ANNALES
POLONICI MATHEMATICI
XXIV (1971)

On some properties of eigenvalues and eigenfunctions of certain differential equations of the fourth order

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Introduction. Let G be a bounded Jordan-measurable domain in the space E^m of m variables $X = (x_1, \ldots, x_m)$ which can be approximated by an increasing sequence of domains G_n with regular boundaries (i.e., the boundary ∂G_n of G_n is a surface of class C_σ^1; for the definition of a surface of class C_σ^1 see [3], p. 132). We do not require any regularity properties of the boundary of G.

We shall consider a differential equation of the form

$$\mathscr{E}(u) - \mu u = 0, \tag{1}$$

where $\mathscr{E}(u)$ is a differential operator of the form $\mathscr{E}(u) = L_1[L_0(u)]$ and the operators $L_k(\varphi)$ $(k = 0, 1)$ are selfadjoint differential operators, i.e.,

$$L_k(\varphi) = -\sum_{i,j=1}^{m} \frac{\partial}{\partial x_i}\left[a_{ij}^k(X)\frac{\partial \varphi}{\partial x_j}\right] + q^k(X)\varphi \qquad (k = 0, 1),$$

μ is a real parameter. We make the following assumptions: $a_{ij}^k(X) = a_{ji}^k(X)$ $(i, j = 1, \ldots, m)$ are of class C^{3-2k} in $\bar{G}$ $(k = 0, 1)$, $q^k(X) \geqslant 0$ are of class C^{2-2k} in $\bar{G}$ $(k = 0, 1)$, and the quadratic forms $\sum_{i,j=1}^{m} a_{ij}^k(X)\xi_i\xi_j$ $(k = 0, 1)$ are positive definite in $\bar{G}$.

We shall also consider the generalized boundary condition (cf. [1] and [2]) which in the case where the boundary ∂G is regular may be written in the form

$$R_k(\varphi^k) = 0 \quad \text{on } \partial G \quad (k = 0, 1); \qquad \varphi^0(X) = u(X), \quad \varphi^1(X) = L_0(u), \tag{2}$$

where $R_k(u) = 0$ on ∂G means

$$\frac{du}{d\nu_k} - h^k(X)u = 0 \quad \text{on} \quad \partial G - \Gamma_k, \quad u = 0 \quad \text{on } \Gamma_k \quad (k = 0, 1) \tag{3}$$

and Γ_k denote the $(m-1)$-dimensional parts of ∂G (Γ_k being connected or not); in extreme cases Γ_k may be the whole boundary of G or an empty set. Here $h^k(X)$ $(k=0,1)$ are non-negative continuous functions in $\bar{G}$, and $d\varphi/d\nu_k$ $(k=0,1)$ are the transversal derivatives of φ with respect to the operators L_k $(k=0,1)$, respectively, i.e.,

$$\frac{d\varphi}{d\nu_k} = \sum_{i,j=1}^{m} a_{ij}^k(X)\frac{\partial\varphi}{\partial x_i}\cos(n, x_j) \qquad (k=0,1),$$

n being the interior normal to ∂G.

1. EIGENVALUES AND EIGENFUNCTIONS OF PROBLEM (1), (2)

The object of the following considerations are some properties of eigenvalues and eigenfunctions corresponding to equation (1) and condition (2) (we shall shortly say: eigenvalues and eigenfunctions of problem (1), (2)).

DEFINITION. We shall say that a real number λ is an *eigenvalue* of problem (1), (2) if there exists a function $u(X) \not\equiv 0$ belonging to $C^4(G) \cap \cap \mathscr{L}^2(G)$, and satisfying the boundary condition (2) (in a generalized sense) and equation (1) for $\mu = \lambda$. This function $u(X)$ we shall call the *eigenfunction* of problem (1), (2) corresponding to the eigenvalue λ.

The existence of eigenvalues and eigenfunctions of problem (1), (2) will be reduced to the following auxiliary problems. Namely, we consider the equations

$$L_k(u) - \mu u = 0 \qquad (k=0,1) \tag{4}$$

and the equation

$$L_0(u) - \mu K(u) = 0 \tag{5}$$

with the boundary conditions

$$R_k(u) = 0 \quad \text{on } \partial G \qquad (k=0,1) \tag{6}$$

and

$$R_0(u) = 0 \quad \text{on } \partial G, \tag{7}$$

respectively. The operators L_k $(k=0,1)$ in equations (4) and (5) are the operators defined in the introduction of this paper, and the operator K in (5) satisfies the following conditions:

1° $K\colon \mathscr{L}^2(G) \to \mathscr{L}^2(G)$, is a linear bounded operator,

2° the subspace $\mathscr{L}^2(G) \cap C(G)$ of continuous functions is invariant with respect to K, 3° K is symmetric, i.e.,

$$\big(\varphi, K(\psi)\big) = \int_G \varphi(X) K(\psi)\, dX = \int_G \psi(X) K(\varphi)\, dX = \big(\psi, K(\varphi)\big)$$

for $\varphi, \psi \in \mathscr{L}^2(G)$,

4° is positive, i.e., $\big(\varphi, K(\varphi)\big) > 0$ for $\varphi \not\equiv 0$.

The boundary conditions $R_k(u)$ $(k = 0, 1)$ are defined by (3).

The eigenvalues and eigenfunctions of problems (4), (6) $(k = 0, 1)$ are defined as in [1], and eigenvalues and eigenfunctions of problem (5), (7) are defined as in [2].

We shall need the following assumptions:

Hypothesis Z_k. *Given* (4) *and* (6) *there exist sequences of eigenvalues*

$$0 \leqslant \varkappa_1^k \leqslant \varkappa_2^k \leqslant \varkappa_3^k \leqslant \ldots \quad (k = 0, 1) \tag{8}$$

and corresponding sequences of eigenfunctions

$$w_1^k(X), w_2^k(X), w_3^k(X), \ldots \quad (k = 0, 1) \tag{9}$$

which belong to $\mathscr{F}$ (¹).

Hypothesis Z. *Given* (5) *and* (7) *there exists a sequence of eigenvalues*

$$0 \leqslant \mu_1 \leqslant \mu_2 \leqslant \mu_3 \leqslant \ldots \tag{10}$$

and a corresponding sequence of eigenfunctions

$$v_1(X), v_2(X), v_3(X), \ldots \tag{11}$$

which belong to $\mathscr{F}$.

Lemma 1. *If Γ_1 is not an empty set or if the function $h^1(X) > 0$ in $\bar{G}$, then the first eigenvalue $\varkappa_1^1 > 0$.*

Proof. From the assumption on the coefficients of (1) and on $h^1(X)$ it follows that $\varkappa_1^1 \geqslant 0$ (cf. [1]). Suppose that $\varkappa_1^1 = 0$. Therefore (see [1]), the function $w_1^1(X)$ satisfies the boundary condition $R_1(w_1^1) = 0$ on ∂G and equation

$$L_1(w_1^1) = 0. \tag{12}$$

From the well-known theorems on the uniqueness of the solutions of the elliptic equations it follows that $w_1^1(X) \equiv 0$ in G (cf. [3], p. 166). On the other hand, by the definition of $w_1^1(X)$, we have $w_1^1(X) \not\equiv 0$ in G, and thus we get contradiction.

Lemma 2. *Under the assumptions of Lemma 1 the restriction of the operator L_1 to the space $\mathscr{F}_{h^1, \Gamma_1}(G)$ (²) admits a bounded inverse operator.*

(¹) For the definition of the space $\mathscr{F}$ see [1].

(²) For the definition of the space $\mathscr{F}_{h, \Gamma}(G)$ see [1].

Proof. As we know (see [1]),

$$\varkappa_1^1 = \min_{\varphi \in \mathring{\mathscr{D}}} \frac{D_1(\varphi)}{H_1(\varphi)}, \tag{13}$$

where $\mathring{\mathscr{D}}$ is the space of functions φ of class C_σ^1 in G, $\varphi = 0$ on Γ_1 (in the generalized sense) for which $D_1(\varphi) < \infty$ and $H_1(\varphi) < \infty$, where

$$D_1(\varphi) = \int_G \Big[\sum_{i,j=1}^{m} a_{ij}^1 \frac{\partial\varphi}{\partial x_i} \frac{\partial\varphi}{\partial x_j} + q^1\varphi^2 \Big] dX + \int_{\partial G - \Gamma_1} h^1 \varphi^2 dS, \tag{14}$$

$$H_1(\varphi) = \int_G \varphi^2 dX = \|\varphi\|^2 \tag{15}$$

(see [1]).

It is easy to verify that if $\varphi(X) \in \mathscr{F}_{h^1,\Gamma_1}(G)$ ($\mathscr{F}_{h^1,\Gamma_1}(G)$ is a subclass of $\mathring{\mathscr{D}}$) consisting of the functions φ of class $C^2(G)$ and satisfying boundary condition $R_1(\varphi) = 0$ on ∂G (see [1]), then

$$D_1(\varphi) = \int_G \varphi(X) L_1(\varphi) dX = \big(\varphi, L_1(\varphi)\big). \tag{16}$$

From (13), (15) and (16) it follows that the operator L_1 is positive definite on $\mathscr{F}_{h^1,\Gamma_1}(G)$ and its range is the space $\mathscr{L}^2(G) \cap C(G)$.

From this it follows that $K = L_1^{-1}$ exists and K satisfies the following conditions (cf. [4], p. 563):

1° $K: \mathscr{L}^2(G) \cap C(G) \to \mathscr{F}_{h^1,\Gamma_1}(G)$ is a linear bounded operator ([3]);

2° K is symmetric, i.e.,

$$\big(\varphi, K(\psi)\big) = \big(\psi, K(\varphi)\big) \quad \text{for } \varphi, \psi \in L^2(G) \cap C(G),$$

3° K is positive, i.e., $\big(\varphi, K(\varphi)\big) > 0$ for $\varphi \not\equiv 0$.

LEMMA 3. *The function $u(X) \not\equiv 0$ in G belonging to $C^4(G) \cap \mathscr{L}^2(G)$ is an eigenfunction of problem* (1), (2) *corresponding to the eigenvalue λ if and only if the function $u(X)$ is an eigenfunction of problem* (5), (7) *corresponding to the eigenvalue λ.*

Proof. Suppose now, that $u(X)$ is an eigenfunction of problem (1), (2) corresponding to the eigenvalue λ. This means that $u(X)$ satisfies equation (1) with $\mu = \lambda$ and the boundary condition (2). From this it follows that $u(X)$ satisfies the boundary condition (7), and

$$L_1[L_0(u)] - \lambda u = 0.$$

Since $L_1 K = I$, the last equation may be written in the form

$$L_1\{L_0(u) - \lambda K(u)\} = 0.$$

([3]) Since $\mathscr{L}^2(G) \cap C(G)$ is dense in $\mathscr{L}^2(G)$, we may assume that K is bounded on $\mathscr{L}^2(G)$.

From this it follows that $u(X)$ satisfies equation (5) with $\mu = \lambda$, and hence $u(X)$ is an eigenfunction of problem (5), (7) corresponding to the eigenvalue λ.

If $u(X)$ is an eigenfunction of problem (5), (7), then $u(X)$ satisfies the boundary condition (7) and equation (5) with $\mu = \lambda$, and $u(X) \not\equiv 0$ in G. By the definition of the operator K and from (5) we have $L_0(u) \in R[K]$. This means that the function $v_0(X) = L_0(u)$ satisfies the boundary condition $R_1(v_0) = 0$ on ∂G. By (5) we have

$$L_1[L_0(u)] - \lambda L_1[K(u)] = 0.$$

From this, by the definition of operator K, it follows that $u(X)$ satisfies equation (1) with $\mu = \lambda$. Since $u(X)$ satisfies also the boundary condition (2), $u(X)$ is an eigenfunction of problem (1), (2) corresponding to the eigenvalue λ.

Using Lemmas 1, 2 and 3 and using the results of [2], we shall prove the following theorem:

THEOREM 1. *Under assumption* Z *if the functions* $v_n(X)$ $(n = 1, 2, 3, \ldots)$ *are of class* C^4 *in* G, *then sequence* (10) *contains all the eigenvalues of problem* (1), (2), *and every eigenfunction of problem* (1), (2) *is a suitable linear combination of eigenfunctions of sequence* (11).

Proof. It follows from Lemma 3 that each eigenvalue of problem (5), (7) is an eigenvalue of problem (1), (2), and conversely, each eigenvalue of problem (1), (2) is an eigenvalue of problem (5), (7). On the other hand, we know (see [2]) that the sequence of eigenfunctions of problem (5), (7) is a complete system in $\mathscr{L}^2(G)$ with respect to the scalar product $H(u, v) = (u, K(v))$. From this it follows that sequence (10) contains all eigenvalues of problem (1), (2). This completes the proof.

Under the assumptions of Theorem 1, from Theorem 1 and from the results of paper [2], we have the following statements:

COROLLARY 1. *The sequence of eigenfunctions of problem* (1), (2) *is a complete system in* $\mathscr{L}^2(G)$ *with respect to the scalar product* $H(u, v)$.

COROLLARY 2. *Every eigenvalue of problem* (1), (2) *has finite multiplicity.*

COROLLARY 3. *Every function* $f(X) \in \mathscr{L}^2(G)$ *can be expanded in a series of eigenfunctions* $\{u_n(X)\}$ *of problem* (1), (2) *which converges in the mean, i.e.,*

$$\lim_{n\to\infty} H\Big(f - \sum_{k=1}^{n} c_k u_k(X)\Big) = 0.$$

2. SOME PROPERTIES OF THE FIRST EIGENVALUE AND FIRST EIGENFUNCTION OF PROBLEM (1), (2)

We shall prove the following

LEMMA 4. *The operator K defined in Lemma 2 satisfies the condition: if $\varphi(X) \geqslant 0$ in G, then $K(\varphi) \geqslant 0$ in G.*

Proof. Let $\psi(X) = K(\varphi)$. Then, by the definition of K,

$$L_1(\psi) = \varphi(X) \quad \text{and} \quad R_1(\psi) = 0 \quad \text{on } \partial G.$$

Since $\varphi(X) \geqslant 0$ in G, in virtue of the maximum principle for the solutions of elliptic equations, we have $\psi(X) \geqslant 0$ in G (see for instance [3], Chapter V).

In the sequel we shall need the following assumption:

HYPOTHESIS A. *No eigenfunction of* (1), (2) *can vanish identically in any subdomain of domain G.*

Remark 1. In the case $m > 1$ Hypothesis A is satisfied under the assumption that the coefficients of (1) are analytic; however, in the case $m = 1$ Hypothesis A is satisfied under the previous assumptions (see the introduction).

If Hypothesis A is satisfied, then Lemma 4 implies that the operator K defined in Lemma 2 satisfies all the assumptions of paper [2]. Therefore, from Theorem 1 and from the results of paper [2] we get the following theorems:

THEOREM 2. *The first eigenfunction $u_1(X)$ of problem* (1), (2) *does not vanish at any point of the domain G.*

THEOREM 3. *If there exists a function $\varphi(X) \neq 0$ of class C^4 in G satisfying the boundary condition* (2) *and equation* (1) *with $\mu = t$, then $t = \lambda_1$ and $\varphi(X) = cu_1(X)$, where $c = \text{const} \neq 0$.*

THEOREM 4. *The first eigenvalue of problem* (1), (2) *is a single eigenvalue, i.e. each function $\varphi(X) \not\equiv 0$ of class C^4 in G satisfying the boundary condition* (2) *and equation* (1) *with $\mu = \lambda_1$ is equal to the first eigenfunction of* (1), (2) *multiplied by a constant $c \neq 0$, whence $\lambda_1 < \lambda_2$.*

Remark 2. All the results of this paper may be generalized without essential changes to the case of a more general equation of the form

$$L_1^p[L_0(u)] - \mu u = 0, \tag{17}$$

where p is any integer and L_0, L_1 are the operators defined in the introduction of this paper, with the boundary conditions

$$\begin{aligned} R_0(u) &= 0 \quad \text{on } \partial G, \\ R_1(\varphi^k) &= 0 \quad \text{on } \partial G \quad (k = 1, \ldots, p), \end{aligned} \tag{18}$$

where $\varphi^k = L_0(\varphi^{k-1})$, $k = 1, \ldots, p$, $\varphi^0 = u$.

Finally, I would like to point out that the method used in this paper cannot be applied when the operator $\mathscr{E}(u)$ in the equation (1) is of the form

$$\mathscr{E}(u) = A_p A_{p-1} \ldots A_1 A_0(u), \tag{19}$$

where the operators $A_0, A_1, \ldots, A_p$ are operators of the same form as the operators L_0 and L_1, because then the operator

$$K = A_1^{-1} A_2^{-1} \ldots A_p^{-1}, \tag{20}$$

is not necessarily positive. And its being positive is essential in the method used in this paper.

References

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Reçu par la Rédaction le 12. 7. 1967