ZASTOSOWANIA MATEMATYKI
APPLICATIONES MATHEMATICAE
Hugo Steinhaus Jubilee Volume
X (1969)

M. KALECKI (Warszawa)

ON THE CHOICE OF TECHNIQUES OF PRODUCTION IN A SOCIALIST ECONOMY

1. The present essay deals with the problem mentioned in the title under rather severely restrained conditions. Nevertheless the results we arrive at are not devoid of practical significance. They bring out the essentials of the problem and may serve as a point of departure for establishing more complicated methods applicable directly to the actual choice of techniques.

We make the following assumptions. The economy consists of a number of branches. For each of these branches there is fixed a plan of continuous expansion of production at a constant rate r for the same long period $(0, T)$. The rate of growth r is in general different for each branch. In addition the plan includes for each branch a constant coefficient of scrapping a: the production P_t is reduced in the interval $(t, t+dt)$ due to scrapping of old equipment characterised by the highest unit costs of labour by $aP_t dt$. We also assume that in the existing equipment no changes either in production or in labour costs occur other than the scrapping referred to above. Thus production out of new investment in a given branch in the interval $(t, t+dt)$ is equal to $(r+a)P_t dt$.

The increment $(r+a)P_t dt$ may be realized by investment variants representing different techniques, i.e. by various combinations of investment outlays (valued at constant prices) and labour costs (valued at constant wages). The variants to choose from are, of course, subject to change as a result of technical progress: while new variants come into existence those becoming absolutely ineffective, i.e. involving higher investment but no lower labour costs or vice versa, are discarded. With regard to the use of raw materials we abstract from technical progress. We also assume that the variants considered at a given time do not differ as far as raw material costs are concerned. This is in line with the fixed program of all branches of production some of which produce raw materials for the others.

It should be noted, however, that the problem re-emerges with regard to different volumes of investment resulting from the choice of techniques.

We shall make the assumption that within certain limits shifts between output of investment goods and that of consumption goods may be made without disturbing significantly the branch program of production. (In practice to solve this problem a method of successive approximations shall have to be applied.)

Let us denote by i_t investment outlays per unit of the production $(r+a)P_t dt$ and the unit labour cost by c_t. The variants at time t are represented by various combinations (i_t, c_t). Let us denote the highest unit labour cost in the existing equipment by x_t and the aggregate cost of labour entering the branch considered by $Y_t dt$. We have

$$(r+a)P_t dt\cdot c_t - aP_t dt\cdot x_t = Y_t dt. \tag{1}$$

The net increment of production in the interval $(t, t+dt)$ being $rP_t dt$ the investment outlay per unit of this increment is $i_t(r+a)/r$ and the addition to the labour cost of the branch per such a unit is according to formula (1) equal to

$$\frac{Y_t}{r\cdot P_t} = c_t\frac{r+a}{r} - x_t\cdot\frac{a}{r}.$$

It should be noticed here that as long as the scrapping of obsolete equipment concerns only the production capacities existing at time 0, the value x_t is determined by the rate of expansion of production r of the branch considered and by the scrapping coefficient a. On the contrary, (i_t, c_t) is to be chosen from the variants existing at time t. However, when equipment installed in the period $(0, T)$ begins to be scrapped, x_t is not fully determined by r and a, because it depends on the past choice of techniques as well.

We denote the sum of Y_t taken for all branches by W_t. This is nothing else but the demand for the additional labour by all branches of production expressed in terms of additional aggregate labour costs (valued at constant wages).

2. We shall now try to solve the problem of minimizing the aggregate investment in all branches of production $I_t = \Sigma(r+a)P_t i_t$ with a given new supply of labour at each moment of the period $(0, T)$. In the first stage we shall treat the highest unit labour cost in the old equipment x_t as given as well. In the second stage of the argument we shall show that on certain assumptions the condition that x_t is given may be dropped, and that as a result the level of investment arrived at by our procedure is likely to constitute an aproximate minimum at any time t of the interval $(0, T)$ with postulated r and a for each branch of production and a given time curve of the aggregate increment of the supply of labour to all branches.

We start by introducing the concept of the index of efficiency of investment E_t, being a linear function of investment and labour cost per unit of the increment of production $r \cdot P_t dt$:

$$E_t = \varepsilon_{i_t} \frac{r+a}{a} + c_t \frac{r+a}{r} - x_t \frac{a}{r}, \tag{2}$$

where ε is a positive parameter whose value is the same for all branches. We proceed by choosing for a given ε the technique (i_t, c_t) on the basis of the criterion

$$E_t = \text{minimum}, \tag{3}$$

adopting in the case of the equality of E_t for two techniques that one for which i_t is lower. It should be noticed that a, r and x_t being given this criterion is equivalent to

$$\varepsilon_{i_t} + c_t = \text{minimum}. \tag{4}$$

Let us denote the variant selected by (i'_t, c'_t), and the respective E_t by E'_t. We have thus by definition:

$$E'_t \leqslant E_t. \tag{5}$$

Consequently, taking the sum over all branches we have

$$\sum E'_t r P_t dt \leqslant \sum E_t r P_t dt$$

but, according to (1) and (2), we have

$$\sum E_t r P_t dt = \varepsilon \sum (r+a) P_t i_t dt + \sum Y_t dt = \varepsilon (I_t + W_t). \tag{6}$$

We obtain thus

$$\varepsilon I'_t + W'_t \leqslant \varepsilon I_t + W_t, \tag{7}$$

where I'_t and W'_t correspond to the variants selected in each branch on the basis of the criteria (3) or (4).

3. A change in the parameter ε influences, obviously, the selection of the variants of investment. If to ε there corresponds in some branch an E'_t which is equal to the E_t of another variant with a higher i_t, even an arbitrarily small decline in ε, which we shall denote by $\Delta\varepsilon$, will cause a shift to the latter variant. Indeed, the indices of the efficiency of investment will be now $E'_t - \Delta\varepsilon \cdot i'_t$ and $E_t - \Delta\varepsilon \cdot i_t$, the latter expression being less because $i_t > i'_t$.

Thus, at certain values of ε_k (which we range in increasing order) a change in (I'_t, W'_t) will occur in the sense that to ε_k there corresponds still $(I'_{t,k}, W'_{t,k})$, but to $\varepsilon_k - \Delta\varepsilon$ already $I'_{t,k-1} > I'_{t,k}$ and $W'_{t,k-1} < W'_{t,k}$. It follows that $I'_{t,k}$ is a decreasing sequence, and $W'_{t,k}$ an increasing one.

It follows directly from (7) that

$$(\varepsilon_k - \Delta\varepsilon) I'_{t,k-1} + W'_{t,k-1} \leqslant (\varepsilon_k - \Delta\varepsilon) I'_{t,k} + W'_{t,k}$$

and

$$\varepsilon_k I'_{t,k-1} + W'_{t,k-1} \geqslant \varepsilon_k I'_{t,k} + W'_{t,k}.$$

Hence, taking into consideration $I'_{t,k-1} - I'_{t,k} > 0$, we have

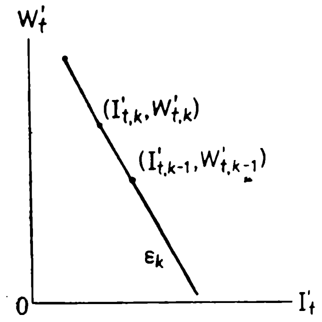

Fig. 1

$$\varepsilon_k - \Delta\varepsilon \leqslant \frac{W'_{t,k} - W'_{t,k-1}}{I'_{t,k-1} - I'_{t,k}} \leqslant \varepsilon_k,$$

wherefrom we obtain

$$\frac{W'_{t,k} - W'_{t,k-1}}{I'_{t,k-1} - I'_{t,k}} = \varepsilon_k \tag{8}$$

because of the arbitrariness of $\Delta\varepsilon$.

It follows that $(I'_{t,k}, W'_{t,k})$ may be represented by vertices of a downward sloping concave polygonal line. The slope of the side $(I'_{t,k}, W_{t,k}) \to (I'_{t,k-1}, W'_{t,k-1})$ is equal (in absolute value) to ε_k (see Fig. 1).

4. Let us still ponder over the significance of "inside" points of the sides of the polygonal line, e.g. over the point C (see Fig. 2). Denoting the ratio BC/AB by λ we have

$$OC_1 = \lambda \cdot OA_1 + (1-\lambda) \cdot OB_1 \quad \text{and} \quad CC_1 = \lambda \cdot AA_1 + (1-\lambda) \cdot BB_1.$$

It will be easily seen that if in all branches there are introduced "mixed variants" in the sense of achieving the new production $\lambda(r+a)P_t dt$ by means of the variant $(i'_{t,A}, c'_{t,A})$, corresponding to ε_A, and the new production $(1-\lambda)(r+a)P_t dt$ by means of the variant $(i'_{t,B}, c'_{t,B})$, corresponding to ε_B, then the aggregate investment is equal to OC_1 and the aggregate additional demand for labour (valued at constant wages) is equal to C_1C. Thus, after the introduction of "mixed variants" the relationship between I'_t and W'_t is represented by the polygonal line which is a continuous curve.

W'_t is the *demand* for new labour corresponding to the aggregate investment I'_t. Now, as assumed above, at every t of the period $(0, T)$ we face a definite supply of new labour (valued at constant wages) which we shall denote by S_t. Drawing a horizontal at a distance S_t from the abscissa axis we find at its point of intersection G with the polygonal line the aggregate investment required OG_1 (see Fig. 2). This just happens to be the lowest level of aggregate investment at time t compatible with the new labour supply S_t, given for each branch the rate of expansion of production r, the coefficient of scrapping a, and the highest unit cost of labour in the existing equipment x_t. We denote it by I'_t.

The slope of EF yields the value of the parameter ε_E, the slope of FH — that of the parameter ε_F, and the ratio EG/EF — the value of λ. Thus the choice of variants in particular branches of production is also fully determined. (If the polygonal line is constructed precisely, ε_E and ε_F are actually not necessary for the purpose because the branch variants corresponding to ε_E and ε_F are known directly as components of $(I'_{t,E}, W'_{t,E})$ and $(I'_{t,F}, W'_{t,F})$. If, however, in practice an approximate polygonal line is applied, e.g. that established a few years back, ε_E and ε_F will still be required; indeed, while we face new variants, ε_E and ε_F will still be approximately correct from the point of view of the balance of labour.)

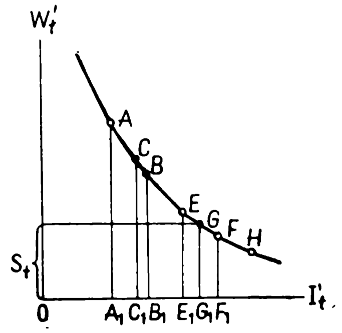

Fig. 2

5. We have now come to the end of the first stage of the argument where we assumed the highest unit cost of labour in the old equipment as given. At present we are embarking on the second stage where we shall show that this condition may be dropped on certain assumptions. We assume now that as a result of technical progess

(i) c'_t of the variants selected in the period $(0, T)$ is lower than any unit cost of labour in the establishments existing at time 0;

(ii) c'_t actually selected is a diminishing function of time in that period;

(iii) the ratio $\frac{1}{r} \log \frac{r+a}{a}$ is the same for all branches.

It follows from the assumption (i) that the equipment existing at time 0 will be scrapped prior to any piece of equipment put to use in the period $(0, T)$. The time τ it will take is easy to determine. We have first of all $P_t = P_0 e^{rt}$.

Next, since in the period $(0, \tau)$ we scrap the productive capacity P_0, we have

$$(9) \qquad P_0 = \int_0^\tau aP_0 e^{rt} dt = P_0 \frac{a}{r}(e^{r\tau}-1).$$

Hence

$$(10) \qquad (r+a)/a = e^{r\tau}$$

and

$$(11) \qquad \tau = \frac{1}{r} \log \frac{r+a}{a}.$$

In the period $(0, \tau)$ the value of x_t is fully determined by the rate of expansion of production r and by the coefficient of scrapping a. Thus in this period the polygonal line at time t is determined by the plan of production, the coefficients of scrapping and the variants of productive techniques available at time t.

In period (τ, T) the position is more complicated because x_t depends here on the choice of the variants in the past. We shall prove, however, on the basis of assumption (ii) that the polygonal line at time t will also here be determined by r, a, the investment variants at this time, and by the accretions to the supply of labour in the past.

It follows from (10) that

$$\frac{P_t}{P_{t-\tau}} = \frac{r+a}{a}$$

or

$$aP_t = (r+a)P_{t-\tau}. \tag{12}$$

Thus, the productive capacities scrapped in the interval $(t, t+dt)$, where $t > \tau$, are equal to the production out of equipment installed in the interval $(t-\tau, t-\tau+dt)$. Since, according to assumption (ii), c'_t actually selected is a diminishing function of t, the highest unit cost of labour x_t is also a diminishing function of t in the period (τ, T). From this and equation (12) it follows that in the interval $(t, t+dt)$ of that period there is scrapped exactly this piece of equipment that was installed in the interval $(t-\tau, t-\tau+dt)$, and thus

$$x_t = c'_{t-\tau} \tag{13}$$

where $c'_{t-\tau}$ corresponds to the new supply of labour at time $t-\tau$, i.e. to $S_{t-\tau}$.

Now, taking into consideration formula (1), we have

$$W'_t = \Sigma Y'_t = \Sigma(r+a)P_t c'_t - \Sigma a P_t x_t$$

but, according to (12) and (13)

$$\Sigma a P_t x_t = \Sigma(r+a)P_{t-\tau}c'_{t-\tau}$$

and according to (1) is

$$\Sigma a P_t x_t = \Sigma(r+a)P_{t-\tau}c'_{t-\tau} = \Sigma a P_{t-\tau}x_{t-\tau} + S_{t-\tau}. \tag{14}$$

Thus, finally,

$$W'_t = \Sigma(r+a)P_t c'_t - S_{t-\tau} - \Sigma a P_{t-\tau}x_{t-\tau}. \tag{15}$$

It follows from this formula that for t in the interval the polygonal line is determined by r and a for all branches, by the choice of the variants (i'_t, c'_t) according to the criterion (4), by the accretion to the supply of labour $S_{t-\tau}$ at time $t-\tau$, and finally by the value of $x_{t-\tau}$. But the latter, $t-\tau$ being in the interval $(0, \tau)$, is determined by r and a. Thus the polygonal line at time t is fully determined by the production plan, by the scrapping coefficients, by the variants of techniques available at time t, and by the accretion to the supply of labour at time $t-\tau$.

By means of equations (14) and (15) it is easy to show in a similar fashion that the polygonal line at time t in the interval $(k\tau, (k+1)\tau)$ is fully determined by the production plan, the scrapping coefficients, the variants available at time t, and the accretions to the supply of labour at times $t-k\tau, t-(k-1)\tau, \ldots, t-\tau$, i.e. by $S_{t-k\tau}, S_{t-(k-1)\tau}, \ldots, S_{t-\tau}$.

Thus the polygonal line at any time t in the period $(0, T)$ is determined by the rates r of expansion of production of the branches, by the coefficients of scrapping, by the time curve of accretions to the supply of labour in the period $(0, T)$, and by the variants of the techniques of production available at time t. It is, however, independent of the past choice of the investment variants. The same is obviously true of the point (I'_t, S'_t) of the line actually chosen at time t.

In this way we have proved that the level of investment I'_t arrived at by the procedure described in sections 2, 3 and 4 is a minimum not only in the situation given at time t, but also the lowest level compatible with the production plan for the period $(0, T)$ and the scrapping coefficients as well as the time curve of accretions to the supply of labour and the variants of productive techniques available throughout the same period. In other words, our method (based on the assumptions (i), (ii) and (iii) of this section) secures minimum investment at any t of the period $(0, T)$. It is true that these assumptions, especially the third one, will not be in fact generally fulfilled; however, the proof of the theorem makes it likely that minimum investment will be approximately achieved throughout the period $(0, T)$.

6. In fact, we have now solved the problem which was posed. We shall, however, make still some observations on the subject of coefficients of scrapping a. Indeed, there arises the problem how these are fixed. One alternative is to maintain a at the level existing just before the time 0; another is the government's decision on a new list of these coefficients.

In the first instance if the rates of expansion r are unchanged, to the investment variants existing at time 0 there corresponds in the interval $(0, dt)$ the same polygonal line as in the interval $(-dt, 0)$.

In the second instance this will not be in general the case. It should be noticed that even if the shift of the polygonal line will lead to a lower aggregate investment I'_0, this may not be the case for some $t > 0$ of the period $(0, T)$ because the change in a affects directly the time cruve of the highest unit labour cost x_t in the old equipment. To each list of scrapping coefficients there corresponds an optimum time curve of investment in the period $(0, T)$ arrived at by the method outlined above. But in general these curves intersect each other so that there need not be among them one that lies for all t of the interval $(0, T)$ below the other ones, thus constituting a clear-cut *optimum optimorum*. Thus a government decision on the subject may become necessary whenever there is a conflict between the short-period and long-run level of investment, and thus that of consumption.

7. Another question — also associated actually with the concept of scrapping coefficients being given for a branch of production — may be asked. As follows from section 2, the choice of investment variants at time t is made in fact on the strength of the criterion (4), being determined (according to sections 3 and 4) by the polygonal line and the accretion to the supply of labour at time t. (The problem is somewhat complicated by the necessity of the application of "mixed variants", but this does not affect the basic issue.) As will be seen, the whole process of the selection of variants is not influenced by the factor of "durability" of equipment. If e. g. we consider two variants characterized by the same value of $\varepsilon i_t + c_t$, we choose by the convention adopted in section 2 that one where i_t is lower — without the factor of "durability" coming into the picture. It should be kept in mind, however, that the problem of "durability" is mainly associated with obsolescence resulting from technical progress. Thus "durability" in this sense is rather an economic than a technical parameter to be chosen one way or another. If we adopt for a branch as a whole a coefficient of scrapping a, we scrap per annum the same proportion of productive capacities whether the stock of equipment is more or less "durable" according to habitual ideas. In this way the "paradox" of disregarding "durability" is rooted in the postulated *modus operandi* of the authorities which consists in scrapping the equipment characterized by highest labour cost constituting a given proportion of the total productive capacity of a branch.

Received on 18. 9. 1967