = 1/P'_{i-1}(0).$$

The first equation of the system (4) is of the form $P'_{i-1}(t) = i\lambda P_i(t)$, thus

$$F_{i,n}(x) = 1-\frac{P_i(x)}{P_i(0)}.$$

The last formula may be written using Laplace transforms. If we denote $\mathscr{L}\big(F_{i,n}(x)\big) = f^*_{i,n}(s)/s$ then

(10) $$\frac{1-f^*_{i,n}(s)}{s} = \frac{P_i^*(s)}{P_i(0)}.$$

The solution of the system (5) with respect to $P_i^*(s)$ is

(11) $$P_i^*(s) = P_i(0)\det V_n^{(i)}(s)/\det W_n^{(i)}(s)$$

where

$$V_n^{(i)}(s) = \begin{pmatrix} 1 & -(i+1)\lambda & 0 & \dots & 0 & 0 \\ p_{i+1}/p_i & s+(i+1)\lambda+(n-i-1)\mu & -(i+2)\lambda & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ p_{n-1}/p_i & 0 & 0 & \dots & s+(n-1)\lambda+\mu & -n\lambda \\ p_n/p_i & 0 & 0 & \dots & -\mu & s+n\lambda \end{pmatrix}. \tag{12}$$

and $W_n^{(i)}(s)$ is given in (8).

We shall now prove that

$$\det V_n^{(i)}(s) = \det W_{n-1}^{(i)}(s+a), \qquad a = \lambda+\mu. \tag{13}$$

Proof. Developing $\det V_n^{(i)}(s)$ with respect to the first row we obtain

$$\det V_n^{(i)}(s) = \det W_n^{(i+1)}(s) + (i+1)\lambda \frac{p_{i+1}}{p_i} \det V_n^{(i+1)}(s)$$
$$= \det W_n^{(i+1)}(s) + (n-i)\mu \det V_n^{(i+1)}(s).$$

Repeating this procedure we obtain

$$\det V_n^{(i)}(s) = \det W_n^{(i+1)}(s) + (n-i)\mu \det W_n^{(i+2)}(s) +$$
$$+ (n-i-1)(n-i)\mu^2 \det W_n^{(i+3)}(s) +$$
$$+ \dots + 2\cdot 3 \cdot \dots \cdot (n-i)\mu^{n-i-1} \det W_n^{(n)}(s) + (n-i)!\,\mu^{n-i}$$
$$= \det W_n^{(i+1)}(s) + (n-i)\mu \{\det W_n^{(i+2)}(s) +$$
$$+ \dots + 3\mu [\det W_n^{(n-1)}(s) + 2\mu (\det W_n^{(n)}(s) + \mu)] \dots\}$$

$$= \begin{vmatrix} s+(i+1)\lambda+(n-i-1)\mu+(n-i)\mu & (n-i)\mu-(i+2)\lambda & (n-i)\mu & \dots & (n-i)\mu \\ -(n-i-1)\mu & s+(i+2)\lambda+(n-i-2)\mu & -(i+3)\lambda & \dots & 0 \\ 0 & -(n-i-2)\mu & s+(i+3)\lambda+(n-i-3)\mu & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & -n\lambda \\ 0 & 0 & 0 & \dots & s+n\lambda \end{vmatrix}.$$

Adding to every row in this determinant, starting with the first one, all following rows and then subtracting from every column the preceding column, starting with the last one, we get

$$\det V_n^{(i)}(s) = \det W_{n-1}^{(i)}(s+a).$$

This ends the proof of formula (13).

Making use of formulae (10), (11) and (13) we obtain

$$\frac{1-f_{i,n}^*(s)}{s} = \frac{\det W_{n-1}^{(i)}(s+a)}{\det W_n^{(i)}(s)}. \tag{14}$$

It is easy to prove that

$$\det W_n^{(i)}(s) - s\det W_{n-1}^{(i)}(s+a) = i\lambda \det W_n^{(i-1)}(s) \tag{15}$$

hence

$$f_{i,n}^*(s) = \frac{i\lambda \det W_n^{(i+1)}(s)}{\det W_n^{(i)}(s)}. \tag{16}$$

This is a particular case of the formula (4.1.32) from [5]. It has been obtained from the system of equations (5) with the initial condition $P_i(t) = 1$ which means that at the moment $t = 0$ the minimum number of facilities is at work required for the system to be in working order. This may be the case if at $t = 0$ there is a transition from the breakdown order into the working order. Probability $P_{i-1}(t)$ is a distribution function of the working time for the system, although the initial condition mentioned above does not guarantee that the system is in the state i for the first time in the actual working time of the system.

Proof of formula (15). Let $O(W)$ denote the matrix W transformed in the following way. To every row, starting with the first one, we add all following rows, and them from each column, starting with the last one, we subtract the preceding column. In the reverse transformation, which we denote by $O^{-1}(W)$, all preceding columns have been added first to each column, starting with the second one, and then the previous row has been substracted from every row, starting with the last one. Of course $\det O(W) = \det O^{-1}(W) = \det W$.

Developing det $O\big(W_n^{(i)}(s)\big)$ with respect to the first row (or the first column) we obtain

$$\det W_n^{(i)}(s) = (s+i\lambda)\det W_{n-1}^{(i)}(s+a) - i(n-i)\lambda\mu \det W_{n-1}^{(i+1)}(s+a). \tag{17}$$

Thus

$$\begin{aligned}
&\det W_n^{(i)}(s) - s\det W_{n-1}^{(i)}(s+a)\\
&\qquad = i\lambda \det W_{n-1}^{(i)}(s+a) - i(n-i)\lambda\mu \det W_{n-1}^{(i+1)}(s+a)\\
&\qquad = \det O^{-1}\begin{vmatrix} i\lambda & -i\lambda \quad 0 \quad 0 \\ \cdots & \cdots \\ -(n-i)\mu & \\ 0 & \\ \cdots & W_{n-1}^{(i+1)}(s+a) \\ 0 & \end{vmatrix} = i\lambda \det W_n^{(i+1)}(s).
\end{aligned}$$

This ends the proof of formula (15).

Developing $\det W_n^{(i)}(s)$ with respect to the first line and using (17) we obtain two recurrent formulae

$$(18)\quad \det W_n^{(i-1)}(s) = [s+(i-1)\lambda+(n-i+1)\mu]\det W_n^{(i)}(s)- \\ -i(n-i+1)\lambda\mu\det W_n^{(i+1)}(s),$$

$$(19)\quad \det W_n^{(i)}(s) = (s+i\lambda)\det V_n^{(i)}(s)-i(n-i)\lambda\mu\det V_n^{(i+1)}(s),$$

moreover, from the definition we get

$$(20)\quad \det W_n^{(n)}(s) = s+n\lambda,$$

$$(21)\quad \det W_n^{(n-1)}(s)= s^2+\big((2n-1)\lambda+\mu\big)s+n(n-1)\lambda^2,$$

$$(22)\quad \det V_n^{(n-1)}(s) = s+n\lambda+\mu,$$

$$(23)\quad \det V_n^{(n-2)}(s) = s^2+[(2n-1)\lambda+3\mu]s+n(n-1)\lambda^2+2n\lambda\mu+2\mu^2.$$

Let us express $\det W_n^{(i)}(s)$ and $\det V_n^{(i)}(s)$ in the form of polynomials of variable s:

$$\det W_n^{(i)}(s) = w_{n,0}^{(i)}+w_{n,1}^{(i)}s+\dots$$
$$\det V_n^{(i)}(s) = v_{n,0}^{(i)}+v_{n,1}^{(i)}s+\dots$$

It is easy to calculate

$$(24)\quad \begin{aligned} w_{n,0}^{(i)} &= \det W_n^{(i)}(0) = i(i+1)\dots n\lambda^{n-i+1}, \\ v_{n,0}^{(i)} &= \det V_n^{(i)}(0) = \Big(\sum_{j=i}^{n} p_j\Big)(i+1)(i+2)\dots n\lambda^{n-i}/p_i, \end{aligned}$$

since the matrices $W_n^{(i)}(0)$ and $V_n^{(i)}(0)$, after having added to each row (starting with the first one) all the following rows, have zeros above the diagonal. To calculate $w_{n,1}^{(i)}$ and $v_{n,1}^{(i)}$ it is enough to compare the coefficients in formulae (18) and (19). We obtain then recurrent formulae

$$(25)\quad w_{n,1}^{(i-1)} = w_{n,0}^{(i)}+[(i-1)\lambda+(n-i+1)\mu]w_{n,1}^{(i)}-i(n-i+1)\lambda\mu w_{n,1}^{(i+1)},$$

$$(26)\quad w_{n,1}^{(i)} = v_{n,0}^{(i)}+i\lambda v_{n,1}^{(i)}-i(n-i)\lambda\mu v_{n,1}^{(i+1)},$$

where

$$(27)\quad w_{n,1}^{(n)} = 1,\quad w_{n,1}^{(n-1)} = (2n-1)\lambda+\mu,$$

$$(28)\quad v_{n,1}^{(n-1)} = 1,\quad v_{n,1}^{(n-2)} = (2n-1)\lambda+3\mu.$$

As we have found the transform

$$\frac{1-f_{i,n}^{*}(s)}{s} = \det V_n^{(i)}(s)/\det W_n^{(i)}(s)$$

it is now easy to find

$$EX_{i,n} = \frac{1-f^*_{i,n}(s)}{s}\bigg|_{s=0} = \frac{v^{(i)}_{n,0}}{w^{(i)}_{n,0}}, \tag{29}$$

$$EX^2_{i,n} = 2\,\frac{EX_{i,n}-\dfrac{1-f^*_{i,n}(s)}{s}}{s}\Bigg|_{s=0} = 2\,\frac{v^{(i)}_{n,0}w^{(i)}_{n,1}-w^{(i)}_{n,0}v^{(i)}_{n,1}}{(w^{(i)}_{n,0})^2}. \tag{30}$$

Thus

$$EX_{i,n} = \Big(\sum_{j=i}^{n} p_j\Big)\Big/ i\lambda p_i. \tag{31}$$

4. In [3] a different method has been used to find the working and the breakdown time distributions for the system of facilities in parallel and for $i = 1$ (in this case we say that the facilities are arranged in series). It has been then proved that

$$\frac{1}{s+n\mu\big(1-f^*_{1,n}(s)\big)} = \sum_{j=0}^{n}\binom{n}{j}p^j q^{n-j}\frac{1}{s+ja},$$

thus

$$\frac{1-f^*_{1,n}(s)}{s} = \frac{m^{(1)}_{n-1}(s+a)}{m^{(1)}_n(s)}, \tag{32}$$

where

$$m^{(1)}_n(s) = \sum_{j=0}^{n}\binom{n}{j}p^j q^{n-j}\frac{s(s+a)\ldots(s+na)}{s+ja}. \tag{33}$$

We shall prove now that

$$m^{(1)}_n(s) = \det W^{(1)}_n(s). \tag{34}$$

Developing $O\big(W^{(0)}_n(s)\big)$ with respect to the first row we shall indeed, obtain the recurrent formula

$$\det W^{(0)}_n(s) = s\det W^{(0)}_{n-1}(s+a),$$

thus

$$\det W^{(0)}_n(s) = s(s+a)\ldots(s+na). \tag{35}$$

From (13) and (19) we obtain

$$\det W^{(1)}_n(s) = (s+\lambda)\det W^{(1)}_{n-1}(s+a)-(n-1)\lambda\mu\det W^{(2)}_{n-1}(s+a),$$

from (18), however, we have

$$\det W^{(0)}_n(s) = (s+n\mu)\det W^{(1)}_n(s)-n\lambda\mu\det W^{(2)}_n(s).$$

From the last three formulae we obtain equation

$$\det W_n^{(1)}(s) = (s+a)\ldots(s+na) - n\mu \det W_{n-1}^{(1)}(s+a), \tag{36}$$

the solution of which is $\det W_n^{(1)}(s) = m_n^{(1)}(s)$ which is easy to check by substitution.

5. Formula (34) is a particular case of the formula

$$\det W_n^{(i)}(s) = \sum_{j=0}^{n-i+1} \binom{n-i+1}{j} p^j q^{n-i+1-j} \frac{s(s+a)\ldots(s+na)}{(s+ja)\ldots\big(s+(j+i-1)a\big)}, \tag{37}$$

or

$$\det W_n^{(i)}(s) = \sum_{j=0}^{n-i+1} \binom{n-i+1}{j} \mu^j \lambda^{n-i+1-j} \frac{\frac{s}{a}\left(\frac{s}{a}+1\right)\ldots\left(\frac{s}{a}+n\right)}{\left(\frac{s}{a}+j\right)\left(\frac{s}{a}+j+1\right)\ldots\left(\frac{s}{a}+j+i-1\right)}. \tag{37'}$$

This formula, in the form of (37′), comes from (18). It fulfills (18) identically which can easily be checked by transforming all terms on one side and by ordering the expressions according to powers of μ^i ($i = 1, 2, \ldots, n-i+1$, making no use of the fact that $a = \lambda+\mu$).

From the formula (37′) we easily obtain

$$w_{n,1}^{(i)} = \frac{1}{\lambda+\mu}\left(\sum_{j=i}^{n} \frac{i(i+1)\ldots n}{j} \lambda^{n-i+1} + \sum_{j=1}^{n-i+1} \binom{n-i+1}{j} \mu^j \lambda^{n-i+1-j} \frac{n!}{j(j+1)\ldots(j+i-1)}\right), \tag{38}$$

$$w_{n,2}^{(i)} = \frac{1}{(\lambda+\mu)^2}\left(\sum_{j=i}^{n} \sum_{\substack{k=i \\ k\neq j}}^{n} \frac{i(i+1)\ldots n}{jk} \lambda^{n-i+1} + \sum_{j=1}^{n-i+1} \binom{n-i+1}{j} \mu^j \lambda^{n-i+1-j} \sum_{\substack{k=1 \\ k\neq j,j+1,\ldots,j+i-1}}^{n-i+1} \frac{n!}{j(j+1)\ldots(j+i-1)k}\right). \tag{39}$$

From the formulae (37′) and (13) we, however, obtain

$$v_{n,1}^{(i)} = \frac{1}{\lambda+\mu}\left(\sum_{j=i+1}^{n} \frac{(i+1)(i+2)\ldots n}{j} \lambda^{n-i} + \sum_{j=1}^{n-i} \binom{n-i}{j} \mu^j \lambda^{n-i-j} \sum_{\substack{k=1 \\ k\neq j+1,\ldots,j+i}}^{n-i} \frac{n!}{(j+1)(j+2)\ldots(j+i)k}\right). \tag{40}$$

From this we obtain moments of the random variables $X_{i,n}$ in an explicit form without the necessity of using recurrent formulae (25)-(28). They may be more convenient of we are studying one distribution. It will be easy now to find the moment of the working time of the new system. As from formulae (9) and (24) we have det $U_n^{(i)}(s) = w_{n,0}^{(i)}$ so, denoting by $\check{X}_{i,n}$ the working time of the new system, from formula (6) we have

$$E\check{X}_{i,n} = \frac{w_{n,1}^{(i)}}{w_{n,0}^{(i)}}, \tag{41}$$

$$D^2\check{X}_{i,n} = \frac{(w_{n,1}^{(i)})^2 - w_{n,0}^{(i)} w_{n,2}^{(i)}}{(w_{n,0}^{(i)})^2}. \tag{42}$$

6. To give an example the mean values $EX_{i,n}$ and the variances $D^2X_{i,n}$ for the case $\lambda = \mu = 1$ are presented in Tables 1 and 2. The variances for the random variables $\tilde{X}_{i,n} = X_{i,n}/EX_{i,n}$ are presented in Table 3. The values of the first column in Table 3 can be found in the second column of table 1 in [4]. It has been proved in [3] that if n is growing to infinity the distributions of the random variables $X_{1,n}$ tend to an exponential distribution with parameter 1. We may expect that this convergence is taking place also in the case $i > 1$. This is based on the

TABLE 1. Expected values of the working time $X_{i,n}$ for $\lambda = \mu = 1$

n \ i	1	2	3	4	5	6	7	8	9	10
1	1.0000									
2	1.5000	0.5000								
3	2.3333	0.6667	0.3333							
4	3.7500	0.9167	0.4167	0.2500						
5	6.2000	1.3000	0.5333	0.3000	0.2000					
6	10.5000	1.9000	0.7000	0.3667	0.2333	0.1667				
7	18.1429	2.8571	0.9429	0.4571	0.2762	0.1905	0.1429			
8	31.8750	4.4107	1.3036	0.5821	0.3321	0.2202	0.1607	0.1250		
9	56.7778	6.9722	1.8492	0.7579	0.4063	0.2579	0.1825	0,1389	0.1111	
10	102.3000	11.2556	2.6889	1.0095	0.5063	0.3063	0.2095	0.1556	0.1222	0.1000

TABLE 2. The variances of the working time $X_{i,n}$ for $\lambda = \mu = 1$

n \ i	1	2	3	4	5	6	7	8	9	10
1	1.0000									
2	2.7500	0.2500								
3	7.4444	0.5556	0.1111							
4	20.2708	1.2292	0.2153	0.0625						
5	56.1733	2.7433	0.4178	0.1100	0.0400					
6	159.439	6.2278	0.8189	0.1944	0.0656	0.0278				
7	464.916	14.4621	1.6306	0.3472	0.1080	0.0431	0.0204			
8	1393.62	34.4894	3.3124	0.6290	0.1797	0.0672	0.0303	0.0156		
9	4290.41	84.6873	6.8877	1.1596	0.3030	0.1058	0.0452	0.0224	0.0123	
10	13535.1	214.404	14.6991	2.1818	0.5187	0.1685	0.0681	0.0322	0.0172	0.0100

hehaviour of the values in the second column of the Table 3 where the variances are decreasing for $n = 9$ and $n = 10$. However, the approximation of the distributions of random variables $X_{i,n}$ by the exponential distribution does not seem good for practically important values of n and i.

TABLE 3. The variances of the reduced working time $X_{i,n}/EX_{i,n}$ for $\lambda = \mu$

n \ i	1	2	3	4	5	6	7	8	9
1	1.0000								
2	1.2222	1.0000							
3	1.3673	1.2500	1.0000						
4	1.4415	1.4628	1.2400	1.0000					
5	1.4613	1.6233	1.4687	1.2222	1.0000				
6	1.4462	1.7251	1.6712	1.4463	1.2041	1.0000			
7	1.4124	1.7716	1.8342	1.6615	1.4162	1.1875	1.0000		
8	1.3717	1.7728	1.9493	1.8559	1.6294	1.3857	1.1728	1.0000	
9	1.3308	1.7421	2.0142	2.0186	1.8350	1.5903	1.3573	1.1600	1.0000
10	1.2933	1.6924	2.0330	2.1409	2.0233	1.7955	1.5508	1.3316	1.1488

7. To find the breakdown time distributions in process $a_{i,n}(t)$ it is enough to notice that they are equivalent to the working time distributions in a system of n identical facilities arranged in parallel, with independent breakdown processes $\tilde{a}^{(j)}(t), j = 1, 2, \ldots, n$ where the working time for each facility is an exponential random variable with parameter $\tilde{\lambda} = \mu$, the breakdown time for each facility is also exponential with parameter $\tilde{\mu} = \lambda$, and the breakdown process of the system is defined by

$$\tilde{a}_{n-i+1,n}(t) = \begin{cases} 1 & \text{for} \quad \sum_{j=1}^{n} \tilde{a}^{(j)}(t) \geqslant n-i+1, \\ 0 & \text{for} \quad \sum_{j=1}^{n} \tilde{a}^{(j)}(t) < n-i+1. \end{cases}$$

The breakdown time distribution of the system, therefore, may be found by applying the same methods as those used in finding the working time distribution in the previous parts of the paper. In particular the expected value of the breakdown time is

$$EY_{i,n} = \frac{\sum_{j=n-i+1}^{n} \binom{n}{j} q^j p^{n-j}}{(n-i+1)\mu \binom{n}{n-i+1} q^{n-i+1} p^{i-1}}.$$

From this as well as from the formulae (3) and (31) we obtain the obvious conclusion

$$P(a_{i,n}(t) = 1) = \frac{EX_{i,n}}{EX_{i,n}+EY_{i,n}} = \sum_{j=i}^{n} \binom{n}{j} p^j q^{n-j} = P(\beta_n(t) \geqslant i).$$

References

[1] А. Д. Епифанов, *Надежность автоматических систем*, Машиностроение, Москва 1964.

[2] Б. В. Гнеденко, Ю. К. Беляев, А. Д. Соловьев, *Математические методы в теории надежности*, Наука, Москва 1965.

[3] B. Kopociński, *Breakdown processes of systems in series*, Zastosow. Matem. 9 (1967), pp. 149-160.

[4] B. Kopociński and E. Trybusiowa, *Breakdown time distributions of systems in series*, Zastosow. Matem. 9 (1968), pp. 227-237.

[5] Б. А. Козлов, И. А. Ушаков, *Краткий справочник по расчету надежности радиоэлектронной аппаратуры*, Советское Радио, Москва 1966.

[6] Gerald H. Sandler, *System reliability engineering* (Russian ed. *Техника надежности систем*, Наука, Москва 1966).

[7] Н. М. Седякин, *Элементы теории случайных импульсных потоков*, Советское Радио, Москва 1965.

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PROCESY AWARII SYSTEMÓW URZĄDZEŃ O UKŁADZIE RÓWNOLEGŁYM

STRESZCZENIE

W pracy rozpatruje się systemy urządzeń złożone z n jednakowych urządzeń o niezależnych procesach awarii z wykładniczymi rozkładami czasu pracy i czasu awarii dla każdego urządzenia, połączonych w ten sposób, że system znajduje się w stanie pracy wtedy i tylko wtedy, gdy co najmniej i urządzeń jest w stanie pracy. Symbolicznie proces awarii urządzenia definiujemy w następujący sposób:

$$a^{(j)}(t) = \begin{cases} 1 & \text{jeśli urządzenie znajduje się w stanie pracy w chwili } t, \\ 0 & \text{jeśli urządzenie znajduje się w stanie awarii w chwili } t, \end{cases}$$

natomiast proces awarii systemu definiujemy przez

$$a_{i,n}(t) = \begin{cases} 1 & \text{jeśli } \beta_n(t) \geqslant i, \\ 0 & \text{jeśli } \beta_n(t) < i, \end{cases}$$

gdzie

$$\beta_n(t) = \sum_{j=1}^{n} a^{(j)}(t)$$

jest ilością urządzeń znajdujących się w stanie pracy w chwili t.

W niniejszej pracy znaleziono rozkłady czasu pracy i czasu awarii systemu w stacjonarnym procesie działania systemu, tzn. rozkłady czasu pracy systemu od zakończenia awarii do początku następnej awarii i rozkłady czasu awarii systemu od zakończenia pracy do rozpoczęcia następnej pracy.

Do obliczenia średnich i wariancji tych rozkładów znaleziono proste równania rekurencyjne i efektywne wzory. Parametry te obliczone przykładowo dla systemu urządzeń o jednakowych (jednostkowych) średnich czasach pracy i awarii urządzeń wskazują, że narzucająca się aproksymacja rozkładów czasu pracy i awarii systemu przez rozkład wykładniczy nie jest doskonała.

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ПРОЦЕССЫ АВАРИИ СИСТЕМ УСТРОЙСТВ СОЕДИНЕННЫХ ПАРАЛЛЕЛЬНО

РЕЗЮМЕ

В работе рассматриваются системы состоящие из n одинаковых устройств имеющих независимые процессы аварии с показательными распределениями времени работы и времени аварии для каждого устройства. Эти устройства соединены таким образом, что система исправна тогда и только тогда, когда по крайней мере i устройств исправных. Символически, процесс аварии устройства определяется следующим образом:

$$a^{(j)}(t) = \begin{cases} 1, & \text{если устройство исправно в момент } t, \\ 0, & \text{если устройство неисправно в момент } t. \end{cases}$$

Процесс аварии системы определен, как следует:

$$a_{i,n}(t) = \begin{cases} 1, & \text{если } \beta_n(t) \geqslant i, \\ 0, & \text{если } \beta_n(t) < i, \end{cases}$$

где

$$\beta_n(t) = \sum_{j=1}^{n} a^{(j)}(t)$$

есть число устройств исправных в момент t.

В настоящей работе найдены распределения времени работы и времени аварии системы в стационарном процессе работы системы, т.е. найдены распределения времени работы системы с конца аварии до начала следующей аварии и распределения времени аварии системы с конца работы до начала следующей работы системы.

Даны простые рекуррентные уравнения и эффективные формулы для определения математических ожиданий и дисперсий этих распределений. В качестве примера эти параметры вычислены для системы устройств с одинаковыми (единичными) средними временами работы и аварии. Вычисления указывают на то, что натуральная аппроксимация распределения времени работы и аварии показательным распределением не очень точна, даже для сравнительно больших n.
