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An analogue of a problem of J. Balázs and P. Turán, II
(Inequalities)

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1. Many theorems of the theory of approximation depend upon the fact that a polynomial of degree n cannot change too rapidly. Markov's inequality asserts that if $P_n(x)$ is an arbitrary algebraic polynomial of degree $\leqslant n$ and if $|P_n(x)| \leqslant M$ on $[-1,+1]$, then $|P_n'(x)| \leqslant Mn^2$. It is also known that the extremal polynomial turns out to be a Tchebycheff polynomial of the first kind. One can sometimes improve the estimate of the derivative of algebraic polynomials. In 1940 Erdös [5] proved that if an algebraic polynomial $P_n(x)$ has only real roots that are outside the interval $[-1,+1]$, then for $-1 \leqslant x \leqslant +1$

$$\max_{-1\leqslant x\leqslant +1} |P_n'(x)| \leqslant \tfrac{1}{2} en \max_{-1\leqslant x\leqslant +1} |P_n(x)|.$$

In this connection the results of Lax [11] and Lorentz [8] deserve to be mentioned. A beautiful theorem due to Turán [17] asserts that if $f(x)$ is an algebraic polynomial of n-th degree at most having all roots in $[-1,+1]$, then

$$\max_{-1\leqslant x\leqslant +1} |f'(x)| > \frac{n^{1/2}}{6} \max_{-1\leqslant x\leqslant +1} |f(x)|.$$

In 1958, Balázs and Turán [4] obtained interesting inequalities which arise from their consideration of a $(0,2)$ interpolation on π-abscissas, $\pi_n(x) = (1-x^2)P_{n-1}'(x)$, where $P_n(x)$ is the Legendre polynomial of degree n. By $(0,2)$ interpolation they mean the problem of finding interpolatory polynomials $R_n(x)$ of degree $\leqslant 2n-1$ for which

(1.1) $$R_n(x_k) = a_k, \quad R_n''(x_k) = b_k, \quad k = 1,2,\ldots,n$$

are prescribed. Here x_k's are the zeros of $\pi_n(x)$.

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THEOREM 1.1 (J. Balázs and P. Turán). *Let n be even, and if $Q_{2n-1}(x)$ is an arbitrary polynomial of degree $\leqslant 2n-1$ in x satisfying*

$$(1.2) \qquad |Q_{2n-1}(x_k)| \leqslant A, \quad |Q''_{2n-1}(x_k)| \leqslant B, \quad k = 1, 2, \ldots, n,$$

then for $-1 \leqslant x \leqslant +1$ we have

$$(1.3) \qquad |Q_{2n-1}(x)| \leqslant \pi^6 nA + \frac{\pi^5 B}{n}$$

and

$$(1.4) \qquad |Q'_{2n-1}(x)| \leqslant \pi^8 n^{5/2} A + \pi^5 n^{1/2} B.$$

They showed also that (1.3) and (1.4) are the best possible in a certain sense. Results (1.3) and (1.4) are very significant. (1.3) gives the estimate of $|Q_{2n-1}(x)|$ uniformly in $[-1,+1]$, where (1.4) gives the estimate of $|Q'_{2n-1}(x)|$ uniformly in $[-1,+1]$, which is far better than what one can obtain from Markov's inequality applied to (1.3).

The object in this paper is to establish similar inequalities to those obtained by J. Balázs and P. Turán. In our case we will take x_k's as the zeros of $(1-x^2)P_n(x)$, $P_n(x)$ being the Legendre polynomial of degree $\leqslant n$. It is of interest to point out that we shall start with a weaker assumption than in (1.2) and yet the estimates of the polynomial and its derivative are similar to those given in (1.3) and (1.4) as far as the order of n is concerned. The details of our result will be given in a next article.

2. Here we continue the investigation of the interpolatory polynomials $R_n(x)$ of degree $\leqslant 2n+1$ in x for which

$$(2.1) \qquad \begin{aligned} R_n(x_i) &= a_i \quad (i = 1, 2, \ldots, n+2), \\ R''_n(x_i) &= b_i \quad (i = 2, 3, \ldots, n+1) \end{aligned}$$

are prescribed in advance. Here x_k's are the zeros of $(1-x^2)P_n(x)$ given by

$$(2.2) \qquad 1 = x_1 > x_2 > \ldots > x_{n+1} > x_{n+2} = -1.$$

As mentioned in an earlier work [12], the interpolatory polynomials satisfying (2.1) are unique provided n is even. We shall prove the following theorem:

THEOREM 2.1. *Let $Q_{2n+1}(x)$ be an arbitrary algebraic polynomial in x of degree $\leqslant 2n+1$ satisfying (n even)*

$$(2.3) \qquad |Q_{2n+1}(x_i)| \leqslant A \quad (i = 1, 2, \ldots, n+2)$$

and

$$(2.4) \qquad |Q''_{2n+1}(x_i)| \leqslant \frac{B}{1-x_i^2} \quad (i = 2, 3, \ldots, n+1);$$

then for $-1 \leqslant x \leqslant +1$ *we have*

$$|Q_{2n+1}(x)| \leqslant \left(1894\, nA + \frac{288}{n} B\right) d^{-3} \tag{2.5}$$

and

$$|Q'_{2n+1}(x)| \leqslant (2632\, An^{5/2} + 910\, n^{1/2} B)\, d^{-3}. \tag{2.6}$$

Here x_i's are the same as those given in (2.2), d is a positive constant independent of n and x and is given by (4.7).

Also (2.6) is the best possible in the sense that there exists a polynomial $f_1(x)$ of degree $\leqslant 2n+1$ in x satisfying (2.3) and (2.4) for which

$$|f_1'(1)| > d_3(An^{5/2} + Bn^{1/2}) \tag{2.7}$$

Here d_3 is also a positive constant independent of n and x.

We may point to a result of the first author [18] which is very closely concerned with this theorem. Let $Q_{2n+1}(x)$ be an arbitrary polynomial of degree $\leqslant 2n+1$ in x satisfying (2.3) and (2.4) but we choose x_i's as the zeros of $(1-x^2)T_n(x)$, $T_n(x)$ being the Tchebycheff polynomials of the first kind; then we have for $-1 \leqslant x \leqslant +1$

$$|Q_{2n+1}(x)| \leqslant d_4\left(An^{3/2} + \frac{B}{n^{1/2}}\right) \tag{2.8}$$

and

$$|Q'_{2n+1}(x)| \leqslant d_5(An^{5/2} + Bn^{1/2}). \tag{2.9}$$

Moreover, (2.8) and (2.9) are both the best possible. In comparison with theorem 2.1 we observe here that the growth of the derivative is constant n times only, though (2.6) and (2.9) are similar. Similar results on other nodes were given in [19]. The results of Fejer [6] and those of Landau [10] deserve to be mentioned.

3. Preliminaries. The explicit form of the polynomials $R_n(x)$ obtained in an earlier work [12] satisfying (2.1) is given by

$$R_n(x) = \sum_{i=1}^{n+2} a_i r_i(x) + \sum_{i=2}^{n+1} b_i \varrho_i(x), \tag{3.1}$$

where

$$\varrho_i(x) = \frac{(1-x^2)P_n(x)q_{n-1}(x)}{2P_n'(x_i)} \quad (i = 2, 3, \ldots, n+1), \tag{3.2}$$

where $q_{n-1}(x)$ is a polynomial in x of degree $\leqslant n-1$ and is given by

$$(1-x^2)^{1/2} q_{n-1}(x) = A_i \int_{-1}^{x} P_n(t)w(t)\,dt + \int_{-1}^{x} l_i(t)w(t)\,dt, \tag{3.3}$$

$$A_i \int_{-1}^{+1} P_n(t)w(t)\,dt = -\int_{-1}^{+1} l_i(t)w(t)\,dt. \tag{3.4}$$

Here $l_i(t)$ denotes the fundamental polynomial of the Lagrange interpolation and is given by

$$(3.5)\qquad l_i(t) = \frac{P_n(t)}{(t-x_i)P_n'(x_i)}, \quad w(t) = (1-t^2)^{-1/2}.$$

We note that $q_{n-1}(x)$ also depends on i. The fundamental polynomials of the first kind are given by

$$(3.6)\qquad r_1(x)$$
$$= \frac{1+x}{2}P_n^2(x) - \left(1-\frac{1}{2}x^2\right)P_n(x)P_n'(x) - \frac{1}{2}(1-x^2)^{1/2}P_n(x)\int_{-1}^{x} P_n'(t)w(t)\,dt,$$

$$(3.7)\qquad r_{n+2}(x)$$
$$= \frac{1-x}{2}P_n^2(x) + \frac{1}{2}(1-x^2)P_n(x)P_n'(x) - \frac{1}{2}(1-x^2)^{1/2}P_n(x)\int_{-1}^{x} P_n'(t)w(t)\,dt,$$

and for $i = 2, 3, \ldots, n+1$ we have

$$(3.8)\qquad r_i(x) = \frac{(1-x^2)}{2(1-x_i^2)}\left[l_i^2(x) + \frac{l_i(x)P_n'(x)}{P_n'(x_i)} + \frac{P_n(x)s_{n-1}(x)}{P_n'(x_i)}\right] + c_i\varrho_i(x).$$

Here $s_{n-1}(x)$ is a polynomial in x of degree $\leqslant n-1$ (it depends on i also) and is given by

$$(3.9)\qquad c_i = \frac{n(n+1)}{1-x_i^2} - \frac{x_i^2}{(1-x_i^2)^2},$$

$$(3.10)\qquad (1-x^2)^{1/2}s_{n-1}(x) = B_i\int_{-1}^{x} P_n(t)w(t)\,dt - \int_{-1}^{x} t l_i'(t)w(t)\,dt.$$

Here B_i is given by

$$(3.11)\qquad B_i\int_{-1}^{+1} P_n(t)w(t)\,dt = \int_{-1}^{+1} t l_i'(t)w(t)\,dt.$$

We remark that throughout this paper n will be even, $w(t) = (1-t^2)^{-1/2}$, and the constants $d_1, d_2, \ldots$ will be independent of x and n and will represent some positive numbers.

4. Some known results concerning Legendre polynomials. We know that for $-1 \leqslant x \leqslant +1$ we have

$$(4.1)\qquad (1-x^2)^{1/4}|P_n(x)| \leqslant \left(\frac{2}{n\pi}\right)^{1/2},$$

$$(4.2)\qquad (1-x^2)^{3/4}|P_{n-1}'(x)| \leqslant (2n)^{1/2},$$

$$(4.3)\qquad |P_n'(x)| \leqslant \frac{n(n+1)}{2},$$

$$|P_n(x)| \leqslant 1, \tag{4.4}$$

$$|(1-x^2)^{1/2}P_n'(x)| \leqslant n, \tag{4.5}$$

$$\sum_{i=2}^{n+1} \frac{(1-x^2)l_i^2(x)}{1-x_i^2} = 1-P_n^2(x) < 1, \tag{4.6}$$

$$|P_n'(x_i)| \geqslant dn^2(i-1)^{-3/2}, \quad i = 2, 3, \ldots, \tfrac{1}{2}n+1, \tag{4.7}$$

$$|P_n'(x_i)| \geqslant dn^2(n-i+2)^{-3/2}, \quad i = \tfrac{1}{2}n+2, \ldots, n+1, \tag{4.8}$$

$$(1-x_i^2) > (i-1)^2 n^{-2}, \quad i = 2, 3, \ldots, \tfrac{1}{2}n+1, \tag{4.9}$$

$$> (n-i+2)^2 n^{-2}, \quad i = \tfrac{1}{2}n+2, \ldots, n+1.$$

Let $x_i = \cos\theta_i$, $i = 2, 3, \ldots, n+1$, be the zeros of $P_n(x)$; then it follows from Bruns inequalities

$$\frac{(i-\frac{1}{2})\pi}{(n+\frac{1}{2})} < \theta_i < \frac{(i-1)\pi}{(n+\frac{1}{2})}, \quad i = 2, 3, \ldots, n+1. \tag{4.10}$$

From (4.2) and (4.5) it follows immediately that

$$|(1-x_i^2)^{5/8}P_r'(x_i)| \leqslant 2^{1/4}r^{3/4}. \tag{4.11}$$

5. Here we will obtain certain results concerning Legendre polynomial. From Sansone [13], p. 200, we have

$$P_{2r}(x) = 2\sum_{j=0}^{r-1} a_j a_{2r-j}\cos(2r-2j)\theta + a_r^2, \quad x = \cos\theta, \tag{5.1}$$

where

$$a_j = \left(\frac{2}{\pi(2j+l)}\right)^{1/2}, \quad 0 < l < 1,\ a_0 = 1. \tag{5.2}$$

Using (5.1), (5.2) and observing that Tchebycheff polynomials $T_n(x)$ are orthogonal with weight function $(1-x^2)^{-1/2}$ in $[-1, +1]$, we obtain

$$\int_{-1}^{+1} [P_{2r}(x) - P_{2r+2}(x)]w(x)\,dx = \pi(a_r^2 - a_{r+1}^2) < r^{1/2}. \tag{5.3}$$

From the Christoffel formula (see [13], 197) we have

$$l_i(x) = (1-x_i^2)^{-1}[P_n'(x_i)]^{-2}\Big[\sum_{r=1}^{n-1} P_r'(x_i)\{P_{r-1}(x) - P_{r+1}(x)\} + \tag{5.4}$$

$$+P_n'(x_i)P_{n-1}(x) + P_{n+1}'(x_i)P_n(x)\Big],$$

$$l_i(x) = (1-x_i^2)^{-1}[P_n'(x_i)]^{-2}\Big[1 + \sum_{r=1}^{n-1}(2r+1)P_r(x_i)P_r(x)\Big]. \tag{5.5}$$

We need the differential equation for $P_r(x)$ in the form

$$(5.6)\qquad (1-x^2)P_r'(x) = -r(r+1)\int_{-1}^{x} P_r(t)\,dt, \qquad r = 1, 2, \ldots, n.$$

It follows from (5.5) and (5.6) that

$$(5.7)\qquad \int_{-1}^{x} l_i(t)\,dt$$
$$= (1-x_i^2)^{-1}[P_n'(x_i)]^{-2}\left[1+x-\sum_{r=1}^{n-1}\frac{(2r+1)}{r(r+1)}(1-x^2)P_r'(x)P_r(x_i)\right];$$

using (4.4) and (4.2) we obtain

$$(5.8)\qquad \left|\int_{-1}^{x} l_i(t)\,dt\right| \leqslant (1-x_i^2)^{-1}[P_n'(x_i)]^{-2}\left[2+2\sqrt{2}(1-x^2)^{1/4}\sum_{r=1}^{n-1} r^{-1/2}\right].$$

Therefore, using (4.1) and (5.8), we get

$$(5.9)\qquad \left|P_n(x)\int_{-1}^{x} l_i(t)\,dt\right| \leqslant 7(1-x_i^2)^{-1}[P_n'(x_i)]^{-2}.$$

We shall very often need the estimate

$$(5.10)\qquad \sum_{k=2}^{m} k^{-\delta} \leqslant \frac{m^{1-\delta}-1}{1-\delta}, \qquad \delta \neq 1.$$

From (5.8), (4.5) and (4.2) we have

$$(5.11)\qquad \left|(1-x^2)^{1/2}P_n'(x)\int_{-1}^{x} l_i(t)\,dt\right| \leqslant 10n(1-x_i^2)^{-1}[P_n'(x_i)]^{-2}.$$

From a result of Saxena [14] we have

$$(5.12)\qquad \left|P_n(x)\int_{-1}^{x} l_i(t)\,dt\right| \leqslant n^{-3/2}|l_i(x)|, \qquad i = 2, 3, \ldots, n+1,$$

and as a consequence of a beautiful inequality (4.6) due to Egervary and Turán [4] we obtain

$$(5.13)\qquad |(1-x^2)^{1/2} l_i(x)| \leqslant (1-x_i^2)^{1/2}.$$

Therefore, using (5.12) and (5.13), we obtain

$$(5.14)\qquad \left|(1-x^2)^{1/2}P_n(x)\int_{-1}^{x} l_i(t)\,dt\right| \leqslant (1-x_i^2)^{1/2} n^{-3/2}.$$

Let us denote $t = \cos\gamma$, $x = \cos\theta$, let $x_i = \cos\theta_i$ be the zeros of $P_n(x)$, for $i = 2, 3, \ldots, n+1$. Since

$$\sin\theta_i \leqslant \sin\theta_i + \sin\gamma \leqslant 2\sin\frac{\theta_i+\gamma}{2}, \quad 0 \leqslant \gamma \leqslant \pi, 0 < \theta_i < \pi,$$

and similarly

$$\sin\gamma \leqslant 2\sin\frac{\theta_i+\gamma}{2}.$$

From these inequalities we obtain

$$(5.15) \qquad (\sin\theta_i)^{5/8}(\sin\gamma)^{3/8} \leqslant 2\sin\frac{\theta_i+\gamma}{2}.$$

From (4.1) and (5.15) we have

$$(5.16) \qquad \left|\int_\theta^\pi \cot\frac{(\theta_i+\gamma)}{2} P_n(\cos\gamma)\,d\gamma\right| \leqslant 2\sqrt{\frac{2}{\pi n}}(\sin\theta_i)^{-5/8}\int_0^\pi \frac{1}{(\sin\gamma)^{7/8}}\,d\gamma$$
$$\leqslant 42n^{-1/2}(\sin\theta_i)^{-5/8}.$$

Here we have used the fact that

$$\int_0^{\pi/2} \frac{d\gamma}{(\sin\gamma)^{7/8}} \leqslant \left(\frac{\pi}{2}\right)^{7/8} \int_0^{\pi/2} \gamma^{-7/8}\,d\gamma = 4\pi.$$

By a simple computation it follows from (5.4) that

$$(5.17) \qquad |(1-x^2)^{1/4} l_i'(x)| \leqslant \frac{4n^2}{(1-x_i^2)^{7/4}[P_n'(x_i)]^2}.$$

From a known identity [4] we have

$$(5.18) \qquad \sum_{i=2}^{n+1} \frac{1}{(1-x_i^2)[P_n'(x_i)]^2} = 1.$$

On using Cauchy's inequality, (5.18) and (4.5) we obtain

$$(5.19) \qquad \sum_{i=2}^{n+1} \frac{(1-x^2)^{1/2}|l_i(x)|}{(1-x_i^2)|P_n'(x_i)|} \leqslant 1.$$

Since $|(1-x^2)^{1/2}P_n'(x)| \leqslant n$, we have

$$(5.20) \qquad \sum_{i=2}^{n+1} \frac{|(1-x^2)P_n'(x)l_i(x)|}{|(1-x_i^2)P_n'(x_i)|} \leqslant n.$$

6. Here we will prove some lemmas which will be directly applied to the estimates of the fundamental polynomials.

LEMMA 6.1. *The following estimates are valid:*

$$\frac{2}{n+1} < \Big| \int_{-1}^{+1} P_n(t)w(t)\,dt \Big| < \frac{2}{n} \quad (n \text{ even}), \tag{6.1}$$

$$\Big| \int_{-1}^{x} P_n(t)w(t)\,dt \Big| < \frac{13}{n+1}, \quad -1 \leqslant x \leqslant +1, \tag{6.2}$$

$$\Big| \int_{-1}^{x} P'_n(t)w(t)\,dt \Big| \leqslant 26n, \quad -1 \leqslant x \leqslant 1. \tag{6.3}$$

This lemma was proved earlier in our work [12] (see Lemmas 4.2, 4.3 and 5.1).

LEMMA 6.2. *For* $i = 2, 3, \ldots, n+1$ *the following estimates are valid:*

$$\Big| \int_{-1}^{+1} l_i(t)w(t)\,dt \Big| \leqslant 12(1-x_i^2)^{-13/8}[P'_n(x_i)]^{-2}, \tag{6.4}$$

$$\Big| \int_{-1}^{x} t l'_i(t)w(t)\,dt \Big| \leqslant 162 n^{3/2}(1-x_i^2)^{-7/4}[P'_n(x_i)]^{-2}, \tag{6.5}$$

$$\Big| \int_{-1}^{+1} t l'_i(t)w(t)\,dt \Big| \leqslant 14 n^{3/2}(1-x_i^2)^{-7/4}[P'_n(x_i)]^{-2}. \tag{6.6}$$

Proof of (6.5) and (6.6) were given in Lemmas 5.4 and 5.3 (see [12]). Here (6.4) is in a slightly improved version of that given in [12]. From (5.3), (6.1) and (5.4) we have

$$\Big| \int_{-1}^{+1} l_i(t)w(t)\,dt \Big| \leqslant (1-x_i^2)^{-1}[P'_n(x_i)]^{-2}\Big[\sum_{r=1}^{n-1} \frac{|P'_r(x_i)|}{r^2} + \frac{2}{n}|P'_{n+1}(x_i)|\Big]$$
$$\leqslant 12(1-x_i^2)^{-13/8}[P'_n(x_i)]^{-2}.$$

The last estimate follows from (4.11) and (5.10). This proves the lemma.

LEMMA 6.3. *For* $i = 2, 3, \ldots, n+1$, $-1 \leqslant x \leqslant +1$, *the following estimates are valid:*

$$\Big| P_n(x) \int_{-1}^{x} l_i(t)w(t)\,dt \Big| \leqslant 70(1-x_i^2)^{-25/16}[P'_n(x_i)]^{-2}, \tag{6.7}$$

$$\Big| (1-x^2)^{1/2} P'_n(x) \int_{-1}^{x} l_i(t)w(t)\,dt \Big| \leqslant 73n(x-1_i^2)^{-25/16}[P'_n(x_i)]^{-2}, \tag{6.8}$$

$$\Big| (1-x^2)^{1/2} P_n(x) \int_{-1}^{x} l_i(t)w(t)\,dt \Big| \leqslant 65 n^{-1/2}(1-x_i^2)^{-25/16}[P'_n(x_i)]^{-2}. \tag{6.9}$$

The proof of this lemma depends on the following observation, which can be verified by a simple computation,

(6.10)
$$\int_{-1}^{x} l_i(t)w(t)\,dt = (1-x_i^2)^{-1/2}\left[\int_{\theta}^{\pi} \frac{\cot\frac{(\theta_i+\gamma)}{2}P_n(\cos\gamma)\,d\gamma}{P_n'(x_i)} + \int_{-1}^{x} l_i(t)\,dt\right].$$

Here we put $t = \cos\gamma$, $\cos\theta = x$ and use the fact that $\sin\theta = (\sin\theta - \sin\theta_i)+\sin\theta_i$.

From (5.16) and (5.9) we have

$$(6.11)\qquad \left|P_n(x)\int_{-1}^{x} l_i(t)w(t)\,dt\right| \leqslant \frac{42n^{-1/2}(1-x_i^2)^{-13/16}}{|P_n'(x_i)|} + \frac{7(1-x_i^2)^{-3/2}}{[P_n'(x_i)]^2}.$$

But, using (4.2), we have

$$(6.12)\qquad \frac{1}{|P_n'(x_i)|} \leqslant \frac{|P_n'(x_i)|}{[P_n'(x_i)]^2} \leqslant \frac{(2n)^{1/2}(1-x_i^2)^{-3/4}}{[P_n'(x_i)]^2}.$$

Combining (6.11) and (6.12) we have (6.7). Using (4.5), (6.10) and (5.11), we have

$$\left|(1-x^2)^{1/2}P_n'(x)\int_{-1}^{x} l_i(t)w(t)\,dt\right| \leqslant \frac{42n^{1/2}(1-x_i^2)^{-13/16}}{|P_n'(x_i)|} + \frac{10n(1-x_i^2)^{-3/2}}{[P_n'(x_i)]^2},$$

using (6.12) we get (6.8). To prove (6.9) we again use (6.10), (5.14) and we get

$$\left|(1-x^2)^{1/2}P_n(x)\int_{-1}^{x} l_i(t)w(t)\,dt\right| \leqslant \frac{42n^{-1}(1-x_i^2)^{-13/16}}{|P_n'(x_i)|} + n^{-3/2}.$$

Again applying (6.12) to both terms on the right-hand side, we get (6.9).

7. Estimation of the fundamental polynomials of the second kind.

LEMMA 7.1. *For* $-1 \leqslant x \leqslant +1$ *we have*

$$(7.1)\qquad \sum_{i=2}^{n+1} \frac{|\varrho_i(x)|}{1-x_i^2} \leqslant 288n^{-1}d^{-3},$$

where d *is the constant given in* (4.7).

Proof. First we remark that this inequality is slightly improved with regard to that obtained in [12]. From (3.4), (6.1) and (6.4) we have

$$(7.2)\qquad |A_i| \leqslant 6(n+1)(1-x_i^2)^{-13/8}[P_n'(x_i)]^{-2}, \qquad i = 2, 3, \ldots, n+1.$$

Therefore, from (3.3), (6.2), (4.1) and (6.9) we obtain

$$\left|\frac{(1-x^2)P_n(x)q_{n-1}(x)}{(1-x_i^2)}\right| \leqslant \frac{39(1-x_i^2)^{-21/8}n^{-1/2}+33n^{-1/2}(1-x_i^2)^{-41/16}}{|P_n'(x_i)|^3} \leqslant 72(1-x_i^2)^{-21/8}n^{-1/2}|[P_n'(x_i)]^{-3}|; \tag{7.3}$$

using (4.7)—(4.9) and (5.10) we have

$$\sum_{i=2}^{n+1}\frac{1}{(1-x_i^2)^{21/8}|[P_n'(x_i)]^3|} \leqslant 4n^{-1/2}d^{-3}. \tag{7.4}$$

From (3.2), (7.3) and (7.4) we have (7.1).

8. Estimation of the fundamental polynomials of the first kind.

LEMMA 8.1. *For* $-1 \leqslant x \leqslant +1$ *the following estimates are valid*:

$$|r_1(x)| \leqslant 19n^{1/2}, \qquad |r_{n+2}(x)| \leqslant 19n^{1/2}, \tag{8.1}$$

$$\sum_{i=1}^{n+2}|r_i(x)| \leqslant 39n+1855nd^{-3} \leqslant 1894nd^{-3}, \tag{8.2}$$

where d is positive constant given in 4.7 *and* $d \leqslant 1$.

Proof. (8.1) was proved in [12] (see Lemma 5.2). We also remark that inequality (8.2) is in an improved from in comparison to what we obtained in [12]. Using (4.10) and the inequality $\sin\theta \geqslant \frac{2}{\pi}\theta$, $0 \leqslant \theta \leqslant \frac{\pi}{2}$, we obtain

$$(1-x_i^2)^{-1} < 4n^2. \tag{8.3}$$

Hence, from (3.9) we have

$$|C_i| \leqslant 6n^2(1-x_i^2)^{-1}. \tag{8.4}$$

Therefore, using (7.1) we have

$$\sum_{i=2}^{n+1}|C_i\varrho_i(x)| \leqslant 1728nd^{-3}. \tag{8.5}$$

From (3.11), (6.1) and (6.6) we obtain

$$|B_i| \leqslant 7n^{3/2}(n+1)(1-x_i^2)^{-7/4}[P_n'(x_i)]^{-2}. \tag{8.6}$$

Starting from (3.10) and using (8.6), (6.2) and (6.5), we obtain

$$|(1-x^2)^{1/2}s_{n-1}(x)| \leqslant 253n^{3/2}(1-x_i^2)^{-7/4}[P_n'(x_i)]^{-2}. \tag{8.7}$$

From (4.7)—(4.9) it follows that

$$\sum_{i=2}^{n+1}(1-x_i^2)^{-11/4}|[P_n'(x_i)]^{-3}| \leqslant d^{-3}n^{-1/4}. \tag{8.8}$$

Now, using (4.1), (8.7), (8.8) we obtain

$$(8.9)\qquad \sum_{i=2}^{n+1}\left|\frac{(1-x^2)P_n(x)s_{n-1}(x)}{2(1-x_i^2)P_n'(x_i)}\right| \leqslant \frac{253}{2}\, n^{3/2}n^{-1/2}d^{-3}n^{-1/4} = 127n^{3/4}d^{-3}.$$

Combining results (8.5), (8.9), (8.1), (4.6) and (5.20) we obtain

$$\sum_{i=1}^{n+2}|r_i(x)| \leqslant 38n^{1/2}+\frac{1}{2}+\frac{n}{2}+127n^{3/4}d^{-3}+1728nd^{-3},$$

from which the lemma follows.

9. Estimation of the derivative of the fundamental polynomial of the second kind.

LEMMA 9.1. *For* $-1\leqslant x\leqslant +1$, *we have*

$$(9.1)\qquad \sum_{i=2}^{n+1}\frac{|\varrho_i'(x)|}{1-x_i^2}\leqslant 910n^{1/2}d^{-3},$$

where d *is a constant given in* (4.7).

Proof. From the representation of $\varrho_i(x)$ as given in (3.2) and (3.3) we obtain

$$(9.2)\qquad \frac{\varrho_i'(x)}{1-x_i^2} = \frac{[(1-x^2)P_n'(x)-xP_n(x)]q_{n-1}(x)+A_iP_n^2(x)+P_n(x)l_i(x)}{2(1-x_i^2)P_n'(x_i)};$$

using (3.3), (7.2), (6.8) and (6.2) we have

$$(9.3)\qquad \left|\frac{(1-x^2)P_n'(x)q_{n-1}(x)}{2(1-x_i^2)P_n'(x_i)}\right|$$
$$\leqslant 39n(1-x_i^2)^{-21/8}|[P_n'(x_i)]^{-3}|+37n(1-x_i^2)^{-41/16}|[P_n'(x_i)]^{-3}|$$
$$\leqslant 76n(1-x_i^2)^{-21/8}|P_n'(x_i)|^{-3}.$$

Hence, using (7.4) we have

$$(9.4)\qquad \sum_{i=2}^{n+1}\left|\frac{(1-x^2)P_n'(x)q_{n-1}(x)}{2(1-x_i^2)P_n'(x_i)}\right|\leqslant 304n^{1/2}d^{-3}.$$

From (3.3), (6.2), (6.7) and (7.2) we obtain

$$(9.5)\qquad \left|\frac{(1-x^2)^{1/2}xP_n(x)q_{n-1}(x)}{2(1-x_i^2)P_n'(x_i)}\right|\leqslant 74(1-x_i^2)^{-21/8}|P_n'(x_i)|^{-3}.$$

Here we shall use a well-known Lemma due to Schur (see [9]), which states that if an arbitrary polynomial of degree $\leqslant p$ in x satisfies

$|(1-x^2)^{1/2}t_p(x)| \leqslant A$, then $|t_p(x)| \leqslant Ap$. Thus applying the above lemma to (9.5) we obtain

$$\left|\frac{xP_n(x)q_{n-1}(x)}{2(1-x_i^2)P_n'(x_i)}\right| \leqslant 148n(1-x_i^2)^{-21/8}|P_n'(x_i)|^{-3} \tag{9.6}$$

Using (7.4) we have

$$\sum_{i=2}^{n+1}\left|\frac{xP_n(x)q_{n-1}(x)}{2(1-x_i^2)P_n'(x_i)}\right| \leqslant 148n\, 4n^{-1/2}d^{-3} = 592n^{1/2}d^{-3} \tag{9.7}$$

From (4.4), (7.2) and (7.4) we obtain

$$\sum_{=2}^{n+1}\frac{|A_iP_n^2(x)|}{|2P_n'(x_i)(1-x_i^2)|} \leqslant 3n\,4n^{-1/2}d^{-3} = 12n^{1/2}d^{-3}. \tag{9.8}$$

Since

$$1 = \frac{(1-x^2)+(x^2-x_i^2)}{1-x_i^2},$$

we can write

$$\frac{P_n(x)l_i(x)}{2(1-x_i^2)P_n'(x_i)} = \frac{(1-x^2)P_n(x)l_i(x)+(x^2-x_i^2)P_n(x)l_i(x)}{2(1-x_i^2)^2P_n'(x_i)}. \tag{9.9}$$

From (4.6) it follows that

$$|(1-x^2)^{1/2}l_i(x)| \leqslant (1-x_i^2)^{1/2} \tag{9.10}$$

and from (4.1)

$$|(1-x^2)^{1/2}P_n(x)| \leqslant n^{-1/2}. \tag{9.11}$$

Therefore, using (4.7)—(4.9) we have

$$\sum_{i=2}^{n+1}\frac{|(1-x^2)P_n(x)l_i(x)|}{2(1-x_i^2)^2|P_n'(x_i)|} \leqslant \frac{1}{2}n^{-1/2}\sum_{i=2}^{n+1}\frac{1}{(1-x_i^2)^{3/2}|P_n'(x_i)|} \tag{9.12}$$

$$\leqslant \frac{n^{1/2}}{2d}\sum_{i=2}^{n+1} i^{-3/2} = n^{1/2}d^{-1}.$$

Since $|P_n(x)| \leqslant 1$ we have from (4.7)—(4.9)

$$\sum_{i=2}^{n+1}\left|\frac{(x^2-x_i^2)P_n(x)l_i(x)}{2(1-x_i^2)^2P_n'(x_i)}\right| \leqslant \sum_{i=2}^{n+1}\frac{1}{(1-x_i^2)^2[P_n'(x_i)]^2} \leqslant d^{-2}n^{1/2}. \tag{9.13}$$

Therefore, from (9.9), (9.12) and (9.13) we obtain

$$\sum_{i=2}^{n+1}\left|\frac{P_n(x)l_i(x)}{2(1-x_i^2)P_n'(x_i)}\right| \leqslant n^{1/2}d^{-1}[1+d^{-1}]. \tag{9.14}$$

Now combining the estimates from (9.4), (9.7), (9.8) and (9.14) and substituting them in (9.2), we have

$$\sum_{i=2}^{n+1} \frac{|\varrho_i'(x)|}{(1-x_i^2)} \leqslant 304n^{1/2}d^{-3}+592n^{1/2}d^{-3}+12n^{1/2}d^{-3}+n^{1/2}d^{-1}(1+d^{-1})$$
$$= (908d^{-3}+d^{-2}+d^{-1})n^{1/2} \leqslant 910d^{-3}n^{1/2}, \quad \text{since } d \leqslant 1.$$

This proves the lemma.

10. Investigation of the estimate of the derivative of the fundamental polynomials of the first kind.

In order to obtain the above estimate, we follow another idea due to Balázs and Turán [3]. In our case we shall require the corresponding results based on a quasi-Hermite-Fejér interpolation [15] introduced by Professor P. Szász. He considered the problem of finding the interpolation polynomials of degree $\leqslant 2n+1$ whose values are prescribed at the points given by (2.2) and whose first derivatives are prescribed only at the zeros of $P_n(x)$. These polynomials exist uniquely and have the form

$$Q_n(x) = \sum_{k=1}^{n+2} y_k E_k(x) + \sum_{k=2}^{n+1} y_k' F_k(x). \tag{10.1}$$

Here $Q_n(x_i) = y_i$, $i = 1, 2, \ldots, n+2$, $Q_n'(x_i) = y_i'$, $i = 2, 3, \ldots, n+1$. Further, owing to the uniqueness of these polynomials, if $\chi(x)$ is an arbitrary polynomial of degree $\leqslant 2n+1$, we have

$$\chi(x) = \sum_{k=1}^{n+2} \chi(x_k) E_k(x) + \sum_{k=2}^{n+1} \chi'(x_k) F_k(x). \tag{10.2}$$

Further, we observe from the results of Szász [15] that

$$E_k(x) \geqslant 0 \quad \text{and} \quad \sum_{k=1}^{n+2} E_k(x) = 1.$$

Let us consider in our case $\chi(x) = r_i'(x)$, which is a polynomial in x of degree $\leqslant 2n$ and such that $r_i''(x_j) = 0$, $j = 2, 3, \ldots, n+1$, $i = 1, 2, \ldots, n+2$. Therefore from (10.2) we have

$$r_i'(x) = \sum_{k=1}^{n+2} r_i'(x_k) E_k(x).$$

Since it was pointed out above that $E_k(x) \geqslant 0$ and $\sum_{k=1}^{n+2} E_k(x) \equiv 1$, we have

$$|r_i'(x)| \leqslant \max_k |r_i'(x_k)|. \tag{10.3}$$

Actually, we have even

$$\max |r_i'(x)| = \max_k |r_i'(x_k)|, \tag{10.4}$$

i. e. all polynomials $r_i'(x)$ attain their absolute maxima with respect to $[-1,+1]$ at one of the x_j points.

LEMMA 10.1. *For* $-1 \leqslant x \leqslant +1$ *we have*

$$|r_1'(x)| \leqslant 14n^2, \quad |r_{n+2}'(x)| \leqslant 14n^2. \tag{10.5}$$

Proof. From (3.6) we have for $j = 2, 3, \ldots, n+1$,

$$r_1'(x_j) = -\frac{1}{2}(1-x_j^2)[P_n'(x_j)]^2 - \frac{1}{2}(1-x_j^2)P_n'(x_j)\int_{-1}^{x} P_n'(t)w(t)\,dt.$$

Using (4.5) and (6.3) we obtain

$$|r_1'(x_j)| \leqslant \frac{n^2}{2} + 13n^2 \leqslant 14n^2, \quad j = 2, 3, \ldots, n+1.$$

Again from (3.6) we have $r_1'(1) = 3P_n'(1) + \frac{1}{2}$. Therefore, using (4.3) we get $|r_1'(1)| \leqslant 4n^2$. Similarly $|r_1'(-1)| \leqslant 4n^2$. Hence, using (10.4) we obtain $|r_1'(x)| \leqslant 14n^2$.

This proves the lemma.

LEMMA 10.2. *For* $-1 \leqslant x \leqslant +1$ *we have*

$$\sum_{i=2}^{n+1} |r_1'(x)| \leqslant 2604 d^{-3} n^{5/2}, \tag{10.6}$$

where d is a positive constant given in (4.7).

Proof. Denote the terms on the right-hand side of (3.8) by $I_1(x, x_i)$, $I_2(x, x_i)$, $I_3(x, x_i)$ and $I_4(x, x_i)$, respectively. Since

$$l_i(x_j) = \begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases} \quad \text{and} \quad l_1'(x_i) = \frac{x_i}{(1-x_i^2)}, \tag{10.7}$$

we easily obtain

$$I_i'(x_j, x_i) = 0 \quad \text{for } i, j = 2, 3, \ldots, n+1. \tag{10.8}$$

Here and in what follows the dash denotes differentiation with respect to x. Using (10.7) and the differential equation for $P_n(x)$ we obtain

$$I_2'(x_i, x_i) = 0, \quad i = 2, 3, \ldots, n+1. \tag{10.9}$$

Using (5.17) and (4.2), we obtain for $i, j = 2, 3, \ldots, n+1$

$$|I_2'(x_j, x_i)| = \left| \frac{(1-x_j^2)P_n'(x_j)l_i'(x_j)}{2(1-x_i^2)P_n'(x_i)} \right|, \quad i \neq j,$$

$$|I_2'(x_j, x_i)| \leqslant 3n^{5/2}(1-x_i^2)^{-11/4}|P_n'(x_i)|^{-3}. \tag{10.10}$$

Similarly, using (4.5) and (8.7) we obtain

$$(10.11)\quad |I_3'(x_j, x_i)| = \left|\frac{(1-x_j^2)P_n'(x_j)s_{n-1}(x_j)}{2(1-x_i^2)P_n'(x_i)}\right|, \qquad i,j = 2,3,\ldots,n+1,$$

$$\leqslant 127n^{5/2}(1-x_i^2)^{-11/4}|P_n'(x_i)|^{-3}, \quad i,j = 2,3,\ldots, n+1.$$

Since

$$I_4'(x_j, x_i) = C_i\varrho_i'(x_j) = \frac{C_i(1-x_j^2)P_n'(x_j)q_{n-1}(x_j)}{2P_n'(x_i)},$$

using (8.4) and (9.3) we have

$$(10.12)\qquad |I_4'(x_j, x_i)| \leqslant 6n^2\cdot 76n(1-x_i^2)^{-21/8}|P_n'(x_i)|^{-3} = 456n^3(1-x_i^2)^{-21/8}|P_n'(x_i)|^{-3}.$$

Therefore, from (10.8)—(10.12) we have

$$|r_i'(x_j)| \leqslant 130n^{5/2}(1-x_i^2)^{-11/4}|P_n'(x_i)|^{-3} + 456n^3(1-x_i^2)^{-21/8}|P_n'(x_i)|^{-3}.$$

From (8.3) we have $(1-x_i^2)^{-1/8} \leqslant \frac{3}{2}n^{1/4}$. So we get

$$|r_i'(x_j)| \leqslant 195(1-x_i^2)^{-21/8}n^{11/4}|P_n'(x_i)|^{-3} + 456n^3(1-x_i^2)^{-21/8}|P_n'(x_i)|^{-3}$$

$$\leqslant 651n^3(1-x_i^2)^{-21/8}|P_n'(x_i)|^{-3}, \quad i,j = 2,3,\ldots,n+1.$$

But a simple computation shows that

$$|r_i'(\pm 1)| \leqslant 300n^3(1-x_i^2)^{-21/8}|P_n'(x_i)|^{-3}.$$

Hence using (10.4) we have

$$\max|r_i'(x)| = \max_k |r_i'(x_k)| \leqslant 651n^3(1-x_i^2)^{-21/8}|P_n'(x_i)|^{-3}.$$

Using (7.4) we have finally

$$\sum_{i=2}^{n+1}|r_i'(x)| \leqslant 651n^3 4n^{-1/2}d^3 = 2604n^{5/2}d^{-3}.$$

This proves the lemma.

11. Proof of Theorem 2.1. The above estimates very easily prove our theorem. From the uniqueness theorem [12] we have for an arbitrary polynomial $Q_{2n+1}(x)$ of degree $\leqslant 2n+1$ the representation

$$(11.1)\qquad Q_{2n+1}(x) = \sum_{i=1}^{n+2} Q_{2n+1}(x_i)r_i(x) + \sum_{i=2}^{n+1} Q_{2n+1}''(x_i)\varrho_i(x).$$

Here $r_i(x)$ and $\varrho_i(x)$ are fundamental polynomials stated in Article 3. Further, differentiating both sides of (11.1) with respect to x we obtain

$$(11.2)\qquad Q_{2n+1}'(x) = \sum_{i=1}^{n+2} Q_{2n+1}(x_i)r_i'(x) + \sum_{i=2}^{n+1} Q_{2n+1}''(x_i)\varrho_i'(x).$$

Using (11.1), (2.3) and the estimate of the fundamental polynomials given by (8.2) and (7.1), we obtain

$$|Q_{2n+1}(x)| \leqslant 1894nd^{-3}A + \frac{288}{n}d^{-3}B,$$

which proves assertion (2.5). To prove (2.6) we start with (11.2), (2.3) and the estimates of the derivatives of the fundamental polynomials given by (9.1), (10.5) and (10.6) and we obtain

$$|Q'_{2n+1}(x)| \leqslant (28n^2 + 2604n^{5/2}d^{-3})A + 914n^{1/2}d^{-3}B;$$

but $d < 1$ and therefore we get

$$|Q'_{2n+1}(x)| \leqslant 2632n^{5/2}d^{-3}A + 914n^{1/2}d^{-3}B,$$

which in turn proves (2.6). This proves the theorem.

12. Here we will show that our theorem 2.1 is essentially the best possible. For this purpose we need the following Lemma:

LEMMA 12.1. *There exist positive constants d_1 and d_2 such that*

$$\sum_{i=2}^{n+1} \frac{|\varrho_i'(1)|}{1-x_i^2} \geqslant d_1 n^{1/2} \tag{12.1}$$

and

$$\sum_{i=1}^{n+2} |r_i'(1)| \geqslant d_2 n^{5/2} \tag{12.2}$$

for n sufficiently large.

Proof. From (3.2) and (3.3) it follows that

$$\frac{\varrho_i'(1)}{1-x_i^2} = \frac{A_i + l_i(1)}{(1-x_i^2)P_n'(x_i)}; \tag{12.3}$$

using (4.7)—(4.9) we have

$$\sum_{i=2}^{n+1} \frac{|l_i(1)|}{|(1-x_i^2)P_n'(x_i)|} \leqslant 2\sum_{i=2}^{n+1} \frac{1}{(1-x_i^2)^2[P_n'(x_i)]^2} \leqslant 2d^{-2}\log n. \tag{12.4}$$

Here we have used the fact that

$$|l_i(1)| = \left|\frac{1}{(1-x_i)P_n'(x_i)}\right| \leqslant \frac{|(1+x_i)|}{|(1-x_i^2)P_n'(x_i)|} \leqslant \frac{2}{|(1-x_i^2)P_n'(x_i)|}.$$

From the definition of A_i as given in (3.4) and using (6.1) we obtain

$$E \equiv \sum_{i=2}^{n+1} \frac{|A_i|}{(1-x_i^2)|P_n'(x_i)|} \geqslant \frac{n}{2}\sum_{i=2}^{n+1} \frac{1}{(1-x_i^2)|P_n'(x_i)|}\left|\int_{-1}^{+1} l_i(t)w(t)\,dt\right|; \tag{12.5}$$

using (4.2) and the fact that $(1-x_i^2)^{1/4} \leqslant 1$, we obtain

$$E \geqslant \frac{n}{2\sqrt{2n}} \sum_{i=2}^{n+1} \left| \int_{-1}^{+1} l_i(t) w(t)\, dt \right|.$$

$\sum_{i=2}^{n+1} l_i(t) = 1$ and the observation

$$\left| \int_{-1}^{+1} l_i(t) w(t)\, dt \right| \geqslant \int_{-1}^{+1} l_i(t) w(t)\, dt$$

lead us to state that

$$E \geqslant \frac{1}{2^{3/2}} n^{1/2} \int_{-1}^{+1} (1-t^2)^{1/2}\, dt \geqslant \frac{\pi}{2^{3/2}} n^{1/2}. \tag{12.6}$$

Hence using (12.3)—(12.6) we obtain (12.1). To prove (12.2) we use the notation adopted in Article 10 and using (4.7)—(4.9), we easily obtain

$$\sum_{i=2}^{n+1} |I_1'(1, x_i)| = \sum_{i=2}^{n+1} \frac{l_i^2(1)}{(1-x_i^2)} \leqslant 4 \sum_{i=2}^{n+1} \frac{1}{(1-x_i^2)^3 |P_n'(x_i)|^2} \leqslant 12 n^2 d^{-2}. \tag{12.7}$$

Similarly, using (4.3) and (12.4)

$$\sum_{i=2}^{n+1} |I_2'(1, x_i)| = \sum_{i=2}^{n+1} \left| \frac{P_n'(1) l_i(1)}{(1-x_i^2) P_n'(x_i)} \right| \leqslant n(n+1) d^{-2} \log n. \tag{12.8}$$

Lastly from (3.8)—(3.11) we have

$$I_3'(1, x_i) = \frac{|s_{n-1}(1)|}{|(1-x_i^2) P_n'(x_i)|} \leqslant \frac{2[|B_i| + |l_i(1)|]}{|(1-x_i^2) P_n'(x_i)|}. \tag{12.9}$$

Using (8.6) and (5.17) we obtain

$$\begin{aligned} &|I_3'(1, x_i)| \\ &\leqslant 14 (1-x_i^2)^{-11/4} |P_n'(x_i)|^{-3} (n+1)^{5/2} + 8n^2 (1-x_i^2)^{-11/4} |P_n'(x_i)|^{-3} \\ &\leqslant 22 (1-x_i^2)^{-11/4} |P_n'(x_i)|^{-3} (n+1)^{5/2}. \end{aligned} \tag{12.10}$$

Therefore, using (4.7)—(4.9) we have

$$\begin{aligned} \sum_{i=2}^{n+1} |I_3'(1, x_i)| &\leqslant 22 (n+1)^{5/2} \sum_{i=2}^{n+1} \frac{1}{(1-x_i^2)^{11/4} |P_n'(x_i)|^3} \\ &\leqslant 22 (n+1)^2 d^{-3} \log n. \end{aligned} \tag{12.11}$$

Combining (12.7)—(12.11) and using (12.1) we at once obtain (12.1). This proves the lemma.

Now we will explicitly exhibit the polynomial $f_1(x)$ which has properties (2.3) and (2.4) and satisfies (2.7), as remarked earlier at the end of the statement of the theorem. Consider

$$f_1(x) = \sum_{k=1}^{n+2} A \operatorname{sign} r_k'(1) r_k(x) + \sum_{k=2}^{n+1} \frac{B}{1-x_k^2} \operatorname{sign} \varrho_k'(1) \varrho_k(x);$$

then

$$f_1(x_i) = A, \quad i = 1, 2, \ldots, n+2,$$

and

$$f_1''(x_i) = \frac{B}{1-x_i^2}, \quad i = 1, 2, \ldots, n+1.$$

Therefore $f_1(x)$ satisfies (2.3) and (2.4). Using the above lemma we obtain (2.7) as desired. This proves the result completely.

References

[1] J. Balázs and P. Turán, *Notes on interpolation II*, Acta Math. Acad. Sci. Hung. 8 (1957), p. 201–215.

[2] — — *Notes on interpolation III* (Convergence problem), ibidem 9 (1958), p. 195–214.

[3] — — *Notes on interpolation IV* (inequalities), ibidem 9 (1958), p. 243–258.

[4] E. Egervary and P. Turán, *Notes on interpolation V*, ibidem 9 (1958), p. 259–267.

[5] P. Erdös, *Extremal properties of derivatives of polynomials*, Ann. of Math. (2) 41 (1940), p. 310–313.

[6] L. Fejer, *Die Abschätzung eines Polynoms in einem Intervalle, wenn Schranken für seine werte und ersten Ableitungswerte in einzelnen Punkten des Intervalles gegeben sind, undihre Anwendung auf die konvergenzfrage Hermitescher Interpolationsreihen*, Math. Zeitschr. 32 (1930), p. 426–457.

[7] — *Lagrangesche Interpolation und die zugehörigen konjugierten Punkte*, Math. Ann. 106 (1932), p. 1–55.

[8] G. G. Lorentz, *The degree of approximation by polynomials with positive coefficients*, ibidem 151 (1963), p. 239–251.

[9] — *Approximation of functions*, Holt Rinehart and Winston, Inc., 1966.

[10] E. Landau, *Einige Ungleichungen für zweimal differenzierbare Funktionen*, Proc. London Math. Soc. Ser. 2, 13 (1913), p. 43–49.

[11] P. Lax, *Proof of a conjecture of P. Erdös*, B.A.M.S. 50 (1944), p. 509–513.

[12] J. Prasad and A. K. Varma, *An analogue of a problem of J. Balázs and P. Turán*, Canadian Jour. of Math. 21 (1969), p. 54–63.

[13] G. Sansone, *Orthogonal functions*, Interscience Publishers, Inc., vol. 9, New York 1959.

[14] R. B. Saxena, *On some interpolatory properties of Legendre polynomials III*, Bull. de L'institute de Math. vol. 8, (1964), p. 63–93.

[15] P. Szász, *On quasi-Hermite-Fejer interpolation*, Acta Math. Acad. Sci. Hung. 10 (1959), p. 413–439.

[16] G. Szegö, *Orthogonal polynomials*, Amer. Math. Soc. Colloquium Pub. vol. 23 (1959).

[17] P. Turán, *Über die Ableitung von Polynomes,* Compos. Math. 7 (1939–1940), p. 80–95.
[18] A. K. Varma, *Inequalities for polynomials and their derivatives I,* Acta Sci. Math. Szeged Hungary 29 (1968), p. 137–146.
[19] — *Inequalities for polynomials and their derivatives II,* Tokohu Math. Journal (Japan) (3) 20 (1968), p. 265–288.

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