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ADMISSIBLE OPERATIONS ON SAMPLE SPACES OVER THE FREE ORTHOGONALITY MONOID

BY

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1. Introduction. A sample space is a fundamental tool in the study, by C. H. Randall and D. J. Foulis, of the mathematical structure of operational statistics (see [1] through [5]). Such a sample space consists of an orthogonality space, representing the outcome set of some well defined scientific procedure, together with an identification of the operations which may be carried out in the various possible executions of this procedure. These are called the *admissible operations* for the sample space.

Upon a given sample space may be constructed the free orthogonality monoid (FOM) representing the possible outcomes of finite collections of "primitive" operations executed in sequence. Viewed in this light, the FOM becomes a sample space provided that the admissible operations are specified. In this paper, three collections of operations, $\mathscr{B}$, $\mathscr{B}_s$, and $\mathscr{B}_0$, are exhibited over the FOM, and it is shown that each forms a set of admissible operations. $\mathscr{B}$ will be shown to be too general in that it allows no dispersion-free weight functions to be defined on the FOM. This leaves $\mathscr{B}_s$ and $\mathscr{B}_0$ as the more appropriate choices. Moreover, while $\mathscr{B}_0 \subset \mathscr{B}_s$, it may be shown by example that $\mathscr{B}_0$ is not in general of sufficient size to allow for all operations which should be admissible from a physical standpoint. In the general case, therefore, $\mathscr{B}_s$ becomes the more natural choice. In many applications, $\mathscr{B}_s$ and $\mathscr{B}_0$ coincide. When two elements of a sample space represent distinct possible outcomes of some admissible operation, then we say that these two elements are *orthogonal* to each other, a relationship which is evidently symmetric and anti-reflexive.

An *orthogonality space* $(X, \perp)$ consists of a non-empty set X equipped with a symmetric anti-reflexive binary relation $\perp$. If $W \subset X$, then $W^{\perp} = \{x \in X;\ x \perp w \text{ for every } w \in W\}$, $W^{\perp\perp} = (W^{\perp})^{\perp}$, etc. If $x \in X$, then $x^{\perp} = \{x\}^{\perp}$. A set $V \subset X$ is said to be *orthogonal* if whenever $v, w \in V$ and $v \neq w$, then $v \perp w$. Let $\mathscr{O}(X, \perp) = \{V \subset X;\ V \text{ is orthogonal}\}$, $\mathscr{E}(X, \perp)$

$= \{V \in \mathcal{O}(X, \perp);\ V^{\perp} = \emptyset\}$. Note that $\mathcal{E}(X, \perp)$ is the collection of all *maximal orthogonal sets* (with respect to set inclusion). A subset S of X is called a *scattered set* if $s, t \in S$ imply $s \notin t^{\perp}$. Let $S(X, \perp)$ denote the family of all *maximal scattered sets* (under set inclusion). Notice that $S \in S(X, \perp)$ iff for $x \in X$, $x^{\perp} \cap S = \emptyset$ implies that $x \in S$. $(X, \perp)$ is a *scattered space* if $X \in S(X, \perp)$.

Definition 1 (Foulis-Randall). A *sample space* is a triple $(X, \perp, \mathcal{A})$, where $(X, \perp)$ is an orthogonality space and $\mathcal{A} \subset \mathcal{E}(X, \perp)$ satisfies:

(a) $\bigcup \mathcal{A} = X$.

(b) If $A \subset E \in \mathcal{A}$, $B \subset F \in \mathcal{A}$, and $A \subset B^{\perp}$, then there exists $G \in \mathcal{A}$ such that $A \cup B \subset G$.

Members of $\mathcal{A}$ will be called *admissible operations* for the sample space.

Let $(X, \#)$ be any orthogonality space, and let Γ denote the free monoid (free semigroup with unit 1) over the set X. We call elements a, b of Γ *orthogonal*, and write $a \perp b$, if there exist elements $c, d, e \in \Gamma$, and $x, y \in X$, where $x \# y$ and $a = cxd$, $b = cye$. With this relation (called *lexicographic orthogonality*), $(\Gamma, \perp)$ is an orthogonality space called the *free orthogonality monoid* (*FOM*) *over* $(X, \#)$. Henceforth in this paper, the notation $(\Gamma, \perp)$ will denote the above-given construction. We shall also assume that $(X, \#)$ (and hence also $(\Gamma, \perp)$) is not a scattered space.

Definition 2. Let $A, B \subset \Gamma$ and let $c, d \in \Gamma$. Then:

(a) $AB = \{ab \in \Gamma;\ a \in A, b \in B\}$, $dB = \{d\}B$.

(b) $c \leqslant d$ iff $c \in d\Gamma$. If $c \leqslant d$ and $c \neq d$, then $c < d$.

(c) $A^0 = \{a \in A;\ a < e \Rightarrow e \notin A\}$ and $A^- = \{a \in A;\ e < a \Rightarrow e \notin A\}$.

(d) If $a = x_1 x_2 \ldots x_n \in \Gamma$, where $x_i \in X$ for $i = 1, \ldots, n$, then we define $|a| = n$. If $a = 1$, then $|a| = 0$.

$|a|$ is called the *length* of a. For $A \subset \Gamma$, the *length* of A is given by $|A| = \sup\{|a|;\ a \in A\}$. If $|A| < +\infty$, we say that A is *bounded*.

(e) $c^{-1}A = \{e \in \Gamma; ce \in A\}$, $I(A) = \{b \in \Gamma; b^{-1}A \neq \emptyset\}$, $I'(A) = I(A) \setminus A$, and $i(A) = I(A) \cap X$.

(f) A sequence $\{b_n\}_{n=0}^{\infty} \subset I(A)$ is said to be *deeply penetrating* in A if, for all non-negative integers i, j, we have $b_j < b_i$ whenever $i < j$. If A admits no deeply penetrating sequences, then A is said to be *shallow* in Γ.

2. Sample spaces over the free orthogonality monoid.

Definition 3. Let $(X, \#, \mathcal{A})$ be a sample space. If $(\Gamma, \perp)$ is the FOM over $(X, \#)$, then we define subsets $\mathcal{B}$, $\mathcal{B}_s$, and $\mathcal{B}_0$ of $\mathcal{E}(\Gamma, \perp)$ as follows:

(a) $E \in \mathcal{B}$ iff: (i) $E = E^0 \neq \emptyset$ and (ii) $a \in I'(E) \Rightarrow i(a^{-1}E) \in \mathcal{A}$.

(b) $\mathcal{B}_s = \{E \in \mathcal{B};\ E$ is shallow in $\Gamma\}$.

(c) $\mathcal{B}_0 = \{E \in \mathcal{B};\ E$ is bounded in $\Gamma\}$.

LEMMA 4. *Suppose $A \in \mathcal{O}(\Gamma, \perp)$, $1 \notin A \neq \emptyset$. Suppose also that for each $c \in I'(A)$ we can choose $Q_c \in \mathcal{A}$, where $i(c^{-1}A) \subset Q_c$. Then let $M = \bigcup_{c \in I'(A)} cQ_c$ and define $E = M^-$. Then $A \subset E$ and $E \in \mathcal{B}$. Furthermore, if A is shallow (respectively, bounded), then $E \in \mathcal{B}_s$ (respectively, $E \in \mathcal{B}_0$).*

Proof. Let $E = M^-$. If $a \in A$, then, by assumption, $a \neq 1$, so that we may write $a = bx$, where $b \in I'(A)$ and $x \in i(b^{-1}A) \subset Q_b$. Thus $a \in M$. But $a \notin I'(A)$ so that if $g < a$, then $g \notin M$. Hence $a \in M^- = E$. Thus $A \subset E$. We continue by showing that $E \in \mathcal{B}$. Clearly, $E \neq \emptyset$, since $\emptyset \neq A \subset E$. Also, $E = M^- = (M^-)^0 = E^0$. Suppose now that $g \in I'(E)$. We need only show that $i(g^{-1}E) \in \mathcal{A}$. Now $i(g^{-1}E) \neq \emptyset$ since $g \in I'(E)$. Thus we may choose $x \in i(g^{-1}E)$. Then there exists $b \in \Gamma$ such that

$$gxb \in E = M^- \subset M = \bigcup_{c \in I'(A)} cQ_c.$$

Thus, for some $f \in I'(A)$, $w \in Q_f$, we have $gxb = fw$. Thus $f \leqslant g$, so that $g \in I'(A)$. If $b = 1$, then $gx \in M$ and $x \in Q_g$. If $b \neq 1$, then $b = dw$, where $gxd = f \in I'(A)$. Hence for some $e \in \Gamma$, $gxde \in A$, so that $x \in i(g^{-1}E) \subset Q_g$. In any case then we have shown that $x \in Q_g$, and hence that $i(g^{-1}E) \subset Q_g$. We now must show therefore that $Q_g \subset i(g^{-1}E)$. Let $z \in Q_g$. Then $gz \in gQ_g \subset M$. If $gz \notin M^-$, then for some $d \in \Gamma$ and $y \in X$, $gzdy \in M$. Thus $gzd \in I'(A)$. It follows then that $z \in i(g^{-1}A) \subset i(g^{-1}E)$. On the other hand, if $gz \in M^- = E$, then we have $z \in i(g^{-1}E)$ certainly. In any event we may conclude that $Q_g \subset i(g^{-1}E)$, so that equality holds, and hence $i(g^{-1}E) = Q_g \in \mathcal{A}$ whenever $g \in I'(E)$. Thus, by (3), $E \in \mathcal{B}$. It is easily seen that if A is shallow in Γ, then $E = M^-$ is also; i.e., $E \in \mathcal{B}_s$; and if A is bounded, then so is E, and so $E \in \mathcal{B}_0$.

THEOREM 5. *$(\Gamma, \perp, \mathcal{B})$, $(\Gamma, \perp, \mathcal{B}_s)$ and $(\Gamma, \perp, \mathcal{B}_0)$ are all sample spaces.*

Proof. Certainly $\mathcal{B}$, $\mathcal{B}_s$ and $\mathcal{B}_0$ are subsets of $\mathcal{E}(\Gamma, \perp)$. We will show hat in addition they satisfy conditions (a) and (b) of (1).

(a) Note that $\mathcal{B}_0 \subset \mathcal{B}_s \subset \mathcal{B}$, so that if we can show that $\bigcup \mathcal{B}_0 = \Gamma$, then we have condition (a) of definition 1 for all three. Suppose $a \in \Gamma$. If $a = 1$, then $a \in \{1\} \in \mathcal{B}_0$. Suppose $a \neq 1$. Then let $A = \{a\}$. Now using (4) we see that there exists $E \in \mathcal{B}_0$, where $a \in E$. Thus $\Gamma \subset \bigcup \mathcal{B}_0$, and (since the other inclusion certainly holds) we have equality.

(b) We work first on $\mathcal{B}$. Suppose $C \subset E \in \mathcal{B}$, $D \subset F \in \mathcal{B}$ and $C \subset D^\perp$. We need to find $G \in \mathcal{B}$ such that $C \cup D \subset G$. If $C \cup D \subset \{1\}$, then let $G = \{1\}$, and we have a satisfactory G in this case. Assume $1 \notin C \cup D \neq \emptyset$ and let $k \in I'(C \cup D)$. Then $\emptyset \neq i(k^{-1}(C \cup D)) = i(k^{-1}C) \cup i(k^{-1}D)$. Now, since $C \subset E$ and $D \subset F$, we have $i(k^{-1}C) \subset i(k^{-1}E)$, $i(k^{-1}D) \subset i(k^{-1}F)$ and at least one of $i(k^{-1}E)$, $i(k^{-1}F)$ is non-empty, and hence is in $\mathcal{A}$. Thus by (b) of definition 1 there exists $G_k \in \mathcal{A}$ such that $i(k^{-1}C) \cup i(k^{-1}D) \subset G_k$.

Define such a G_k for each k in $I'(C \cup D)$, and let

$$G = \Big(\bigcup_{k \in I'(C \cup D)} kG_k\Big)^{-}.$$

By (4), $C \cup D \subset G \in \mathscr{B}$, and hence $(\Gamma, \perp, \mathscr{B})$ is a sample space. Note also that if C, D are shallow (respectively, bounded), then by (4) so is G. Thus $(\Gamma, \perp, \mathscr{B}_s)$ and $(\Gamma, \perp, \mathscr{B}_0)$ are sample spaces also.

3. Dispersion properties of FOM sample spaces. Let $(X, \perp, \mathscr{A})$ be any sample space. Then $S \in S(X, \perp)$ is *$\mathscr{A}$-dispersed* if $E \cap S \neq \emptyset$ whenever $E \in \mathscr{A}$. $S_d(X, \perp, \mathscr{A})$ will denote the collection of all $\mathscr{A}$-dispersed sets. Also, we will say that $E \in \mathscr{A}$ is *co-dispersed* if $E \cap S \neq \emptyset$ for all $S \in S(X, \perp)$. $\mathscr{A}_{cd}$ will denote the collection of all co-dispersed admissible operations. $(X, \perp, \mathscr{A})$ is called an *F-sample space* if $\mathscr{A}_{cd} = \mathscr{A}$.

THEOREM 6. *In* $(\Gamma, \perp)$, $\mathscr{B}_{cd} \subset \mathscr{B}_s$.

Proof. Suppose $E \in \mathscr{B}_{cd} \setminus \mathscr{B}_s$. Then there exists a sequence $P = \{b_n\}_{n=0}^{\infty}$ which is deeply penetrating in E. Then P is a scattered set so that we may choose $S \in S(\Gamma, \perp)$ such that $P \subset S$. Since E is co-dispersed, $E \cap S \neq \emptyset$. Let $a \in E \cap S$. Then since P is not bounded, there exists $b \in P \subset S$ such that $|a| < |b|$. But $b \in I(E)$, so that $bc \in E$ for some $c \in \Gamma$. Thus $bc \perp a$, since $a \in E$ and $a \neq bc$. Hence $a \in (bc)^{\perp} = b^{\perp} \cup bc^{\perp}$. Since $|a| < |b|$, we have $a \notin bc^{\perp}$. Hence $a \in b^{\perp}$, which contradicts the assumption that $a, b \in S$. Thus $\mathscr{B}_{cd} \setminus \mathscr{B}_s = \emptyset$, so that $\mathscr{B}_{cd} \subset \mathscr{B}_s$.

LEMMA 7. *Let* $S \in S(\Gamma, \perp)$. *Then*:

(a) $i(S) \subset I(S) = S$.

(b) $i(S) \in S(X, \#)$.

(c) $a \in S \Rightarrow a^{-1}S \in S(\Gamma, \perp)$.

Proof. (a) Clearly, $i(S) \subset I(S)$ and $S \subset I(S)$. If $b \in I(S)$, then there exists $c \in \Gamma$, where $bc \in S$. Suppose $d \in S \cap b^{\perp}$. Then $d \in S$ and $d \in b^{\perp} \subset (bc)^{\perp}$. But $bc \in S$, so we contradict that S is scattered. Hence $S \cap b^{\perp} = \emptyset$, so we must have $b \in S$. Thus $I(S) \subset S$.

(b) $i(S) = I(S) \cap X = S \cap X \subset S$. Hence $i(S)$ is a scattered subset of $(X, \#)$. Suppose now that $w \in X$ and $i(S) \cap w^{\#} = \emptyset$. If $b \in S \cap w^{\perp}$, then $b \neq 1$, and we may factor b, where $b = yc$, $c \in \Gamma$, and $y \in i(S) \cap w^{\#} = \emptyset$, a contradiction. Hence $S \cap w^{\perp} = \emptyset$, so $w \in S \cap X = i(S)$. Thus $i(S) \in S(X, \#)$.

(c) Suppose $a \in S$ but $a^{-1}S \notin S(\Gamma, \perp)$. Then there exists $g \in \Gamma$ such that $g^{\perp} \cap a^{-1}S = \emptyset$ but $g \notin a^{-1}S$. Hence, since $ag \notin S$, we have $(ag)^{\perp} \cap S \neq \emptyset$, so that

$$\emptyset \neq (a^{\perp} \cup ag^{\perp}) \cap S = (a^{\perp} \cap S) \cup (ag^{\perp} \cap S).$$

But $a \in S$, and whence $a^{\perp} \cap S = \emptyset$. Thus $\emptyset \neq ag^{\perp} \cap S$. Letting $b \in ag^{\perp} \cap S$, we have $b = ac$, where $c \in g^{\perp}$ and $c \in a^{-1}S$. Hence $g^{\perp} \cap a^{-1}S \neq \emptyset$, which is a contradiction. Thus $a^{-1}S \in S(\Gamma, \perp)$.

THEOREM 8. *Suppose* $(X, \#, \mathscr{A})$ *is an F-sample space, and let* $\mathscr{I}$ *be any subset of* $\mathscr{B}_s$ *such that* $(\Gamma, \perp, \mathscr{I})$ *is a sample space. Then* $(\Gamma, \perp, \mathscr{I})$ *is an F-sample space also.*

Proof. After (6) we need only prove that $\mathscr{B}_s \subset \mathscr{B}_{cd}$ to show that $(\Gamma, \perp, \mathscr{B}_s)$ forms an F-sample space. Suppose that $E \in \mathscr{B}_s \setminus \mathscr{B}_{cd}$. Then there is an $S \in S(\Gamma, \perp)$, where $S \cap E = \emptyset$. Let $b_0 = 1$. Then $b_0 \in S$, so that $b_0 \notin E \neq \emptyset$. Thus $b_0^{-1}S \neq \emptyset \neq b_0^{-1}E$. Suppose now that, for some integer $n \geqslant 0$, $b_0, b_1, \ldots, b_n$ have been defined, where $b_0 > b_1 > \ldots > b_n$ and $b_n^{-1}S \neq \emptyset \neq b_n^{-1}E$. Then $b_n^{-1}S \in S(\Gamma, \perp)$ and $b_n^{-1}E \in \mathscr{B}$, and so $i(b_n^{-1}S) \in S(X, \#)$, and $i(b_n^{-1}E) \in \mathscr{A} = \mathscr{A}_{cd}$. Thus there exists $x_{n+1} \in X$ such that $x_{n+1} \in i(b_n^{-1}S) \cap i(b_n^{-1}E)$. Therefore there exist $f, g \in \Gamma$, where $b_n x_{n+1} f \in S$, $b_n x_{n+1} g \in E$. Let $b_{n+1} = b_n x_{n+1}$. Then we have $b_{n+1}^{-1}S \neq \emptyset \neq b_{n+1}^{-1}E$ and $b_i > b_{n+1}$ for $i = 0, 1, \ldots, n$. Thus we may continue to define the b_j's inductively arriving at a sequence $\{b_i\}_{i=0}^{\infty}$ which is deeply penetrating in E. This contradicts the assumption that E is shallow in Γ. Therefore we have the inclusion that we need to show that $(\Gamma, \perp, \mathscr{B}_s)$ is an F-sample space. Now if $\mathscr{I} \subset \mathscr{B}_s$ and $(\Gamma, \perp, \mathscr{I})$ forms a sample space, then since all of the operations in $\mathscr{B}_s$ are co-dispersed, certainly they all are in $\mathscr{I}$ also. Thus the theorem is proved.

LEMMA 9. *Suppose* $(X, \perp, \mathscr{A})$ *is any sample space which is not scattered. Then if* $S \in S(X, \perp)$, *there exists* $s \in S$ *such that* $s^{\perp} \neq \emptyset$.

Proof. Suppose for all s in S, $s^{\perp} = \emptyset$. Since X is not scattered, there exists $x \in X$ such that $x^{\perp} \neq \emptyset$. If $s \in S \cap x^{\perp}$, then $x \in s^{\perp} = \emptyset$, a contradiction. Thus $S \cap x^{\perp} = \emptyset$, and since $S \in S(X, \perp)$ we have $x \in S$. But this is a contradiction since $x^{\perp} \neq \emptyset$.

THEOREM 10. *If* $S \in S(\Gamma, \perp)$, *then there exists* $E \in \mathscr{B}$ *such that* $E \cap S = \emptyset$. *Consequently,* $S_d(\Gamma, \perp, \mathscr{B}) = \emptyset$.

Proof. If $S \in S(\Gamma, \perp)$, then there exists a sequence $P = \{b_n\}_{n=0}^{\infty}$ which is deeply penetrating in S. We may assume that $|b_j| = j$ for all $j = 0, 1, \ldots$, and if $b_k = b_{k-1}x_k$, where $x_k \in X$, then we may assume by (9) that $x_k^{\#} \neq \emptyset$ for $k = 1, 2, \ldots$ Now, for each k, choose $V_k \in \mathscr{A}$ such that $x_k \in V_k$, and define $W_k = V_k \setminus \{x_k\}$. Now let $E = \bigcup_{k=1}^{\infty} b_{k-1} W_k$. Clearly $E \in \mathscr{E}(\Gamma, \perp)$, and if $c \in I'(E)$, then for some k, $c = b_k \in P$ and $\emptyset \neq i(c^{-1}E) = V_k \in \mathscr{A}$, so that $E \in \mathscr{B}$. Suppose that $d \in E \cap S$. Then, for some positive integer k, $d = b_{k-1}w$, where $w \in W_k$. Thus $w \# x_k$, so that $d = b_{k-1}w \perp \perp b_{k-1}x_k = b_k$. But $b_k \in I(S) = S$, so that $d \notin S$ since S is scattered, a contradiction. Thus $E \cap S = \emptyset$.

4. Dispersion-free weight functions. If $(X, \perp, \mathscr{A})$ is any sample space, then a real-valued function $\omega\colon X \to [0, 1]$ is called a *weight function* if, for each $E \in \mathscr{A}$, $\sum_{e \in E} \omega(e) = 1$.

Of special interest in the present context are those weight functions which are *dispersion-free*, meaning that their range is contained in the two-element set $\{0, 1\}$. Let $\Omega(X, \perp, \mathscr{A})$ denote the collection of all weight functions on $(X, \perp, \mathscr{A})$ and let $\Omega_{df}(X, \perp, \mathscr{A})$ denote those weight functions which are dispersion-free. It is easily seen that there is a one-to-one correspondence between the sets $S_d(X, \perp, \mathscr{A})$ and $\Omega_{df}(X, \perp, \mathscr{A})$, where each $\mathscr{A}$-dispersed maximal scattered set in X corresponds to its own characteristic function. In the FOM, we make the further observation that if $S \in S_d(\Gamma, \perp, \mathscr{B}_0)$ and if $a \in S$, then $a^{-1}S \in S_d(\Gamma, \perp, \mathscr{B}_0)$. From this it is easily shown that $S_d(\Gamma, \perp, \mathscr{B}_0) = S_d(\Gamma, \perp, \mathscr{B}_s)$, and hence $\Omega_{df}(\Gamma, \perp, \mathscr{B}_0) = \Omega_{df}(\Gamma, \perp, \mathscr{B}_s)$. Further, by (6), any operation in $\mathscr{B} \setminus \mathscr{B}_s$ is not co-dispersed, and hence $\mathscr{B}_s$ is the largest subset of $\mathscr{B}$ on which we can define dispersion-free weight functions. Together with (10), this discussion gives the following result:

THEOREM 11. *$(\Gamma, \perp, \mathscr{B})$ admits no dispersion-free weight functions whatsoever. Furthermore, if $\mathscr{B}_0 \subset \mathscr{I} \subset \mathscr{B}$ and $(\Gamma, \perp, \mathscr{I})$ is a sample space, where $\Omega_{df}(\Gamma, \perp, \mathscr{I}) \neq \emptyset$, then $\mathscr{I} \subset \mathscr{B}_s$ and $\Omega_{df}(\Gamma, \perp, \mathscr{I}) = \Omega_{df}(\Gamma, \perp, \mathscr{B}_0)$.*

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