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On an application of the Laplace-Picone transformation in the theory of partial differential equations

by J. Bochenek (Kraków)

1. First we shall prove some auxiliary theorems concerning bounds for the solutions of elliptic equations and also for the solutions of normal parabolic equations.

Let D be a bounded domain in an m-dimensional Euclidean space E^m. We assume that for every point X of the boundary $F(D)$ of D there exists an open half-line $l(X)$ starting from the point X such that the common part of $l(X)$ with a neighbourhood of X lies in D (we shall say that $l(X)$ "penetrates" into the interior of D). Let

$$L(u) \equiv \sum_{i,j=1}^{m} a_{ij}(X)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{k=1}^{m} b_k(X)\frac{\partial u}{\partial x_k} + c(X)u = f(X) \tag{1}$$

be an elliptic partial differential equation in which $a_{ij}(X) = a_{ji}(X)$, $b_k(X)$ $(i, j, k = 1, \dots, m)$, $c(X)$ and $f(X)$ are real functions defined and bounded in the closure $\bar{D}$ of D and the quadratic form $\sum_{i,j=1}^{m} a_{ij}(X)\lambda_i\lambda_j$ is positively defined in $\bar{D}$.

THEOREM 1. *If*

$$c(X) \leqslant -c_0, \quad |f(X)| \leqslant M, \quad X \in \bar{D}, \tag{2}$$

where $c_0 > 0$ and $M \geqslant 0$ are constants, and if $u(X)$ is a solution of (1) *regular in $\bar{D}$ (i.e. $u(X)$ is of class C^2 in D and continuous in $\bar{D}$) such that the derivative du/dl exists for $X \in F(D)$ and*

$$u(X) = 0, \quad X \in F(D) \tag{3}$$

or

$$\frac{du}{dl} - h(X)u = 0, \quad X \in F(D), \tag{3'}$$

where $h(X) > 0$, $X \in F(D)$, then

$$|u(X)| \leqslant \frac{M}{c_0}, \quad X \in \bar{D}. \tag{4}$$

Proof. Let $w_{\pm}(X) = \frac{M}{c_0} \pm u(X)$. Then

$$L(w_{\pm}) = L\left(\frac{M}{c_0}\right) \pm L(u) = c(X)\frac{M}{c_0} \pm f(X) \leqslant -M \pm f(X) \leqslant 0 . \tag{5}$$

It follows from (3) or (3′) that we have for $X \in F(D)$

$$w_{\pm}(X) \geqslant 0 \quad \text{or} \quad \frac{dw_{+}}{dl} - h(X)w_{\pm} \leqslant 0 , \tag{6}$$

respectively. Inequalities (5) and (6) and the extremum property of the solutions of elliptic equations ([3], Chap. V) imply

$$w_{\pm}(X) \geqslant 0 , \quad X \in \bar{D} . \tag{7}$$

Inequality (7) being equivalent to (4), the proof is completed.

In the case of the boundary condition of the Neumann type we have the following

THEOREM 2. *If the boundary $F(D)$ of D is a hypersurface of class C^2, if (2) is satisfied and if $u(X)$ is a solution of (1) regular in $\bar{D}$ such that the derivative du/dl exists for $X \in F(D)$ and*

$$\frac{du}{dl} = 0 , \quad X \in F(D) , \tag{8}$$

then (4) holds.

The proof of this theorem may be based on Theorem 2 of [1] and it is then analogous to the proof of Theorem 1.

Consider a normal parabolic equation

$$\mathcal{F}(u) \equiv \sum_{i,j=1}^{m} a_{ij}(t, X)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{k=1}^{m} b_k(t, X)\frac{\partial u}{\partial x_k} + c(t, X)u - \frac{\partial u}{\partial t} = f(t, X) . \tag{9}$$

The solutions of (9) may be estimated analogously to the solutions of (1). We assume that $a_{ij}(t, X) = a_{ij}(t, X)$, $b_k(t, X)$ $(i, j, k = 1, \ldots, m)$, $c(t, X)$ and $f(t, X)$ are real functions defined and bounded in the closure $\bar{D}_0$ of domain D_0 and that the quadratic form $\sum_{i,j=1}^{m} a_{ij}(t, X)\lambda_i\lambda_j$ is positively defined in $\bar{D}_0$. Here D_0 denotes a domain of the space $E^m \times (-\infty, \infty)$ which is bounded or unbounded in the direction of the axis t at most. We assume that the boundary $F(D_0)$ of D_0 is composed of an m-dimensional domain S_0 contained in the characteristic $t = 0$ of a hypersurface σ_0 oriented with respect to time and an m-dimensional domain S_{T_0} contained in the characteristic $t = T_0$ $(T_0 > 0)$. We assume that

S_{T_0} is empty if D_0 is unbounded (i.e. $T_0 = +\infty$ and σ_0 is unbounded). In both cases we assume that σ_0 is closed. We assume that a half-line $l(P)$ penetrating into the interior of D_0 is attached to every point P of σ_0. Under all these assumptions we have

THEOREM 3. *If*

$$(10) \qquad c(t, X) \leqslant -c_0 \quad \text{and} \quad |f(t, X)| \leqslant M, \quad (t, X) \in \bar{D}_0,$$

where $c_0 > 0$ and $M \geqslant 0$ are constant, and if $u(t, X)$ is a solution of (9) regular in $\bar{D}_0$ such that the derivative du/dl exists for $(t, X) \in \sigma_0$ and

$$(11) \qquad u(t, X) = 0, \quad (t, X) \in \sigma_0$$

or

$$(11') \qquad \frac{du}{dl} - h(t, X)u = 0, \quad (t, X) \in \sigma_0,$$

where $h(t, X)$ is a positive function defined on σ_0, then

$$(12) \qquad |u(t, X)| \leqslant M_0 = \max(M/c_0, M_1), \quad M_1 = \sup_{X \in S_0} |u(0, X)|.$$

The proof of Theorem 3 may be based on the extremum properties of the solutions of parabolic equations and is analogous to the proof of Theorem 1 (cf. [2], p. 9).

In the case of the boundary condition of the Neumann type we have

THEOREM 4. *If 1° the hypersurface σ_0 is of class C^2, 2° the normal n to the hypersurface σ_0 is not parallel to the axis t at any point of σ_0, 3° (10) is satisfied, then each solution $u(t, X)$ of (9) regular in $\bar{D}_0$, having the derivative du/dl at all $P \in \sigma_0$ (the angle between l and the interior normal to σ_0 at P being acute) and satisfying*

$$(13) \qquad \frac{du}{dl} = 0, \quad P \in \sigma_0,$$

satisfies (12).

Proof. Put $w_\pm(t, X) = M_0 \pm u(t, X)$, $M_0 = \max(M/c_0, M_1)$, $M_1 = \sup_{X \in S_0} |u(0, X)|$. We have

$$(14) \qquad \mathcal{F}(w_\pm) = \mathcal{F}(M_0) \pm f(t, X) = c(t, X)M_0 \pm f(t, X) \leqslant 0.$$

By (13) we have

$$(15) \qquad \frac{dw_\pm}{dl} = 0, \quad P \in \sigma_0.$$

By the definition of M_0 we have

$$(16) \qquad w_\pm(0, X) = M_0 \pm u(0, X) \geqslant 0.$$

By means of Theorem 4 of [1] we shall prove the inequality

$$(17) \qquad w_{\pm}(t, X) \geqslant 0, \quad (t, X) \in \bar{D}_0.$$

Indeed, if $u(t, X)$ is not constant in D_0 and if $w_{\pm}(t, X)$ does not satisfy (17), then $w_{\pm}(t, X)$ attains its negative infimum in $\bar{D}_0$. Because of (16) the infimum cannot be attained at any point of $\bar{S}_0$. In view of (15) and in accordance with Theorem 4 of [1] the infimum cannot be assumed at any point of σ_0. By Theorem 3 of [1] the infimum cannot be attained in $\bar{D}_0 \backslash (S_0 \cup \sigma_0)$ either. Therefore if $u(t, X)$ is not constant, then (17) holds, which is equivalent to (12). If $u(t, X)$ is constant, inequality (12) follows directly from (16).

2. Let $f(y)$ be a function integrable in $\langle 0, h\rangle$ and let

$$(18) \qquad g(\lambda) = \int_0^h f(y) e^{-\lambda y} dy, \quad \lambda > 0.$$

The function $g(\lambda)$ given by (18) is called a *Laplace-Picone transform* of $f(y)$.

THEOREM 5. *If there exist a real number $\lambda_0 > 0$, $K \geqslant 0$ and a p such that*

$$(19) \qquad |e^{\lambda h} g(\lambda)| = \Big|\int_0^h e^{\lambda(h-y)} f(y) dy\Big| \leqslant K\lambda^p, \quad \lambda > \lambda_0,$$

then

$$\int_0^\eta f(y) dy = 0, \quad \eta \in \langle 0, h\rangle$$

(*in particular, if $f(y)$ is continuous, then $f(y) \equiv 0$, $y \in \langle 0, h\rangle$*).

The proof is given in [4], p. 352.

Theorem 5 implies immediately the following

THEOREM 6. *If 1° function $f(t, X, y)$ is continuous and bounded in a domain $\bar{D}_0$ (defined in § 1); here we assume that S_0 is a topological product of an $(m-1)$-dimensional domain Ω and an interval $(0, h)$, 2° there exists a positive number λ_0 and real functions $K(t, X) \geqslant 0$ and $p(t, X)$ defined and bounded in $\bar{D}_0$ such that*

$$(20) \qquad \Big|\int_0^h e^{\lambda(h-y)} f(t, X, y) dy\Big| \leqslant K(t, X) \lambda^{p(t,X)}, \quad (t, X) \in \bar{D}_0, \ \lambda > \lambda_0,$$

then the function $f(t, X, y)$ vanishes identically in $\bar{D}_0$.

3. Let Ω be an m-dimensional domain in the space of m real variables $X = (x_1, \ldots, x_m)$. Put $D = \Omega \times (0, h)$. Suppose that to every point $P \in S = F(\Omega) \times \langle 0, h\rangle$ there exists a half-line $l(P)$ penetrating into the interior of D and perpendicular to the axis y. Consider in D a hyperbolic equation of normal type

$$(21)\quad \mathcal{K}(u) \equiv \sum_{i,j=1}^{m+1} a_{ij}(X)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{k=1}^{m+1} b_k(X)\frac{\partial u}{\partial x_k} + c(X)u = f(X, y)$$
$$(x_{m+1} = y)\,.$$

We assume that

(i) $a_{ij}(X) = a_{ji}(X)$, $b_k(X)$ $(i, j, k = 1, \ldots, m+1)$, $c(X), f(X, y)$ are defined and bounded in the closure $\bar{D}$ of D.

(ii) The quadratic form $\sum_{i,j=1}^{m} a_{ij}(X)\xi_i\xi_j$ is positively defined and $a_{m+1,m+1}(X) < 0$ in $\bar{D}$.

Under all the above assumptions we shall prove the following

THEOREM 7. *If* 1° $u(X, y)$ *is a function biregular in* $\bar{D}$ *(i.e.* $u(X, y)$ *is of class* C^2 *in* D *and of class* C^1 *in* $\bar{D}$*) satisfying* (21) *with* $f(X, y) \equiv 0$ *in* $\bar{D}$, 2° $u(x, y) = u'_y(X, y) = 0$ *in* Ω, *and moreover*

$$(22)\qquad u(X, y) = 0 \quad on\ S$$

or

$$(23)\qquad \frac{du}{dl} - \gamma(X)u = 0 \quad on\ S\,,$$

$\gamma(X)$ *being defined and bounded on* S, *then* $u(X, y) \equiv 0$ *in* $\bar{D}$.

Proof. Put

$$(24)\qquad v(X, \lambda) = \int_0^h e^{\lambda(h-y)} u(X, y)\, dy\,.$$

One can easily check that the function $v(X, \lambda)$ satisfies

$$(25)\qquad \sum_{i,j=1}^{m} a_{ij}(X)\frac{\partial^2 v}{\partial x_i \partial x_j} + \sum_{k=1}^{m} \bar{b}_k(X, \lambda)\frac{\partial v}{\partial x_k} + \bar{c}(X, \lambda)v = \bar{f}(X, \lambda)\,,$$

where

$$\bar{b}_k(X, \lambda) = b_k(X) + \lambda a_{k,m+1}(X)\,,$$
$$\bar{c}(X, \lambda) = c(X) + \lambda b_{m+1}(X) + \lambda^2 a_{m+1,m+1}(X)\,,$$
$$\bar{f}(X, \lambda) = -\left[\sum_{i=1}^{m+1} a_{i,m+1}(X)\frac{\partial u}{\partial x_i} + b_{m+1}(X)u + \lambda a_{m+1,m+1}(X)u\right]_{y=h},$$

and moreover

$$(26)\qquad v(X, \lambda) = 0 \quad \text{on } F(\Omega)$$

or

$$\frac{dv}{dl} - \gamma(X)v = 0 . \tag{27}$$

In virtue of 1° and 2° equation (25) is elliptic and $\bar{c}(X, \lambda) \leqslant -c_0 < 0$ for $\lambda > \bar{\lambda}$, $\bar{\lambda}$ being sufficiently large. Since $\bar{f}(X, \lambda)$ is a linear function with respect to λ and its coefficients are functions of X bounded in $\bar{\Omega}$, then there exist numbers λ_0 and $K \geqslant 0$ such that $|\bar{f}(X, \lambda)| \leqslant K\lambda$ for $\lambda > \lambda_0$ and $X \in \Omega$. By Theorem 1 we have

$$|v(X, \lambda)| \leqslant \frac{K}{c_0}\lambda , \qquad X \in \bar{\Omega},\ \lambda > \max(\bar{\lambda}, \lambda_0) . \tag{28}$$

Hence, in view of Theorem 5, we have $u(X, y) \equiv 0$ in $\bar{D}$.

Similarly Theorem 2 implies the following

THEOREM 8. *If* 1° $u(X, y)$ *is a function biregular in* $\bar{D}$ *satisfying* (21) *with* $f(X, y) \equiv 0$ *in* D, 2° $u(X, y) = u'_y(X, y) = 0$ *in* Ω, *and moreover*

$$\frac{du}{dl} = 0 \qquad on\ S \tag{29}$$

(S *being of class* C^2), *then* $u(X, y) \equiv 0$ *in* $\bar{D}$.

Remark 1. Theorem 7 (Theorem 8) implies the uniqueness of the solution of the first or third (of the second) mixed problem for equation (21) in the cylindrical domain D.

COROLLARY 1. *Theorem 7 or Theorem 8 implies the uniqueness of the solutions of the mixed problems for equation* (21) *also in the case of an unbounded cylindrical domain* $D = \Omega \times (0, \infty)$.

Proof. If (X_0, y_0) is an arbitrary point of D, we choose a number h such that $(X_0, y_0) \in D_h = \Omega \times (0, h)$. Then by Theorem 7 or Theorem 8 we have $u(X_0, y_0) = 0$. Hence $u(X, y) \equiv 0$ in D.

4. We shall now give an application of the Laplace-Picone transformation to problems of the uniqueness of solutions of the following equations (cf. [5])

$$\sum_{i,j=1}^{m+1} a_{ij}(t, X)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{k=1}^{m+1} b_k(t, X)\frac{\partial u}{\partial x_k} + c(t, X)u - \frac{\partial u}{\partial t} = f(t, X, y), \tag{30}$$

$$X = (x_1, \ldots, x_m), \qquad x_{m+1} = y, \qquad y \in \langle 0, h\rangle .$$

Equation (30) will be considered in a domain $D = D_0 \times (0, h)$, D_0 being defined in section 1. Assume that to every point $P \in \sigma_0 \times \langle 0, h\rangle$ a half-line $l(P)$ is attached penetrating into the interior of D and perpendicular to the axis y. Further we assume that

1° $a_{ij}(t, X) = a_{ji}(t, X)$, $b_k(t, X)$ $(i, j, k = 1, \ldots, m+1)$, $c(t, X)$ and $f(t, X, y)$ are defined and bounded in the closure $\bar{D}$ of D;

2° the quadratic form $\sum_{i,j=1}^{m} a_{ij}(t, X)\xi_i\xi_j$ is positively defined in $\bar{D}$ and $a_{m+1,m+1}(t, X) < 0$.

We shall consider the following boundary problem for the equation (30): Find a function $u(t, X, y)$ biregular in $\bar{D}$, fulfilling (30) in D and satisfying the following conditions:

$$
(31) \quad \begin{cases} 1^\circ \;\; u(0, X, y) = \varphi(X, y), \quad X \in S_0, \; y \in \langle 0, h\rangle, \\ 2^\circ \;\; \left.\begin{matrix} u(t, X, 0) = \psi_0(t, X) \\ u'_y(t, X, 0) = \psi(t, X) \end{matrix}\right\} \text{ for } (t, X) \in D_0, \\ \left.\begin{matrix} 3^\circ\mathrm{a} \;\; u(t, X, y) = \Phi_1(t, X, y) \quad \text{or} \\ 3^\circ\mathrm{b} \;\; \dfrac{du}{dl} - \gamma(t, X)u = \Phi_2(t, X, y) \end{matrix}\right\} \text{ for } (t, X, y) \in \sigma_0 \times \langle 0, h\rangle . \end{cases}
$$

We assume that the functions $\varphi, \psi_0, \psi, \Phi_1, \Phi_2$ are of class C^1 and satisfy the consistency conditions on the boundaries of the corresponding sets. The function $\gamma(t, X) > 0$ is defined and bounded in $\sigma_0 \times \langle 0, h\rangle$.

THEOREM 9. *Under all the above-mentioned assumptions there exists at most one function $u(t, X, y)$ biregular in $\bar{D}$ and satisfying* (30) *and* (31).

Proof. It is sufficient to prove that the only biregular solution of a homogeneous equation (30) with the homogeneous conditions (31) is the function $u(t, X, y) \equiv 0$ in $\bar{D}$.

After the Laplace-Picone transformation

$$
(32) \qquad v(t, X, \lambda) = \int_0^h e^{\lambda(h-y)} u(t, X, y)\, dy
$$

the homogeneous equation (30) assumes the form

$$
(33) \quad \sum_{i,j=1}^{m} a_{ij}(t, X)\frac{\partial^2 v}{\partial x_i \partial x_j} + \sum_{k=1}^{m} \bar{b}_k(t, X)\frac{\partial v}{\partial x_k} + \bar{c}(t, X)v - \frac{\partial v}{\partial t} = \bar{f}(t, X, \lambda),
$$

where

$$
\bar{b}_k(t, X) = b_k(t, X) + \lambda a_{k,m+1}(t, X),
$$
$$
\bar{c}(t, X) = c(t, X) + \lambda b_{m+1}(t, X) + \lambda^2 a_{m+1,m+1}(t, X),
$$
$$
\bar{f}(t, X, \lambda) = -\left[\sum_{i=1}^{m+1} a_{i,m+1}(t, X)\frac{\partial u}{\partial x_i} + b_{m+1}(t, X)u + \lambda a_{m+1,m+1}(t, X)u\right]_{y=h},
$$

and the homogeneous boundary conditions (31) assume the form

$$(34)\qquad \begin{cases} 1^\circ\ v(0, X, \lambda) = 0 \quad \text{for} \quad X \in S_0, \\ \left.\begin{aligned} &2^\circ\ v(t, X, \lambda) = 0 \quad \text{or} \\ &3^\circ\ \frac{dv}{dl} - \gamma(t, X)v = 0 \end{aligned}\right\} \quad \text{for } (t, X) \in \sigma_0. \end{cases}$$

However, (33) is a normal parabolic equation with homogeneous boundary conditions (34). By the assumptions on the coefficients of (30) there exists a $\lambda_1 > 0$ such that $\bar{c}(t, X) \leqslant -c_0 < 0$ as $\lambda > \lambda_1$ and $(t, X) \in \bar{D}_0$. Analogously one can verify that there exist a $\lambda_0 > 0$ and a $K \geqslant 0$ such that $|\bar{f}(t, X, \lambda)| \leqslant K\lambda$ for $\lambda > \lambda_0$ in $\bar{D}_0$. In view of Theorem 3 (since $M_1 = 0$) we get the inequality

$$(35)\qquad |v(t, X, \lambda)| \leqslant \frac{K}{c_0}\lambda, \qquad (t, X) \in \bar{D}_0, \ \lambda > \max(\lambda_0, \lambda_1).$$

Hence, by Theorem 5, it follows that $u(t, X, y) \equiv 0$ in $\bar{D}$.

Similarly, using Theorem 4, one can prove the following

THEOREM 10. *If* 1° σ_0 *is a hypersurface of class* C^2, 2° *the coefficients of* (30) *and the domain* D *satisfy the assumptions of Theorem* 9, *then there exists one function* $u(t, X, y)$ *at most biregular in* $\bar{D}$ *and satisfying* (30), *the boundary conditions* 1° *and* 2° *of* (31) *and the condition*

$$(36)\qquad \frac{du}{dl} = \Phi_3(t, X, y) \quad \text{for} \quad (t, X, y) \in \sigma_0 \times \langle 0, h \rangle,$$

$\Phi_3(t, X, y)$ *being a function of class* C^1 *in* $\sigma_0 \times \langle 0, h \rangle$.

COROLLARY 2. *Theorem* 9 (*or Theorem* 10) *implies the uniqueness of the solution of the boundary problem* (31) (*or* 1° *and* 2° *of* (31) *and* (36)) *for equation* (30) *also in the case of unbounded* $D = D_0 \times (0, +\infty)$ (cf. Corollary 1).

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