An analogue of a problem of J. Balázs and P. Turán, II
(Inequalities)

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1. Many theorems of the theory of approximation depend upon the
fact that a polynomial of degree \( n \) cannot change too rapidly. Markov's
inequality asserts that if \( P_n(x) \) is an arbitrary algebraic polynomial of
degree \( \leq n \) and if \( |P_n(x)| \leq M \) on \([-1,+1]\), then \( |P'_n(x)| \leq Mn^2 \). It is
also known that the extremal polynomial turns out to be a Tchebycheff
polynomial of the first kind. One can sometimes improve the estimate
of the derivative of algebraic polynomials. In 1940 Erdős [5] proved
that if an algebraic polynomial \( P_n(x) \) has only real roots that are outside
the interval \([-1,+1]\), then for \(-1 \leq x \leq +1\)

\[
\max_{-1 \leq x \leq +1} |P'_n(x)| \leq \frac{1}{2} \max_{-1 \leq x \leq +1} |P_n(x)|.
\]

In this connection the results of Lax [11] and Lorentz [8] deserve
to be mentioned. A beautiful theorem due to Turán [17] asserts that if
\( f(x) \) is an algebraic polynomial of \( n \)-th degree at most having all roots
in \([-1,+1]\), then

\[
\max_{-1 \leq x \leq +1} |f'(x)| > \frac{n^{1/2}}{6} \max_{-1 \leq x \leq +1} |f(x)|.
\]

In 1958, Balázs and Turán [4] obtained interesting inequalities which
arise from their consideration of a \((0,2)\) interpolation on \( \pi \)-abscissas,
\( \pi_n(x) = (1-x^2)P'_{n-1}(x) \), where \( P_n(x) \) is the Legendre polynomial of
degree \( n \). By \((0,2)\) interpolation they mean the problem of finding interpo-
latory polynomials \( R_n(x) \) of degree \( \leq 2n-1 \) for which

\[
R_n(x_k) = a_k, \quad R_n''(x_k) = b_k, \quad k = 1, 2, \ldots, n
\]

are prescribed. Here \( x_k \)'s are the zeros of \( \pi_n(x) \).

* This author acknowledges financial support for this work from the University of
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Theorem 1.1 (J. Balázs and P. Turán). Let \( n \) be even, and if \( Q_{2n-1}(x) \) is an arbitrary polynomial of degree \( \leq 2n-1 \) in \( x \) satisfying
\[
|Q_{2n-1}(x_k)| \leq A, \quad |Q'_{2n-1}(x_k)| \leq B, \quad k = 1, 2, \ldots, n,
\]
then for \(-1 \leq x \leq +1\) we have
\[
|Q_{2n-1}(x)| \leq \pi^6 nA + \frac{\pi^5 B}{n}
\]
and
\[
|Q'_{2n-1}(x)| \leq \pi^8 n^{5/2} A + \pi^5 n^{1/2} B.
\]

They showed also that (1.3) and (1.4) are the best possible in a certain sense. Results (1.3) and (1.4) are very significant. (1.3) gives the estimate of \( |Q_{2n-1}(x)| \) uniformly in \([-1, +1]\), where (1.4) gives the estimate of \( |Q'_{2n-1}(x)| \) uniformly in \([-1, +1]\), which is far better than what one can obtain from Markov's inequality applied to (1.3).

The object in this paper is to establish similar inequalities to those obtained by J. Balázs and P. Turán. In our case we will take \( x_k \)'s as the zeros of \((1-x^2)P_n(x)\), \( P_n(x) \) being the Legendre polynomial of degree \( \leq n \). It is of interest to point out that we shall start with a weaker assumption than in (1.2) and yet the estimates of the polynomial and its derivative are similar to those given in (1.3) and (1.4) as far as the order of \( n \) is concerned. The details of our result will be given in a next article.

2. Here we continue the investigation of the interpolatory polynomials \( R_n(x) \) of degree \( \leq 2n+1 \) in \( x \) for which
\[
R_n(x_i) = a_i \quad (i = 1, 2, \ldots, n+2),
\]
\[
R'_n(x_i) = b_i \quad (i = 2, 3, \ldots, n+1)
\]
are prescribed in advance. Here \( x_n \)'s are the zeros of \((1-x^2)P_n(x)\) given by
\[
1 = x_1 > x_2 > \ldots > x_{n+1} > x_{n+2} = -1.
\]

As mentioned in an earlier work [12], the interpolatory polynomials satisfying (2.1) are unique provided \( n \) is even. We shall prove the following theorem:

Theorem 2.1. Let \( Q_{2n+1}(x) \) be an arbitrary algebraic polynomial in \( x \) of degree \( \leq 2n+1 \) satisfying (\( n \) even)
\[
|Q_{2n+1}(x_i)| \leq A \quad (i = 1, 2, \ldots, n+2)
\]
and
\[
|Q''_{2n+1}(x_i)| \leq \frac{B}{1-x_i^2} \quad (i = 2, 3, \ldots, n+1);
\]
then for \(-1 \leq x \leq +1\) we have

\[
|Q_{2n+1}(x)| \leq \left(1894 nA + \frac{288}{n}B\right)d^{-3}
\]

and

\[
|Q'_{2n+1}(x)| \leq (2632 An^{5/2} + 910 n^{1/2}B)d^{-3}.
\]

Here \(x_i\)'s are the same as those given in (2.2), \(d\) is a positive constant independent of \(n\) and \(x\) and is given by (4.7).

Also (2.6) is the best possible in the sense that there exists a polynomial \(f_1(x)\) of degree \(\leq 2n+1\) in \(x\) satisfying (2.3) and (2.4) for which

\[
|f_1'(1)| > d_3 (An^{5/2} + Bn^{1/2})
\]

Here \(d_3\) is also a positive constant independent of \(n\) and \(x\).

We may point to a result of the first author [18] which is very closely concerned with this theorem. Let \(Q_{2n+1}(x)\) be an arbitrary polynomial of degree \(\leq 2n+1\) in \(x\) satisfying (2.3) and (2.4) but we choose \(x_i\)'s as the zeros of \((1 - x^2)T_n(x)\), \(T_n(x)\) being the Tchebycheff polynomials of the first kind; then we have for \(-1 \leq x \leq +1\)

\[
|Q_{2n+1}(x)| \leq d_4 \left(An^{5/2} + \frac{B}{n^{1/2}}\right)
\]

and

\[
|Q'_{2n+1}(x)| \leq d_4 (An^{5/2} + Bn^{1/2}).
\]

Moreover, (2.8) and (2.9) are both the best possible. In comparison with theorem 2.1 we observe here that the growth of the derivative is constant \(n\) times only, though (2.6) and (2.9) are similar. Similar results on other nodes were given in [19]. The results of Fejer [6] and those of Landau [10] deserve to be mentioned.

3. Preliminaries. The explicit form of the polynomials \(R_n(x)\) obtained in an earlier work [12] satisfying (2.1) is given by

\[
R_n(x) = \sum_{i=1}^{n+2} a_i r_i(x) + \sum_{i=2}^{n+1} b_i q_i(x),
\]

where

\[
q_i(x) = \frac{(1 - x^2)P_n(x)q_{n-1}(x)}{2F_n'(x_i)} \quad (i = 2, 3, \ldots, n+1),
\]

where \(q_{n-1}(x)\) is a polynomial in \(x\) of degree \(\leq n-1\) and is given by

\[
(1-x^2)^{1/2} q_{n-1}(x) = A_i \int_{-1}^{x} P_n(t) w(t) \, dt + \int_{-1}^{x} l_i(t) w(t) \, dt,
\]

\[
A_i \int_{-1}^{-1} P_n(t) w(t) \, dt = - \int_{-1}^{+1} l_i(t) w(t) \, dt.
\]
Here \( l_i(t) \) denotes the fundamental polynomial of the Lagrange interpolation and is given by

\[
(3.5) \quad l_i(t) = \frac{P_n(t)}{(t-x_i)P_n'(x_i)}, \quad w(t) = (1-t^2)^{-1/2}.
\]

We note that \( q_{n-1}(x) \) also depends on \( i \). The fundamental polynomials of the first kind are given by

\[
(3.6) \quad r_1(x) = \frac{1-x}{2} P_n^2(x) - (1-\frac{1}{x^2}) P_n(x) P_n'(x) - \frac{1}{2} (1-x^2)^{1/2} P_n(x) \int_{-1}^{x} P_n'(t) w(t) dt,
\]

\[
(3.7) \quad r_{n+1}(x) = \frac{1-x}{2} P_n^2(x) + \frac{1}{2} (1-x^2) P_n(x) P_n'(x) - \frac{1}{2} (1-x^2)^{1/2} P_n(x) \int_{-1}^{x} P_n'(t) w(t) dt,
\]

and for \( i = 2, 3, \ldots, n+1 \) we have

\[
(3.8) \quad r_i(x) = \left( \frac{1-x^2}{2(1-x_i^2)} \right) \left[ l_i^2(x) + \frac{l_i(x) P_n'(x)}{P_n'(x_i)} + \frac{P_n(x) s_{n-1}(x)}{P_n'(x_i)} \right] + c_i q_i(x).
\]

Here \( s_{n-1}(x) \) is a polynomial in \( x \) of degree \( \leq n-1 \) (it depends on \( i \) also) and is given by

\[
(3.9) \quad c_i = \frac{n(n-1)}{1-x_i^2} \frac{x_i^2}{(1-x_i^2)},
\]

\[
(3.10) \quad (1-x^2)^{1/2} s_{n-1}(x) = B_i \int_{-1}^{x} P_n(t) w(t) dt - \int_{-1}^{x} l_i'(t) w(t) dt.
\]

Here \( B_i \) is given by

\[
(3.11) \quad B_i \int_{-1}^{1} P_n(t) w(t) dt = \int_{-1}^{1} l_i'(t) w(t) dt.
\]

We remark that throughout this paper \( n \) will be even, \( w(t) = (1-t^2)^{-1/2} \), and the constants \( d_1, d_2, \ldots \) will be independent of \( x \) and \( n \) and will represent some positive numbers.

4. **Some known results concerning Legendre polynomials.** We know that for \(-1 \leq x \leq +1\) we have

\[
(4.1) \quad (1-x^2)^{1/4} |P_n(x)| \leq \left( \frac{2}{n\pi} \right)^{1/2},
\]

\[
(4.2) \quad (1-x^2)^{3/4} |P_{n-1}(x)| \leq (2n)^{1/2},
\]

\[
(4.3) \quad |P_n'(x)| \leq \frac{n(n+1)}{2},
\]
(4.4) \[ |P_n(x)| \leq 1, \]
(4.5) \[ |(1-x^2)^{1/2} P'_n(x)| \leq n, \]
(4.6) \[ \sum_{i=2}^{n+1} \frac{(1-x^2)P_i^2(x)}{1-x_i^2} = 1 - P_n^2(x) < 1, \]
(4.7) \[ |P'_n(x_i)| \geq \frac{dn^2}{i - 1)^{3/2}}, \quad i = 2, 3, \ldots, \frac{1}{2} n + 1, \]
(4.8) \[ |P'_n(x_i)| \geq \frac{dn^2}{n-i+2)^{3/2}}, \quad i = \frac{1}{2} n + 2, \ldots, n+1, \]
(4.9) \[ (1-x_i^2) > (i-1)^2 n^{-2}, \quad i = 2, 3, \ldots, \frac{1}{2} n + 1, \]
\[ > (n-i+2)^2 n^{-2}, \quad i = \frac{1}{2} n + 2, \ldots, n+1. \]

Let \( x_i = \cos \theta_i, i = 2, 3, \ldots, n+1, \) be the zeros of \( P_n(x); \) then it follows from Bruns inequalities

(4.10) \[ \frac{(i-\frac{1}{2}) \pi}{(n+\frac{1}{2})} < \theta_i < \frac{(i-1) \pi}{(n+\frac{1}{2})}, \quad i = 2, 3, \ldots, n+1. \]

From (4.2) and (4.5) it follows immediately that

(4.11) \[ |(1-x_i^2)^{1/2} P'_n(x_i)| \leq 2^{1/4} n^{3/4}. \]

5. Here we will obtain certain results concerning Legendre polynomial. From Sansone [13], p. 200, we have

(5.1) \[ P_{2r}(x) = 2 \sum_{j=0}^{r-1} \alpha_j a_{2r-j} \cos(2r-2j) \theta + a_j^2, \quad x = \cos \theta, \]

where

(5.2) \[ \alpha_j = \left( \frac{2}{\pi(2j+l)} \right)^{1/2}, \quad 0 < l < 1, \quad a_0 = 1. \]

Using (5.1), (5.2) and observing that Tchebycheff polynomials \( T_n(x) \) are orthogonal with weight function \((1-x^2)^{-1/2}\) in \([-1, +1]\), we obtain

(5.3) \[ \int_{-1}^{+1} [P_{2r}(x) - P_{2r+2}(x)] w(x) dx = \pi (a_r^2 - a_{r+1}^2) < \pi^{1/2}. \]

From the Christoffel formula (see [13], 197) we have

(5.4) \[ l_i(x) = (1-x^2)^{-1} [P'_n(x_i)]^{-2} \left[ \sum_{r=1}^{n-1} P'_r(x_i) \{ P_r(x_i) - P_{r+1}(x_i) \} + \right. \]
\[ \left. + \sum_{l=1}^{r} (r+1) \right] \sum_{r=1}^{n-1} P_r(x_i)P_{2r+1}(x_i) - \frac{1}{1+x^2}, \]

(5.5) \[ l_i(x) = (1-x^2)^{-1} [P'_n(x_i)]^{-2} \left[ 1 + \sum_{r=1}^{n-1} (2r+1) P_r(x_i)P_{2r}(x_i) \right]. \]
We need the differential equation for $P_r(x)$ in the form

\begin{equation}
(1 - x^2)P'_r(x) = -r(r+1) \int_{-1}^{x} P_r(t) \, dt, \quad r = 1, 2, \ldots, n. \tag{5.6}
\end{equation}

It follows from (5.5) and (5.6) that

\begin{equation}
\int_{-1}^{x} l_i(t) \, dt = (1 - x_i^2)^{-1} [P'_n(x_i)]^{-2} \left[ 1 + x - \sum_{r=1}^{n-1} \frac{(2r+1)}{r(r+1)} (1 - x^2)P'_r(x)P_r(x_i) \right]; \tag{5.7}
\end{equation}

using (4.4) and (4.2) we obtain

\begin{equation}
\left| \int_{-1}^{x} l_i(t) \, dt \right| \leq (1 - x_i^2)^{-1} [P'_n(x_i)]^{-2} \left[ 2 + 2 \sqrt{2} (1 - x^2)^{1/4} \sum_{r=1}^{n-1} r^{-1/2} \right]. \tag{5.8}
\end{equation}

Therefore, using (4.1) and (5.8), we get

\begin{equation}
\left| P_n(x) \int_{-1}^{x} l_i(t) \, dt \right| \leq 7 (1 - x_i^2)^{-1} [P'_n(x_i)]^{-2}. \tag{5.9}
\end{equation}

We shall very often need the estimate

\begin{equation}
\sum_{k=2}^{m} k^{-\delta} \leq \frac{m^{1-\delta} - 1}{1 - \delta}, \quad \delta \neq 1. \tag{5.10}
\end{equation}

From (5.8), (4.5) and (4.2) we have

\begin{equation}
\left| (1 - x^2)^{1/2} P'_n(x) \int_{-1}^{x} l_i(t) \, dt \right| \leq 10n (1 - x_i^2)^{-1} [P'_n(x_i)]^{-2}. \tag{5.11}
\end{equation}

From a result of Saxena [14] we have

\begin{equation}
\left| P_n(x) \int_{-1}^{x} l_i(t) \, dt \right| \leq n^{-3/2} |l_i(x)|, \quad i = 2, 3, \ldots, n+1, \tag{5.12}
\end{equation}

and as a consequence of a beautiful inequality (4.6) due to Egerváry and Turán [4] we obtain

\begin{equation}
| (1 - x^2)^{1/2} l_i(x) | \leq (1 - x_i^2)^{1/2}. \tag{5.13}
\end{equation}

Therefore, using (5.12) and (5.13), we obtain

\begin{equation}
\left| (1 - x^2)^{1/2} P_n(x) \int_{-1}^{x} l_i(t) \, dt \right| \leq (1 - x_i^2)^{1/2} n^{-3/2}. \tag{5.14}
\end{equation}
Let us denote \( t = \cos \gamma, x = \cos \theta \), let \( x_i = \cos \theta_i \) be the zeros of \( P_n(x) \), for \( i = 2, 3, \ldots, n+1 \). Since

\[
\sin \theta_i \leq \sin \theta + \sin \gamma \leq 2 \sin \frac{\theta + \gamma}{2}, \quad 0 \leq \gamma \leq \pi, \ 0 < \theta_i < \pi,
\]

and similarly

\[
\sin \gamma \leq 2 \sin \frac{\theta + \gamma}{2}.
\]

From these inequalities we obtain

\[
(5.15) \quad (\sin \theta)_{\frac{5}{8}} (\sin \gamma)_{\frac{3}{8}} \leq 2 \sin \frac{\theta + \gamma}{2}.
\]

From (4.1) and (5.15) we have

\[
(5.16) \quad \left| \int_0^{\pi} \cot \frac{\theta + \gamma}{2} P_n(\cos \gamma) \, d\gamma \right| \leq 2 \sqrt{\frac{2}{\pi n}} (\sin \theta_i)^{-\frac{5}{8}} \int_0^{\pi} \frac{1}{(\sin \gamma)^{\frac{7}{8}}} \, d\gamma \leq 42n^{-\frac{1}{2}} (\sin \theta_i)^{-\frac{5}{8}}.
\]

Here we have used the fact that

\[
\int_0^{\pi/2} \frac{d\gamma}{(\sin \gamma)^{\frac{7}{8}}} = \left( \frac{\pi}{2} \right)^{\frac{7}{8}} \int_0^{\pi/2} \gamma^{-\frac{7}{8}} \, d\gamma = 4\pi.
\]

By a simple computation it follows from (5.4) that

\[
(5.17) \quad |(1-x^2)^{1/4} l_i'(x)| \leq \frac{4n^2}{(1-x_i^2)^{\frac{7}{8}} [P_n'(x_i)]^2}.
\]

From a known identity [4] we have

\[
(5.18) \quad \sum_{i=2}^{n+1} \frac{1}{(1-x_i^2) [P_n'(x_i)]^2} = 1.
\]

On using Cauchy’s inequality, (5.18) and (4.5) we obtain

\[
(5.19) \quad \sum_{i=2}^{n+1} \frac{(1-x^2)^{1/2} |l_i'(x)|}{(1-x_i^2) |P_n'(x_i)|} \leq 1.
\]

Since \(|(1-x^2)^{1/2} P_n'(x)| \leq n\), we have

\[
(5.20) \quad \sum_{i=2}^{n+1} \frac{|(1-x^2) P_n'(x) l_i(x)|}{|(1-x_i^2) P_n'(x_i)|} \leq n.
\]

6. Here we will prove some lemmas which will be directly applied to the estimates of the fundamental polynomials.
Lemma 6.1. The following estimates are valid:

\[(6.1)\quad \frac{2}{n+1} < \left| \int_{-1}^{1} P_n(t) w(t) \, dt \right| < \frac{2}{n} \quad (n \text{ even}),\]

\[(6.2)\quad \left| \int_{-1}^{x} P_n(t) w(t) \, dt \right| < \frac{13}{n+1}, \quad -1 \leq x \leq +1,\]

\[(6.3)\quad \left| \int_{-1}^{x} P'_n(t) w(t) \, dt \right| \leq 26n, \quad -1 \leq x \leq 1.\]

This lemma was proved earlier in our work [12] (see Lemmas 4.2, 4.3 and 5.1).

Lemma 6.2. For \(i = 2, 3, \ldots, n+1\) the following estimates are valid:

\[(6.4)\quad \left| \int_{-1}^{1} l_i(t) w(t) \, dt \right| \leq 12 (1-a_i^2)^{-13/8} [P'_n(x_i)]^{-2},\]

\[(6.5)\quad \left| \int_{-1}^{x} l_i(t) w(t) \, dt \right| \leq 162n^{3/2} (1-a_i^2)^{-7/4} [P'_n(x_i)]^{-2},\]

\[(6.6)\quad \left| \int_{-1}^{1} l_i(t) w(t) \, dt \right| \leq 14n^{3/2} (1-a_i^2)^{-7/4} [P'_n(x_i)]^{-2}.\]

Proof of (6.5) and (6.6) were given in Lemmas 5.4 and 5.3 (see [12]). Here (6.4) is in a slightly improved version of that given in [12]. From (5.3), (6.1) and (5.4) we have

\[
\left| \int_{-1}^{1} l_i(t) w(t) \, dt \right| \leq (1-a_i^2)^{-1} [P'_n(x_i)]^{-2} \left[ \sum_{r=1}^{n-1} \frac{|P_r(x_i)|}{r^2} + \frac{2}{n} |P'_{n+1}(x_i)| \right]
\leq 12 (1-a_i^2)^{-13/8} [P'_n(x_i)]^{-2}.
\]

The last estimate follows from (4.11) and (5.10). This proves the lemma.

Lemma 6.3. For \(i = 2, 3, \ldots, n+1, -1 \leq x \leq +1\), the following estimates are valid:

\[(6.7)\quad \left| P_n(x) \int_{-1}^{x} l_i(t) w(t) \, dt \right| \leq 70 (1-a_i^2)^{-25/16} [P'_n(x_i)]^{-2},\]

\[(6.8)\quad \left| (1-a_i^2)^{1/2} P'_n(x) \int_{-1}^{x} l_i(t) w(t) \, dt \right| \leq 73n (x-1/2)^{-25/16} [P'_n(x_i)]^{-2},\]

\[(6.9)\quad \left| (1-a_i^2)^{1/2} P_n(x) \int_{-1}^{x} l_i(t) w(t) \, dt \right| \leq 65n^{-1/2} (1-a_i^2)^{-25/16} [P'_n(x_i)]^{-2}.\]
The proof of this lemma depends on the following observation, which can be verified by a simple computation,

\[(6.10) \quad \int_{-1}^{x} l_i(t) w(t) \, dt = \left(1 - x_i^2\right)^{-1/2} \left[ \int_0^{\theta_i + \gamma} \frac{\cot \frac{\theta_i + \gamma}{2} P_n(\cos \gamma) \, d\gamma}{P_n'(x_i)} + \int_{-1}^{x} l_i(t) \, dt \right].\]

Here we put \(t = \cos \gamma\), \(\cos \theta = x\) and use the fact that \(\sin \theta = (\sin \theta - \sin \theta_i) + \sin \theta_i\).

From (5.16) and (5.9) we have

\[(6.11) \quad \left| P_n(x) \int_{-1}^{x} l_i(t) w(t) \, dt \right| \leq \frac{42 n^{-1/2} (1 - x_i^2)^{-13/16}}{|P_n'(x_i)|} + \frac{7 (1 - x_i^2)^{-3/2}}{|P_n'(x_i)|^2}.\]

But, using (4.2), we have

\[(6.12) \quad \frac{1}{|P_n'(x_i)|} \leq \frac{|P_n'(x_i)|}{|P_n'(x_i)|^2} \leq \frac{(2n)^{1/2} (1 - x_i^2)^{-3/4}}{|P_n'(x_i)|^2}.\]

Combining (6.11) and (6.12) we have (6.7). Using (4.5), (6.10) and (5.11), we have

\[\left| (1 - x^2)^{1/2} P_n'(x) \int_{-1}^{x} l_i(t) w(t) \, dt \right| \leq \frac{42 n^{1/2} (1 - x_i^2)^{-13/16}}{|P_n'(x_i)|} + \frac{10 n (1 - x_i^2)^{-3/2}}{|P_n'(x_i)|^2},\]

using (6.12) we get (6.8). To prove (6.9) we again use (6.10), (5.14) and we get

\[\left| (1 - x^2)^{1/2} P_n(x) \int_{-1}^{x} l_i(t) w(t) \, dt \right| \leq \frac{42 n^{-1} (1 - x_i^2)^{-13/16}}{|P_n'(x_i)|} + n^{-3/2}.\]

Again applying (6.12) to both terms on the right-hand side, we get (6.9).

7. Estimation of the fundamental polynomials of the second kind.

**Lemma 7.1.** For \(-1 \leq x \leq +1\) we have

\[(7.1) \quad \sum_{i=2}^{n+1} \frac{|g_i(x)|}{1-x_i^2} \leq 288 n^{-1} d^{-3},\]

where \(d\) is the constant given in (4.7).

**Proof.** First we remark that this inequality is slightly improved with regard to that obtained in [12]. From (3.4), (6.1) and (6.4) we have

\[(7.2) \quad |A_i| \leq 6(n+1)(1-x_i^2)^{-13/8}|P_n'(x_i)|^{-2}, \quad i = 2, 3, \ldots, n+1.\]
Therefore, from (3.3), (6.2), (4.1) and (6.9) we obtain
\[
(7.3) \quad \left| \frac{(1-x_i^2)P_n(x)g_{n-1}(x)}{(1-x_i^2)} \right| \leq \frac{39(1-x_i^2)^{-21/8}n^{-1/2} + 33n^{-1/2}(1-x_i^2)^{-41/16}}{|P'_n(x_i)|^3} \\
\leq 72(1-x_i^2)^{-21/8}n^{-1/2}|[P'_n(x_i)]^{-3}|;
\]
using (4.7)–(4.9) and (5.10) we have
\[
(7.4) \quad \sum_{t=2}^{n+1} \frac{1}{(1-x_i^2)^{21/8}|[P'_n(x_i)]^3|} \leq 4n^{-1/2}d^{-3}.
\]
From (3.2), (7.3) and (7.4) we have (7.1).

8. Estimation of the fundamental polynomials of the first kind.

Lemma 8.1. For $-1 \leq x \leq +1$ the following estimates are valid:
\[
(8.1) \quad |r_1(x)| \leq 19n^{1/2}, \quad |r_{n+2}(x)| \leq 19n^{1/2},
\]
\[
(8.2) \quad \sum_{t=1}^{n+2} |r_t(x)| \leq 39n + 1855nd^{-3} \leq 1894nd^{-3},
\]
where $d$ is a positive constant given in 4.7 and $d \leq 1$.

Proof. (8.1) was proved in [12] (see Lemma 5.2). We also remark that inequality (8.2) is in an improved from in comparison to what we obtained in [12]. Using (4.10) and the inequality $\sin \theta \leq \frac{2}{\pi} \theta$, $0 \leq \theta \leq \frac{\pi}{2}$, we obtain
\[
(8.3) \quad (1-x_i^2)^{-1} \leq 4n^2.
\]
Hence, from (3.9) we have
\[
(8.4) \quad |C_i| \leq 6n^2(1-x_i^2)^{-1}.
\]
Therefore, using (7.1) we have
\[
(8.5) \quad \sum_{t=2}^{n+1} |C_tg_t(x)| \leq 1728nd^{-3}.
\]

From (3.11), (6.1) and (6.6) we obtain
\[
(8.6) \quad |B_t| \leq 7n^{3/2}(n+1)(1-x_i^2)^{-7/4}|[P'_n(x_i)]^{-2}.
\]
Starting from (3.10) and using (8.6), (6.2) and (6.5), we obtain
\[
(8.7) \quad |(1-x_i^2)^{1/2}g_{n-1}(x)| \leq 253n^{3/2}(1-x_i^2)^{-7/4}|[P'_n(x_i)]^{-2}.
\]
From (4.7)–(4.9) it follows that
\[
(8.8) \quad \sum_{t=2}^{n+1} (1-x_i^2)^{-11/4}|[P'_n(x_i)]^{-3}| \leq d^{-3}n^{-1/4}.
\]
Now, using (4.1), (8.7), (8.8) we obtain

\begin{equation}
\left| \frac{1 - x^2}{2(1 - x_i^2)} P_n(x) q_n-1(x) \right| \leq \frac{253}{2} n^{5/2} n^{-1/2} d^{-3} n^{-1/4} = 127 n^{3/4} d^{-3}.
\end{equation}

Combining results (8.5), (8.9), (8.1), (4.6) and (5.20) we obtain

\begin{equation}
\sum_{i=1}^{n+2} |r_i(x)| \leq 38n^{1/2} + \frac{1}{2} + \frac{n}{2} + 127n^{3/4} d^{-3} + 1728n d^{-3},
\end{equation}

from which the lemma follows.

9. Estimation of the derivative of the fundamental polynomial of the second kind.

**Lemma 9.1.** For $-1 \leq x \leq +1$, we have

\begin{equation}
\sum_{i=2}^{n+1} \frac{|q_i'(x)|}{1 - x_i^2} \leq 910n^{1/2} d^{-3},
\end{equation}

where $d$ is a constant given in (4.7).

**Proof.** From the representation of $q_i(x)$ as given in (3.2) and (3.3) we obtain

\begin{equation}
\frac{q_i'(x)}{1 - x_i^2} = \frac{[(1 - x^2) P_n'(x) - x P_n(x)] q_{n-1}(x) + A_i P_n^2(x) + P_n(x) l_i(x)}{2(1 - x_i^2) P_n'(x_i)};
\end{equation}

using (3.3), (7.2), (6.8) and (6.2) we have

\begin{equation}
\left| \frac{1 - x^2}{2(1 - x_i^2)} P_n'(x) q_{n-1}(x) \right| \leq 39n(1 - x_i^2)^{-21/8} \left[ |P_n'(x_i)|^{-3} \right] + 37n(1 - x_i^2)^{-41/16} \left[ |P_n'(x_i)|^{-3} \right] \leq 76n(1 - x_i^2)^{-21/8} |P_n'(x_i)|^{-3}.
\end{equation}

Hence, using (7.4) we have

\begin{equation}
\sum_{i=2}^{n+1} \left| \frac{1 - x^2}{2(1 - x_i^2)} P_n'(x) q_{n-1}(x) \right| \leq 304n^{1/2} d^{-3}.
\end{equation}

From (3.3), (6.2), (6.7) and (7.2) we obtain

\begin{equation}
\left| \frac{1 - x^2}{2(1 - x_i^2)} P_n'(x) q_{n-1}(x) \right| \leq 74(1 - x_i^2)^{-21/8} |P_n'(x_i)|^{-3}.
\end{equation}

Here we shall use a well-known Lemma due to Schur (see [9]), which states that if an arbitrary polynomial of degree $\leq p$ in $x$ satisfies
\[ |(1 - x^2)^{1/2}t_p(x)| \leq A, \text{ then } |t_p(x)| \leq Ap. \text{ Thus applying the above lemma to (9.5) we obtain} \]

\[ \binom{xP_n(x)q_{n-1}(x)}{2(1 - x_i^2)P'_n(x_i)} \leq 148n(1 - x_i^2)^{-21/4}|P'_n(x_i)|^{-3} \]

Using (7.4) we have

\[ \sum_{i=2}^{n+1} \binom{xP_n(x)q_{n-1}(x)}{2(1 - x_i^2)P'_n(x_i)} \leq 148n \cdot 4n^{-1/2}d^{-3} = 592n^{1/2}d^{-3} \]

From (4.4), (7.2) and (7.4) we obtain

\[ \sum_{i=2}^{n+1} \frac{|A_i P^2_n(x)|}{|2P'_n(x_i)(1 - x_i^2)|} \leq 3n \cdot 4n^{-1/2}d^{-3} = 12n^{1/2}d^{-3}. \]

Since

\[ 1 = \frac{(1 - x^2) + (x^2 - x_i^2)}{1 - x_i^2}, \]

we can write

\[ \frac{P_n(x)l_i(x)}{2(1 - x_i^2)P'_n(x_i)} = \frac{(1 - x^2)P_n(x)l_i(x) + (x^2 - x_i^2)P_n(x)l_i(x)}{2(1 - x_i^2)^2P'_n(x_i)}. \]

From (4.6) it follows that

\[ |(1 - x^2)^{1/2}l_i(x)| \leq (1 - x_i^2)^{1/2} \]

and from (4.1)

\[ |(1 - x^2)^{1/2}P_n(x)| \leq n^{-1/2}. \]

Therefore, using (4.7)-(4.9) we have

\[ \sum_{i=2}^{n+1} \frac{|(1 - x^2)P_n(x)l_i(x)|}{2(1 - x_i^2)^2 |P'_n(x_i)|} \leq \frac{1}{2} n^{-1/2} \sum_{i=2}^{n+1} \frac{1}{(1 - x_i^2)^{3/2}|P'_n(x_i)|} \]

\[ \leq \frac{n^{1/2}}{2d} \sum_{i=2}^{n+1} i^{-3/2} = n^{1/2}d^{-1}. \]

Since \( |P_n(x)| \leq 1 \) we have from (4.7)-(4.9)

\[ \sum_{i=2}^{n+1} \left| \frac{(x^2 - x_i^2)P_n(x)l_i(x)}{2(1 - x_i^2)^2P'_n(x_i)} \right| \leq \sum_{i=2}^{n+1} \frac{1}{(1 - x_i^2)^2 |P'_n(x_i)|^2} \leq d^{-2}n^{1/2}. \]

Therefore, from (9.9), (9.12) and (9.13) we obtain

\[ \sum_{i=2}^{n+1} \left| \frac{P_n(x)l_i(x)}{2(1 - x_i^2)P'_n(x_i)} \right| \leq n^{1/2}d^{-1}[1 + d^{-1}]. \]
Now combining the estimates from (9.4), (9.7), (9.8) and (9.14) and substituting them in (9.2), we have
\[
\sum_{i=2}^{n+1} \frac{|g_i(x)|}{(1-x_i^2)} \leq 304n^{1/2}d^{-3} + 592n^{1/2}d^{-3} + 12n^{1/2}d^{-3} + n^{1/2}d^{-1}(1 + d^{-1}) \\
= (908d^{-3} + d^{-2} + d^{-1})n^{1/2} \leq 910d^{-3}n^{1/2}, \quad \text{since } d \leq 1.
\]

This proves the lemma.

10. Investigation of the estimate of the derivative of the fundamental polynomials of the first kind.

In order to obtain the above estimate, we follow another idea due to Balázs and Turán [3]. In our case we shall require the corresponding results based on a quasi-Hermite-Fejér interpolation [15] introduced by Professor P. Szász. He considered the problem of finding the interpolation polynomials of degree \( \leq 2n+1 \) whose values are prescribed at the points given by (2.2) and whose first derivatives are prescribed only at the zeros of \( P_n(x) \). These polynomials exist uniquely and have the form

\[
Q_n(x) = \sum_{k=1}^{n+2} y_k E_k(x) + \sum_{k=2}^{n+1} y_k E_k(x).
\]

Here \( Q_n(x_i) = y_i, i = 1, 2, \ldots, n+2, Q_n(x_i) = y_i', i = 2, 3, \ldots, n+1. \)

Further, owing to the uniqueness of these polynomials, if \( \chi(x) \) is an arbitrary polynomial of degree \( \leq 2n+1 \), we have

\[
\chi(x) = \sum_{k=1}^{n+2} \chi(x_k) E_k(x) + \sum_{k=2}^{n+1} \chi'(x_k) E_k(x).
\]

Further, we observe from the results of Szász [15] that

\[
E_k(x) \geq 0 \quad \text{and} \quad \sum_{k=1}^{n+2} E_k(x) = 1.
\]

Let us consider in our case \( \chi(x) = r_i'(x) \), which is a polynomial in \( x \) of degree \( \leq 2n \) and such that \( r_i'(x_j) = 0, j = 2, 3, \ldots, n+1, i = 1, 2, \ldots, n+2. \) Therefore from (10.2) we have

\[
r_i'(x) = \sum_{k=1}^{n+2} r_i'(x_k) E_k(x).
\]

Since it was pointed out above that \( E_k(x) \geq 0 \) and \( \sum_{k=1}^{n+2} E_k(x) \equiv 1 \), we have

\[
|r_i'(x)| \leq \max_k |r_i'(x_k)|.
\]

Actually, we have even

\[
\max_k |r_i'(x)| = \max_k |r_i'(x_k)|,
\]
i.e. all polynomials $r'_i(x)$ attain their absolute maxima with respect to $[-1, +1]$ at one of the $x_j$ points.

**Lemma 10.1.** For $-1 \leq x \leq +1$ we have

\begin{equation}
|r'_1(x)| \leq 14n^2, \quad |r'_{n+2}(x)| \leq 14n^2.
\end{equation}

**Proof.** From (3.6) we have for $j = 2, 3, \ldots, n+1$,

\begin{equation}
r'_i(x_j) = -\frac{1}{2} (1-x_j^2)[P'_n(x_j)]^2 - \frac{1}{2} (1-x_j^2)P'_n(x_j) \int_{-1}^{x} P'_n(t) w(t) dt.
\end{equation}

Using (4.5) and (6.3) we obtain

\begin{equation}
|r'_i(x_j)| \leq \frac{n^2}{2} + 13n^4 \leq 14n^2, \quad j = 2, 3, \ldots, n+1.
\end{equation}

Again from (3.6) we have $r'_1(1) = 3P'_n(1) + \frac{1}{2}$. Therefore, using (4.3) we get $|r'_1(1)| \leq 4n^2$. Similarly $|r'_1(-1)| \leq 4n^2$. Hence, using (10.4) we obtain $|r'_i(x)| \leq 14n^2$.

This proves the lemma.

**Lemma 10.2.** For $-1 \leq x \leq +1$ we have

\begin{equation}
\sum_{i=2}^{n+1} |r'_i(x)| \leq 2604d^{-3}n^{5/2},
\end{equation}

where $d$ is a positive constant given in (4.7).

**Proof.** Denote the terms on the right-hand side of (3.8) by $I_1(x, x_i)$, $I_2(x, x_i)$, $I_3(x, x_i)$ and $I_4(x, x_i)$, respectively. Since

\begin{equation}
l_i(x_j) = \begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases} \quad \text{and} \quad l'_i(x_i) = \frac{x_i}{(1-x_i^2)},
\end{equation}

we easily obtain

\begin{equation}
I'_i(x_j, x_i) = 0 \quad \text{for } i, j = 2, 3, \ldots, n+1.
\end{equation}

Here and in what follows the dash denotes differentiation with respect to $x$. Using (10.7) and the differential equation for $P_n(x)$ we obtain

\begin{equation}
I'_2(x_i, x_i) = 0, \quad i = 2, 3, \ldots, n+1.
\end{equation}

Using (5.17) and (4.2), we obtain for $i, j = 2, 3, \ldots, n+1$

\begin{equation}
|I'_2(x_i, x_i)| = \frac{(1-x_i^2)P'_n(x_i) l'_i(x_i)}{2(1-x_i^2)P'_n(x_i)}, \quad i \neq j,
\end{equation}

\begin{equation}
|I'_3(x_j, x_i)| \leq 3n^{5/2}(1-x_i^2)^{-11/4}|P'_n(x_i)|^{-3}.
\end{equation}
Similarly, using (4.5) and (8.7) we obtain
\[
(10.11) \quad |I'_i(x_j, x_i)| = \left| \frac{(1-x_j^2)P'_n(x_i)q_{n-1}(x_j)}{2(1-x_i^2)P'_n(x_i)} \right|, \quad i, j = 2, 3, \ldots, n+1,
\]
\[
\leq 127n^{5/2}(1-x_i^2)^{-11/4}|P'_n(x_i)|^{-3}, \quad i, j = 2, 3, \ldots, n+1.
\]

Since
\[
I'_i(x_j, x_i) = C_iq'_i(x_j) = \frac{C_i(1-x_i^2)P'_n(x_i)q_{n-1}(x_j)}{2P'_n(x_i)},
\]
using (8.4) and (9.3) we have
\[
(10.12) \quad |I'_i(x_j, x_i)| \leq 6n^3 \cdot 76n(1-x_i^2)^{-21/8}|P'_n(x_i)|^{-3}
\]
\[
= 456n^3(1-x_i^2)^{-21/8}|P'_n(x_i)|^{-3}.
\]

Therefore, from (10.8)–(10.12) we have
\[
|r'_i(x_j)| \leq 130n^{5/2}(1-x_i^2)^{-11/4}|P'_n(x_i)|^{-3} + 456n^3(1-x_i^2)^{-21/8}|P'_n(x_i)|^{-3}.
\]
From (8.3) we have $(1-x_i^2)^{-1/8} \leq \frac{3}{2}n^{1/4}$. So we get
\[
|r'_i(x_j)| \leq 195(1-x_i^2)^{-21/8}n^{11/4}|P'_n(x_i)|^{-3} + 456n^3(1-x_i^2)^{-21/8}|P'_n(x_i)|^{-3}
\]
\[
\leq 651n^3(1-x_i^2)^{-21/8}|P'_n(x_i)|^{-3}, \quad i, j = 2, 3, \ldots, n+1.
\]

But a simple computation shows that
\[
|r'_i(\pm 1)| \leq 300n^3(1-x_i^2)^{-21/8}|P'_n(x_i)|^{-3}.
\]

Hence using (10.4) we have
\[
\max |r'_i(x)| = \max_k |r'_i(x_k)| \leq 651n^3(1-x_i^2)^{-21/8}|P'_n(x_i)|^{-3}.
\]

Using (7.4) we have finally
\[
\sum_{i=2}^{n+1} |r'_i(x)| \leq 651n^3 \cdot 4n^{-1/2}d^3 = 2604n^{5/2}d^{-3}.
\]

This proves the lemma.

**11. Proof of Theorem 2.1.** The above estimates very easily prove our theorem. From the uniqueness theorem [12] we have for an arbitrary polynomial $Q_{2n+1}(x)$ of degree $\leq 2n+1$ the representation
\[
(11.1) \quad Q_{2n+1}(x) = \sum_{i=1}^{n+2} Q_{2n+1}(x_i)r_i(x) + \sum_{i=2}^{n+1} Q_{2n+1}'(x_i)q_i(x).
\]

Here $r_i(x)$ and $q_i(x)$ are fundamental polynomials stated in Article 3. Further, differentiating both sides of (11.1) with respect to $x$ we obtain
\[
(11.2) \quad Q'_{2n+1}(x) = \sum_{i=1}^{n+2} Q_{2n+1}(x_i)r'_i(x) + \sum_{i=2}^{n+1} Q_{2n+1}'(x_i)q'_i(x).
\]
Using (11.1), (2.3) and the estimate of the fundamental polynomials given by (8.2) and (7.1), we obtain

\[ |Q_{2n+1}(x)| \leq 1894nd^{-3}A + \frac{288}{n}d^{-3}B, \]

which proves assertion (2.5). To prove (2.6) we start with (11.2), (2.3) and the estimates of the derivatives of the fundamental polynomials given by (9.1), (10.5) and (10.6) and we obtain

\[ |Q'_{2n+1}(x)| \leq (28n^2 + 2604n^{5/2}d^{-3})A + 914n^{1/2}d^{-3}B; \]

but \( d < 1 \) and therefore we get

\[ |Q'_{2n+1}(x)| \leq 2632n^{5/2}d^{-3}A + 914n^{1/2}d^{-3}B, \]

which in turn proves (2.6). This proves the theorem.

12. Here we will show that our theorem 2.1 is essentially the best possible. For this purpose we need the following Lemma:

**Lemma 12.1.** There exist positive constants \( d_1 \) and \( d_2 \) such that

\[
\sum_{i=1}^{n+1} \frac{|q_i'(1)|}{1-x_i^2} \geq d_1 n^{1/2}
\]

and

\[
\sum_{i=1}^{n+2} |r_i'(1)| \geq d_2 n^{5/2}
\]

for \( n \) sufficiently large.

**Proof.** From (3.2) and (3.3) it follows that

\[
\frac{q_i'(1)}{1-x_i^2} = \frac{A_i + l_i(1)}{(1-x_i^2)P_n'(x_i)};
\]

using (4.7)—(4.9) we have

\[
\sum_{i=2}^{n+1} \frac{|l_i(1)|}{|(1-x_i^2)P_n'(x_i)|} \leq 2 \sum_{i=2}^{n+1} \frac{1}{(1-x_i^2)^2[P_n'(x_i)]^2} \leq 2d^{-2}\log n.
\]

Here we have used the fact that

\[
|l_i(1)| = \left| \frac{1}{(1-x_i)P_n'(x_i)} \right| \leq \frac{|(1+x_i)|}{|(1-x_i^2)P_n'(x_i)|} \leq \frac{2}{|(1-x_i^2)P_n'(x_i)|}.
\]

From the definition of \( A_i \) as given in (3.4) and using (6.1) we obtain

\[
E \equiv \sum_{i=2}^{n+1} \frac{|A_i|}{(1-x_i^2)[P_n'(x_i)]} \geq \frac{n}{2} \sum_{i=2}^{n+1} \frac{1}{(1-x_i^2)[P_n'(x_i)]} \int_{-1}^{+1} l_i(t)w(t)dt;
\]
using (4.2) and the fact that \((1 - x_i^2)^{1/4} \leq 1\), we obtain
\[
E \geq \frac{n}{2\sqrt{2}n} \sum_{i=2}^{n+1} \left| \int_{-1}^{+1} l_i(t) w(t) \, dt \right|.
\]
\[
\sum_{i=1}^{n+1} l_i(t) = 1 \quad \text{and the observation}
\]
\[
\left| \int_{-1}^{+1} l_i(t) w(t) \, dt \right| \geq \int_{-1}^{+1} l_i(t) w(t) \, dt
\]
lead us to state
\[
E \geq \frac{1}{2^{3/2}} n^{1/2} \int_{-1}^{+1} (1 - t^2)^{1/2} \, dt \geq \frac{\pi}{2^{3/2}} n^{1/2}.
\]

Hence using (12.3)—(12.6) we obtain (12.1). To prove (12.2) we use the notation adopted in Article 10 and using (4.7)—(4.9) we easily obtain
\[
\sum_{i=2}^{n+1} |I_1'(1, x_i)| = \sum_{i=2}^{n+1} \frac{l_i^2(1)}{(1 - x_i^2)} \leq 4 \sum_{i=2}^{n+1} \frac{1}{(1 - x_i^2)^3 |P_n'(x_i)|} \leq 12n^2d^{-2}.
\]

Similarly, using (4.3) and (12.4)
\[
\sum_{i=2}^{n+1} |I_2'(1, x_i)| = \sum_{i=2}^{n+1} \left| \frac{P_n'(1)l_i(1)}{(1 - x_i^2)P_n'(x_i)} \right| \leq n(n + 1) d^{-2} \log n.
\]

Lastly from (3.8)—(3.11) we have
\[
I_3'(1, x_i) = \frac{|s_{n-1}(1)|}{|(1 - x_i^2)P_n'(x_i)|} \leq \frac{2[|B_i| + |l_i(1)|]}{|(1 - x_i^2)P_n'(x_i)|}.
\]

Using (8.6) and (5.17) we obtain
\[
|I_3'(1, x_i)| \leq 14 (1 - x_i^2)^{-11/4} |P_n'(x_i)|^{-3} (n + 1)^{5/2} + 8n^2 (1 - x_i^2)^{-11/4} |P_n'(x_i)|^{-3} \\
\leq 22 (1 - x_i^2)^{-11/4} |P_n'(x_i)|^{-3} (n + 1)^{5/2}.
\]

Therefore, using (4.7)—(4.9) we have
\[
\sum_{i=2}^{n+1} |I_3'(1, x_i)| \leq 22 (n + 1)^{5/2} \sum_{i=2}^{n+1} \frac{1}{(1 - x_i^2)^{11/4} |P_n'(x_i)|^3} \\
\leq 22 (n + 1)^{2} d^{-3} \log n.
\]

Combining (12.7)—(12.11) and using (12.1) we at once obtain (12.1). This proves the lemma.
Now we will explicitly exhibit the polynomial $f_1(x)$ which has properties (2.3) and (2.4) and satisfies (2.7), as remarked earlier at the end of the statement of the theorem. Consider

$$f_1(x) = \sum_{k=1}^{n+2} A \text{sign } r_k'(1)r_k(x) + \sum_{k=2}^{n+1} \frac{B}{1-x_k^2} \text{sign } q_k'(1)q_k(x);$$

then

$$f_1(x_i) = A, \quad i = 1, 2, \ldots, n+2,$$

and

$$f_1''(x_i) = \frac{B}{1-x_i^2}, \quad i = 1, 2, \ldots, n+1.$$

Therefore $f_1(x)$ satisfies (2.3) and (2.4). Using the above lemma we obtain (2.7) as desired. This proves the result completely.

References


Reçu par la Rédaction le 5. 11. 1968