ANNALES
POLONICI MATHEMATICI
XXXIV (1977)

Continuous extensions of multifunctions

by H. A. ANTOSIEWICZ* and A. CELLINA** (Los Angeles)

Abstract. Let X be a metric space, let Z be a normed space, and denote by $\mathscr{F}(Z)$ the metric space of non-empty bounded closed subsets of Z with Hausdorff distance. The authors prove the following generalization of Tietze's extension theorem. Given any non-empty closed set $A \subset X$ and any continuous mapping $F: A \to \mathscr{F}(Z)$, there exists a continuous mapping $G: X \to \mathscr{F}(Z)$ such that $G(a) = F(a)$ for every $a \in A$ and $G(x) \subset \mathrm{co} \bigcup_{a \in A} F(a)$ for every $x \in X$.

In 1951 Dugundji proved that every continuous mapping of a closed subset of a metric space into a locally convex topological linear space admits a continuous extension to the entire space, and that such an extension can be so constructed that its range lies in the convex hull of the range of the original mapping [1] (cf. also [2] p. 188).

In the present note we shall prove the following analogous assertion for a Hausdorff continuous multifunction.

THEOREM. *Let X be a metric space with distance d, let Z be a normed space, and let $\mathscr{F}(Z)$ be the metric space of non-empty bounded closed subsets of Z with Hausdorff distance h.*

Given any non-empty closed set $A \subset X$ and any continuous mapping $F: A \to \mathscr{F}(Z)$, there exists a continuous mapping $G: X \to \mathscr{F}(Z)$ such that $G(a) = F(a)$ for every $a \in A$ and $G(x) \subset \mathrm{co} \bigcup_{a \in A} F(a)$ for every $x \in X$.

Throughout the sequel we will denote by ϱ the distance in Z induced by the given norm and define, for any $C \in \mathscr{F}(Z)$ and any $r \geqslant 0$,

$$V(C; r) = \{z \in Z: \ \varrho(z, C) \leqslant r\},$$

---

* Department of Mathematics, University of Southern California, Los Angeles.
This work was done with partial support from the U. S. Army Research Office (Durham).

** Istituto Matematico "U. Dini", Firenze. Part of this work was done while the author was Visiting Professor in the Department of Mathematics, University of Southern California, Los Angeles.

where as usual

$$\varrho(z, C) = \inf\{\varrho(z, w):\ w \in C\}.$$

Thus $V(C, r) \in \mathscr{F}(Z)$ for any $C \in \mathscr{F}(Z)$ and any $r \geqslant 0$ and, in particular, $V(C; 0) = C$.

The Hausdorff distance between any two sets B, C in $\mathscr{F}(Z)$ is defined as

$$h(B, C) = \max\{\varrho^*(B, C), \varrho^*(C, B)\},$$

where

$$\varrho^*(B, C) = \sup\{\varrho(z, C):\ z \in B\}.$$

As a result, $h(B, C) \leqslant \varepsilon$ holds for some $\varepsilon > 0$ if and only if both $B \subset V(C; \varepsilon)$ and $C \subset V(B; \varepsilon)$. Moreover, if B_i, C_i, $i = 1, 2$ are any sets in $\mathscr{F}(Z)$ and $h(B_1 \cup B_2, C_1) \leqslant \varepsilon$, then $\varrho^*(C_2, B_1 \cup B_2) < \varepsilon$ implies $h(B_1 \cup B_2, C_1 \cup C_2) \leqslant \varepsilon$, as is readily verified.

For notational convenience, let us put

$$H = \operatorname{co} \bigcup_{a \in A} F(a)$$

and write Y for the subspace $\mathscr{C}A$ of X.

LEMMA 1. *Let $F_i \in \mathscr{F}(Z)$, $i = 1, 2$, be subsets of H and let $r_i \geqslant 0$, $i = 1, 2$, be given constants. If $h(F_1, F_2) \leqslant \varepsilon_1$ and $|r_1 - r_2| \leqslant \varepsilon_2$, then*

$$h\big(V(F_1; r_1) \cap H,\ V(F_2; r_2) \cap H\big) \leqslant \varepsilon_1 + \varepsilon_2.$$

For the proof we need only show that

$$\varrho\big(y,\ V(F_2; r_2) \cap H\big) \leqslant \varepsilon_1 + \varepsilon_2$$

for every $y \in V(F_1; r_1) \cap H$, because of symmetry. Given any $y \in V(F_1; r_1) \cap H$ and any $\eta > 0$, there exist a point $z_1 \in F_1$ with $\varrho(y, z_1) \leqslant r_1 + \eta/2$, a point $z_2 \in F_2$ with $\varrho(z_1, z_2) \leqslant \varepsilon_1 + \eta/2$, and a point $y' \in \operatorname{co}\{y, z_2\}$ with $\varrho(y', z_2) = \min\{r_2, \varrho(y, z_2)\}$. Clearly, $y' \in V(F_2; r_2) \cap H$. If $\varrho(y, z_2) \leqslant r_2$, then $y' = y$ and $\varrho(y, y') = 0$; if $\varrho(y, z_2) > r_2$, then

$$\varrho(y, y') = \varrho(y, z_2) - \varrho(y', z_2) \leqslant \varepsilon_1 + \varepsilon_2 + \eta.$$

Thus, in either case, $\varrho\big(y,\ V(F_2; r_2) \cap H\big) \leqslant \varepsilon_1 + \varepsilon_2 + \eta$, which implies the assertion.

LEMMA 2. *For any function Δ: $Y \to \boldsymbol{R}_+$ and any open covering $(U_y)_{y \in Y}$ of Y, there exist a continous function Λ: $Y \to \boldsymbol{R}_+$ and a mapping z: $Y \to Y$ such that $y \in U_{z(y)}$ and $\Lambda(y) \geqslant \Delta\big(z(y)\big)$ for every $y \in Y$.*

Indeed, let $(U'_y)_{y \in Y}$ be a precise locally finite open refinement of $(U_y)_{y \in Y}$, let $(\Pi_y)_{y \in Y}$ be a continuous partition of unity subordinate to $(U'_y)_{y \in Y}$, and define for every $y \in Y$

$$\Lambda(y) = \sum_{w \in Y} \Pi_w(y) \Delta(w).$$

Since $(\mathrm{Supp}(\Pi_w))_{w\in Y}$ is a locally finite closed covering of Y, it follows that Λ is continuous in Y, and that to each $y \in Y$ there corresponds a point $z(y) \in Y$ such that

$$\Delta(z(y)) = \min\{\Delta(w)\colon\ y \in \mathrm{int}\,\mathrm{Supp}(\Pi_w)\}.$$

Thus, $y \in U_{z(y)}$ and $\Lambda(y) \geqslant \Delta(z(y))$ for every $y \in Y$.

Let us now turn to the proof of the theorem.

Let W_y, for every $y \in Y$, be the open ball (in Y) with center y and radius $\frac{1}{2}d(y, A)$; let $(V_y)_{y\in Y}$ be a precise locally finite open refinement of the covering $(W_y)_{y\in Y}$ of Y; and let $(p_y)_{y\in Y}$ be a continuous partition of unity subordinate to $(V_y)_{y\in Y}$. Hence $(\mathrm{Supp}(p_y))_{y\in Y}$ is a locally finite closed refinement such that $\mathrm{Supp}(p_y) \subset V_y$ for every $y \in Y$. Indeed the local finiteness implies that, for every $y \in Y$, there is an open neighborhood $U_y \subset Y$ and a finite set $L(y) \subset Y$ such that

$$U_y \cap \mathrm{Supp}(p_w) \neq \emptyset$$

if and only if $w \in L(y)$. In particular, the set

$$L_0(y) = \{w \in Y\colon\ y \in \mathrm{int}\,\mathrm{Supp}(p_w)\}$$

is non-empty and finite and we may assume that

$$U_y \subset \bigcap_{w\in L_0(y)} \mathrm{int}\,\mathrm{Supp}(p_w).$$

Hence $L_0(y) \subset L(y)$ for every $y \in Y$ and $L_0(y) \subset L_0(z)$ for any $z \in U_y$. Moreover, $(U_y)_{y\in Y}$ is an open covering of Y.

Next, for every $y \in Y$, select a point $g(y) \in A$ with $d(g(y), y) < 2d(y, A)$, put

$$\Delta(y) = \sup\{h(F\circ g(w_1), F\circ g(w_2))\colon\ w_i \in L(y),\ i = 1, 2\}$$

and use Lemma 2 to define, relative to the covering $(U_y)_{y\in Y}$, the continuous function $\Lambda\colon Y \to \boldsymbol{R}_+$ and the mapping $z\colon Y \to Y$. Let

$$p(y) = \sup\{p_w(y)\colon\ w \in Y\}$$

for every $y \in Y$, which implies that p is continuous in Y and $p(y) > 0$ for every $y \in Y$, and introduce

$$r_w(y) = \Lambda(y)p_w(y)/p(y)$$

for every $w \in Y$ and every $y \in Y$, which implies that the family $(r_w)_{w\in Y}$ is equicontinuous in Y.

We assert that the multifunction $G\colon X \to 2^Z$ defined by setting $G(a) = F(a)$ for every $a \in A$ and

$$G(y) = \bigcup_{w\in L(y)} V(F\circ g(w); r_w(y)) \cap H$$

for every $y \in Y$, is the desired extension of F to X.

Clearly, for each $y \in Y$, $G(y)$ is a non-empty bounded closed subset of H because $L_0(y)$ is finite. Thus G maps X into $\mathscr{F}(Z)$ such that $G(x) \subset H$ for every $x \in X$.

Let us show that G is continuous at any $y_0 \in Y_0$.

Given $\varepsilon > 0$ there is an open neighbourhood $N(y_0) \subset Y$ of y_0 such that $|r_w(y) - r_w(y_0)| \leqslant \varepsilon$ for any $y \in N(y_0)$ whatever $w \in L(y_0)$. Since, by construction, $y_0 \in U_{y_0} \cap U_{z(y_0)}$ we may assume that $N(y_0) \subset U_{y_0} \cap U_{z(y_0)}$. This implies, in particular, that $L_0(y) \subset L(z(y_0))$ for every $y \in N(y_0)$.

We claim that $h(G(y), G(y_0)) \leqslant \varepsilon$ for every $y \in N(y_0)$.

Since $y \in N(y_0)$ implies $L_0(y) \supset L_0(y_0)$ and hence $L_0(y) = L_0(y_0) \cup K_0(y)$ with $K_0(y) = L_0(y) \cap \mathscr{C} L_0(y_0)$, we may write $G(y) = G_1(y) \cup G_2(y)$, where

$$G_1(y) = \bigcup_{w \in L_0(y_0)} V(F \circ g(w);\, r_w(y)) \cap H,$$

$$G_2(y) = \bigcup_{w \in K_0(y)} V(F \circ g(w);\, r_w(y)) \cap H.$$

Thus, by the remarks above, $h(G(y), G(y_0)) \leqslant \varepsilon$ will hold for every $y \in N(y_0)$ if both $h(G_1(y), G(y_0)) \leqslant \varepsilon$ and $\varrho^*(G_2(y), G(y_0)) \leqslant \varepsilon$ hold for every $y \in N(y_0)$.

Since Lemma 2 implies, for each $w \in L_0(y_0)$,

$$h\big(V(F \circ g(w);\, r_w(y)) \cap H,\ V(F \circ g(w);\, r_w(y_0)) \cap H\big) \leqslant \varepsilon$$

we deduce at once that $h(G_1(y), G(y_0)) \leqslant \varepsilon$ for every $y \in N(y_0)$.

To show that $\varrho^*(G_2(y), G(y_0)) \leqslant \varepsilon$ for every $y \in N(y_0)$, choose $\overline{w} \in L_0(y_0)$ such that $p_{\overline{w}}(y_0) = p(y_0)$ and hence $r_{\overline{w}}(y_0) = \Lambda(y_0)$, and observe that $L(z(y_0)) \supset L_0(y)$ for every $y \in N(y_0)$ implies, for every $w \in K_0(y)$,

$$h(F \circ g(w), F \circ g(\overline{w})) < \sup\{h(F \circ g(w_1),\ F \circ g(w_2)):\ w_i \in L(z(y_0)),\ i = 1, 2\}$$
$$= \Delta(z(y_0)) \leqslant \Lambda(y_0).$$

Therefore $F \circ g(w) \subset V(F \circ g(\overline{w});\, r_{\overline{w}}(y_0))$, and so $F \circ g(w) \subset G(y_0)$ and

$$\varrho^*(F \circ g(w), G(y_0)) = 0$$

for every $w \in K_0(y)$. It follows that, for every $y \in N(y_0)$,

$$\varrho^*\big(V(F \circ g(w);\, r_w(y)) \cap H, G(y_0)\big) \leqslant \varepsilon$$

whatever $w \in K_0(y)$, because $w \in K_0(y)$ implies $r_w(y_0) = 0$ and hence $|r_w(y)| \leqslant \varepsilon$, and consequently

$$\varrho^*(G_2(y), G(y_0)) \leqslant \varepsilon$$

for every $y \in N(y_0)$.

Let us now show that G is also continuous at any $y_0 \in \partial A$, which will complete the proof.

Given $\varepsilon > 0$, there exists a $\delta > 0$ such that $h(F(y_1), F(y_2)) \leqslant \varepsilon/4$ for any points $y_i \in A$ with $d(y_i, y_0) < \delta$, $i = 1, 2$. We assert that

$$h(G(y), G(y_0)) \leqslant \varepsilon$$

for any $y \in Y$ with $d(y, y_0) < \delta/5$. Clearly, this will be true if, for any $y \in Y$ with $d(y, y_0) < \delta/5$, both $h(F\circ g(w), F(y_0)) \leqslant \varepsilon/2$ and $|r_w(y)| \leqslant \varepsilon/2$ hold whatever $w \in L_0(y)$.

Suppose $y \in Y$ and $w \in L_0(y)$. Then $d(g(w), w) < 2d(w, A)$ and $d(y, w) < \frac{1}{10} d(w, A)$ by construction so that $d(w, A) < \frac{10}{9} d(y, y_0)$ and $d(g(w), y_0) < \frac{10}{3} d(y, y_0) < \delta$ and hence $h(F\circ g(w), F(y_0)) < \varepsilon/4$.

Evidently, for any $y \in Y$ with $d(y, y_0) < \delta/5$, $|r_w(y)| \leqslant \varepsilon/2$ will hold whatever $w \in L_0(y)$ provided

$$\Delta(y) \leqslant \sup\{\Delta(z)\colon\ y \in U_z\} \leqslant \varepsilon/2 .$$

Thus we need only show that, whatever $y \in Y$ with $d(y, y_0) < \delta/5$,

$$\Delta(z) = \sup\{h(F\circ g(w_1), F\circ g(w_2))\colon\ w_i \in L(z),\ i = 1, 2\} \leqslant \varepsilon/2$$

for every z such that $y \in U_z$.

Suppose $y \in Y$ and $d(y, y_0) < \delta/5$. Then $y \in U_z$ implies $d(y, z) < \frac{1}{10} d(z, A)$ and $d(z, A) < \frac{10}{9} d(y, A)$, and $w \in L(z)$ implies

$$d(w, z) < \tfrac{1}{10} d(w, A) + \tfrac{1}{10} d(z, A)$$

and

$$d(w, A) \leqslant d(w, y_0) \leqslant d(w, z) + d(z, y) + d(y, y_0) < \tfrac{1}{10} d(w, A) + \tfrac{11}{9} d(y, y_0)$$

so that $d(w, A) \leqslant \frac{110}{81} d(y, y_0)$ and $d(w, y_0) \leqslant \frac{110}{81} d(y, y_0)$. Thus $y \in U_z$ implies

$$d(g(w), y_0) \leqslant d(g(w), w) + d(w, y_0) \leqslant (\tfrac{22}{27})\cdot 5\, d(y, y_0)$$

whatever $w \in L(z)$, and hence

$$h(F\circ g(w_1), F\circ g(w_2)) \leqslant \varepsilon/4 + \varepsilon/4 = \varepsilon/2$$

whatever $w_i \in L(z)$, $i = 1, 2$. As a result, for every $y \in Y$ with $d(y, y_0) < \delta/5$, $\Delta(z) \leqslant \varepsilon/2$ holds for any z such that $y \in U_z$.

This completes the proof.

An extension theorem for upper semi-continuous multifunctions with compact convex values may be found in [3].

References

[1] J. Dugundji, *An extension of Tietze's Theorem*, Pacific J. Math. 1 (1951), p. 353–369.

[2] — *Topology*, Allyn and Bacon, Inc., Boston 1966.

[3] Tsoy-Wo Ma, *Topological degree of set-valued compact fields in locally convex spaces*, Dissertationes Math. (Rozprawy Mat.) 92 (1972), p. 1–47.

Reçu par la Rédaction le 18. 1. 1975